Digit Tiling of Euclidean Space

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Abstract. This is an expository paper on digit tiling of Euclidean space, a special kind of self-affine tiling by translates of a single tile. In particular, the following topics are discussed: the construction of digit tiles and the construction of the boundary, the Hausdorff dimension of the boundary, the relation between digit tiles and positional number systems, the self-replicating properties of digit tiling, and lattice and crystallographic digit tiling. In the last sections digit tiling is placed into the broader context of both periodic and nonperiodic self-affine tiling of Euclidean space by a finite set of proto-tiles. In particular, the following topics are discussed: general results on hierarchical tiling, results specific to self-affine and self-similar tiling, the construction of self-affine and self-similar tilings using graph iterated function systems, and some illustrative examples.

1. Introduction

Self-similar tilings of $\mathbb{R}^d$ have attracted the interest of mathematicians in recent years for a variety of reasons that are discussed in this paper. One primary reason, especially relevant in the context of this volume, is that many of these tilings are "quasiperiodic" and serve as models for real quasicrystals. The discovery of quasicrystals in 1984 [**SBGC**] was the impetus for, not just intensified research on tilings, but for much of the recent work on the mathematics of long-range aperiodic order. In this paper there is a shift of emphasis between the first and second parts. Sections 1-7 deal mainly with periodic tilings; sections 8-10 mainly with nonperiodic tilings. It is worth noting that self-similar tilings are a relatively recent addition to the large body of work on the geometry and symmetry of tilings, a topic surveyed, beginning with the mosaics in the Alhambra at Granada in Spain, in the book [**GS**] by Grünbaum and Shephard.

Two compact sets in $\mathbb{R}^d$ are said to be *non-overlapping* if their interiors are disjoint. A *tiling* of $\mathbb{R}^d$ is a decomposition of $\mathbb{R}^d$ into non-overlapping compact sets, each the closure of its interior and each with boundary having Lebesgue measure 0.

This paper is organized as follows. The definitions of self-affine and self-similar tile and, in particular, digit tile are given in §2. Digit tiles possess a self-similar property like that of the "Gosper flowsnake" shown in Figure 1. The union of the seven tiles is similar to each small tile. The construction of digit tiles in §2 is by way of iterated function systems, a standard method for constructing fractals. The boundary of a digit tile is usually fractal.

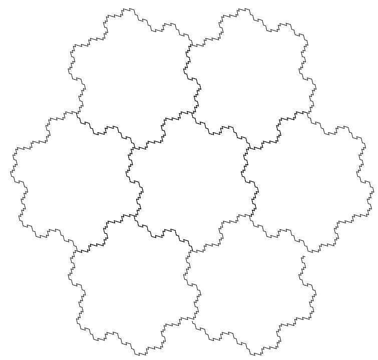


FIGURE 1. Gosper flowsnake tiling.

The construction of a digit tile $T := T(A, D)$ depends only on an expanding matrix A and a finite set D of lattice points in $\mathbb{R}^d$. The terminology "digit tile" comes from this data, which is analogous to the usual base and digits used to represent the integers. This connection to positional number systems (radix systems) is discussed in §3. In particular, the tiling in Figure 1 is related to a certain radix system with applications to image processing.

Sections 2 and 3 concern the digit tiles themselves; §4 concerns tilings by translates of a single digit tile. Every digit tile admits a tiling of $\mathbb{R}^d$ with a strong global property called self-replication. When this self-replicating tiling is a lattice tiling there are applications to the construction of wavelet bases. The main theorem of §4 gives ten conditions, all equivalent to the self-replicating lattice tiling property. One of the conditions is measure theoretic; some concern the behavior of the boundary; some concern unique radix representation of lattice points; and some are algorithmic, allowing for efficient testing procedures. One such condition involves the A-adic numbers, a generalization of the p-adics, p prime.

The boundary of a digit tile is the subject of §5 and §6. In the book *Classics on Fractals* [**E**], Edgar asked what the Hausdorff dimension of the boundary of the Lévy Dragon might be. In general Edgar asked what could be said about the dimension of the boundary of a self-similar tile. In §5 an easily computable formula is provided for the Hausdorff dimension of the boundary of a self-similar digit tile.

Section 6 concerns the construction of the boundary of a digit tile. The recurrent set method for constructing a fractal curve, due to Dekking [**De1, De2**], is related to L-systems, the "L" for Lindenmayer who used the method to model biological growth [**Lin**]. Given an alphabet and a rewriting rule, the idea is to iterate the rule to produce progressively longer strings of symbols. Each symbol is then interpreted geometrically, producing a figure in the plane. The main theorem in §6 gives a bijection between the parameters used to construct a digit tile in $\mathbb{R}^2$ by an iterated function system and the parameters used to construct a curve by the recurrent set method. The bijection is such that the curve constructed by the recurrent set method is the boundary of the tile constructed by the iterated function system. Figure 1 in this paper was constructed by the recurrent set method.

It is an open question as to whether every tile T that admits a tiling of $\mathbb{R}^d$ by translates of T also admits a periodic tiling by translates of T. Related to this question is the *Lattice Tiling Question* of Gröchenig and Haas [**GH**]: every digit tile admits a tiling of $\mathbb{R}^d$ by translates; does every digit tile admit a (not necessarily self-replicating) lattice tiling? This question and its solution by Lagarias and Wang [**LW4**] are discussed in §7. It is also open whether every tile T that admits a tiling of $\mathbb{R}^2$ (not necessarily by translates) also admits a periodic tiling by copies of T. In other words, does there exist an aperiodic proto-tile? Gummelt's solution of the analogous problem for coverings of $\mathbb{R}^2$ [**Gu**], where tiles are allowed to overlap, has received considerable attention recently because of its implications for the structure of quasicrystals. This is also discussed briefly in §7.

Sections 1 - 7 are restricted to digit tiling. The remaining three sections concern generalizations. The intent is to place digit tiling into a broader context. The following topics are briefly discussed: (§8) crystallographic digit tiling; (§9) hierarchical tiling; and (§10) self-affine and self-similar tiling by copies of tiles taken from a finite set of proto-tiles.

Crystallographic digit tiling, due to Gelbrich [**Ge1**], is a generalization from tilings by the image of single tile under the action of a lattice group to tilings by a single tile under the action of a crystallographic group.

Sections 9 and 10 extend the subject from tiling by copies of a single tile to tilings by copies of tiles taken from a finite set of proto-tiles. These include the nonperiodic Penrose tilings [**P1**] and the Pisot tilings of Thurston [**Th**]. We introduce the notion of a hierarchy. Associated with a given hierarchy $\mathcal{P}$ are hierarchical tilings, called $\mathcal{P}$-tilings. General properties of hierarchies and their tilings are discussed in §9. Included are results about codes of $\mathcal{P}$-tilings, number of tilings, nonperiodicity, and quasiperiodicity. Two special types of hierarchies are self-affine and self-similar hierarchies. Miscellaneous properties of their tilings are surveyed in §10. A constructive approach to self-affine and self-similar tiling, by way of graph iterated function systems, is also discussed.

Many of the tilings in §10 are quasiperiodic and serve as models for real quasicrystals. Except for brief comments in §7 and on X-ray diffraction in §10.2, the physics of quasicrystals is not discussed. For an introductory account of quasicrystals we refer the reader to M. Senechal's book [**Se**] and M. Baake's paper [**Ba**]. For current research trends refer to the papers in [**M**] and in this volume.

This paper is basically expository. Proofs of theorems that are readily found elsewhere are omitted. The subject of self-similar sets is vast. The paper is not intended to be comprehensive, and we apologize for any favorite topics or results that are omitted.

2. Digit Tiles

The systematic study of self-similarity properties goes back at least to 1964. Golomb [**Go**] defined a set T in the plane to be *rep-N* if T can be tiled by N congruent similar sets. Three rep-4 figures are shown in Figure 2. These examples are somewhat misleading because the boundary of a rep-N figure is often fractal.

Fractal tiles were constructed early on, for example, by Mandelbrot [**Ma**] and in the replicating superfigures of Giles [**Gil**]. But perhaps the best known method of constructing fractals at this time is by iterated function systems [**Hu**] . Many of the illustrations of fractals in the popular literature use this method; see, for example, the nice expositions by Barnsley [**Bar**] and Falconer [**F1**]. An *iterated*

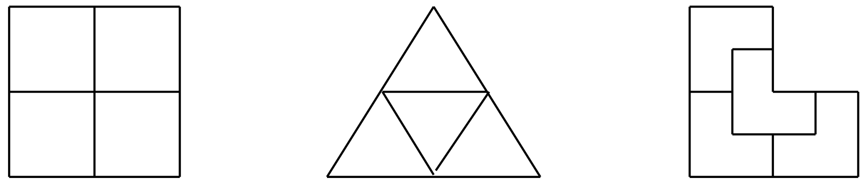


FIGURE 2. Rep-4 figures.

function system (IFS) is a finite set $\{f_1, f_2, \dots, f_N\}$ of contractions:

$$f_i : \mathbb{R}^d \to \mathbb{R}^d.$$

A function $f : \mathbb{R}^d \to \mathbb{R}^d$ is a *contraction* if there is a number c with $0 < c < 1$ such that $|f(x) - f(y)| \le c|x - y|$ for all $x, y \in \mathbb{R}^d$. Let $\mathcal{C}(\mathbb{R}^d)$ denote the collection of all nonempty compact subsets of $\mathbb{R}^d$. The Hausdorff metric h on $\mathcal{C}(\mathbb{R}^d)$ is defined as follows:

$$h(A, B) = \inf\{\epsilon \,|\, A \subset B_\epsilon \text{ and } B \subset A_\epsilon\},$$

where $A_\epsilon = \{x \in \mathbb{R}^d \,:\, |x - y| \le \epsilon \text{ for some } y \in A\}$. With respect to this metric the function

$$F : \mathcal{C}(\mathbb{R}^d) \to \mathcal{C}(\mathbb{R}^d)$$

$$F(X) = \bigcup_{i=1}^{N} f_i(X)$$

is a contraction on the complete metric space $\mathcal{C}(\mathbb{R}^d)$ and thus, by the contraction mapping theorem, has a unique fixed point or *attractor* T that satisfies

$$T = \bigcup_{i=1}^{N} f_i(T). \tag{2.1}$$

There is an alternative representation for the attractor given by

$$T = \lim_{n\to\infty} F^{(n)}(T_0), \tag{2.2}$$

where $F^{(n)}$ denotes the n^{th} iterate of F and T_0 is an arbitrary compact subset of $\mathbb{R}^d$. The limit is with respect to the Hausdorff metric. The set

$$T_n = F^{(n)}(T_0) \tag{2.3}$$

is an n^{th} approximation to T and is easy to express in algorithmic form. It is usually such an algorithm (or a randomized version) that is used to produce the fractal graphics that appear in many books and papers on the subject.

Consider the special case of an IFS where the contractions f_i are affine with the same linear part A^{-1} and with translational parts $D = \{d_1, d_2, \dots, d_N\}$:

$$f_i(x) = A^{-1}(x + d_i). \tag{2.4}$$

Let A be an expanding matrix, where *expanding* means that the modulus of each eigenvalue is greater than 1. With respect to an appropriate metric related to the Euclidean metric [**Li**], A^{-1} is a contraction. The inverse is used merely as a convenience for stating certain results. The functional equation (2.1), for example, is equivalent to

$$A(T) = \bigcup_{i=1}^{N} (T + d_i). \tag{2.5}$$

Let m denote d-dimensional Lebesgue measure and ∂ the boundary. If

1. the attractor T is the closure of its interior and $m(\partial T) = 0$ and
2. the union in Eq. (2.5) is non-overlapping,

then T is called a *self-affine tile*. If, in addition, A is a similarity, then T is called a *self-similar tile*. A linear map is a *similarity* with expansion factor c if $\|Ax\| = c\|x\|$ for some $c > 1$ and for all $x \in \mathbb{R}^d$. The term self-affine refers to the geometric interpretation of Eq. (2.5): the large tile $A(T)$ is the non-overlapping union of translates of the small tile T. The Gosper flowsnake in Figure 1 is a self-similar tile. Figure 3 shows the first 12 approximations to the self-similar "twin dragon," where the IFS is given in Example 2.1 and T_0 in Eq. (2.3) is a unit square.

EXAMPLE 2.1. **Twin dragon.**

$$f_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$$

$$f_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

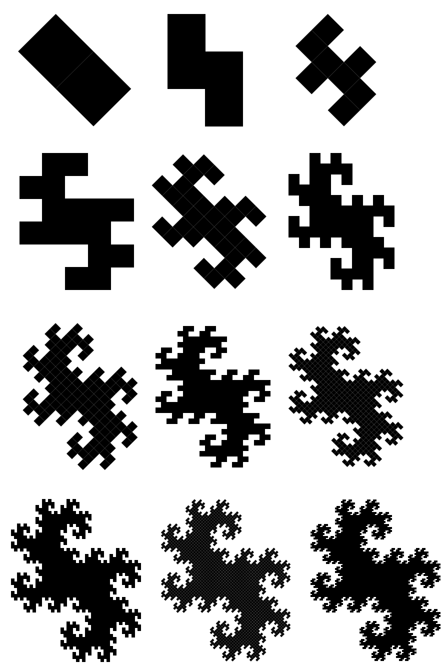


FIGURE 3. Approximations to the twin dragon.

The attractor of the IFS (2.4) usually does not satisfy conditions (1) and (2) in the definition of self-affine tile. One case for which it does is a digit tile. Such tiles have been the subject of research by, among others, Bandt [**B2**], Dekking [**De1, De2**], Gelbrich [**BGe, Ge2**], Gröchenig and Haas [**GH**], Gröchenig and Madych [**GM**], Kenyon [**Ke2**], Lagarias and Wang [**LW2, LW3, LW4**], Solomyak [**So1, So2**], Strichartz [**Str**] and Vince [**V1, V2, V3, V4**]. By a *lattice* in $\mathbb{R}^d$ is meant the set of all integer linear combinations of d linearly independent vectors. If A is a linear map and L is a lattice, we say that L is A-*invariant* if $A(L) \subset L$. If, for some expanding matrix A, there exists a lattice L, invariant under A, then a set D of coset representatives of the quotient $L/A(L)$ is called a *digit set*. It is assumed that $0 \in D$. By standard results in algebra, for D to be a digit set it is necessary that

$$|D| = |\det A|.$$

If A is expanding and $D = \{d_1, \ldots, d_N\}$ is a digit set, then the attractor of the affine IFS in Eq. (2.4) is called a *digit tile*. Note that a digit tile is completely determined by the pair (A, D) and will be denoted $T(A, D)$. Theorem 2.5 below states that a digit tile is indeed a self-affine tile. Figures 1 and 3 are self-similar digit tiles, based on the hexagonal and integer lattices, respectively. Both of these tiles are homeomorphic to a disk. Topologically more complicated self-similar digit tiles (Examples 2.2, 2.3, 2.4) appear in Figure 4. The last example in this figure shows the large tile as the non-overlapping union of the nine small tiles.

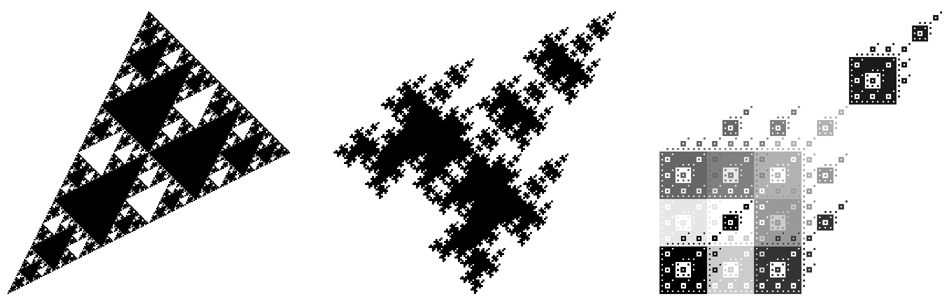


Figure 4. Gasket, rocket and shooter.

Example 2.2. **Gasket.**

$$\begin{aligned} A &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ D &= \{(0,0), (1,0), (0,1), (-1,-1)\} \end{aligned}$$

Example 2.3. **Rocket.**

$$\begin{aligned} A &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\ D &= \{(0,0), (1,1), (2,2), (-1,0), (-2,0), (-1,1), (0,-1), (0,-2), (1,-1)\} \end{aligned}$$

EXAMPLE 2.4. **Shooter.**

$$
\begin{aligned}
A &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\
D &= \{(0,0),(1,0),(2,0),(0,1),(0,2),(2,2),(4,4),(2,1),(1,2)\}
\end{aligned}
$$

What we call a digit set D, Lagarias and Wang [**LW3**] call a *standard* digit set. They call D *nonstandard* if $|D| = |\det A|$ but D is not a set of coset representatives of $L/A(L)$ for any lattice L. For example, in 1-dimension $D = \{0,1,8,9\}$ is nonstandard for matrix $A = (4)$. The attractor of the corresponding IFS is $[0,1] \cup [2,3]$. For most nonstandard digit sets D, however, the attractor T has Lebesgue measure 0; in particular, the interior of T is empty. For example, if $|\det A|$ is prime, then this is always the case [**LW3**]. In general, it seems a nontrivial problem to determine whether a nonstandard digit tile has positive Lebesgue measure. For (standard) digit tiles this is not an issue; a proof of the follow result can be found in [**GH, LW2, V1**].

THEOREM 2.5. *A digit tile T is a self-affine tile. Namely T is compact; T is the closure of its interior; $m(\partial T) = 0$; and the union in Eq. (2.5) is non-overlapping.*

For ease of exposition and with essentially no loss of generality, we will often make the following three assumptions concerning digit tiles. A pair (A,D), consisting of an expanding matrix A and a digit set D, will be called *basic* if the following three statements hold. In this case the tile $T(A,D)$ is also called *basic*.

1. A is an integer matrix and D is a set of coset representatives of $\mathbb{Z}^d/A(\mathbb{Z}^d)$.
2. (A,D) is pure.
3. (A,D) is primitive.

By assumption (1) the invariant lattice is the integer lattice. By *pure* is meant that 0 is contained in the interior of $T(A,D)$. By *primitive* is meant that D is contained in no proper A-invariant sublattice of $\mathbb{Z}^d$. Example 2.6 is a pair (A,D) in $\mathbb{R}^2$ that is not primitive, and Figure 5 shows the first three approximations to the corresponding digit tile $T(A,D)$, which is a square. Note that the sublattice of $\mathbb{Z}^2$ consisting of all lattice points with even coordinate sum is a proper A-invariant sublattice.

EXAMPLE 2.6. Non-primitive digit tile.

$$
\begin{aligned}
A &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\
D &= \{(0,0),(2,0),(1,-1),(-1,1),(1,1),(3,1),(0,2),(2,2),(1,3)\}
\end{aligned}
$$

That little loss of generality is incurred by restricting to basic digit tiles is the statement of the following result [**LW2, V1**].

THEOREM 2.7. *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a linear expanding map, L an A-invariant lattice, and D a digit set. There exists a basic pair (A',D') such that A' is similar to some power of A and $T(A',D') = \phi(T(A,D))$, where ϕ is an invertible affine map. Moreover, if $L = \mathbb{Z}^d$, then "similar to" can be replaced by "equal to", and if (A,D) is already primitive, then ϕ is just a translation.*

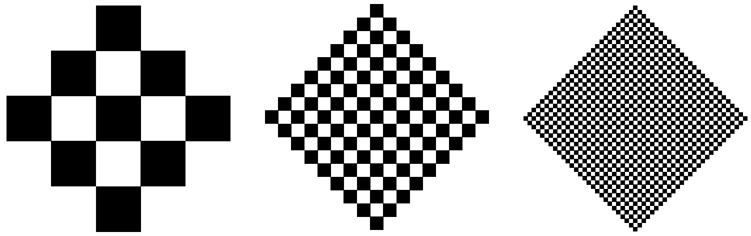


FIGURE 5. Non-primitive digit tile.

It is usually the case that a digit tile is not homeomorphic to a ball. Bandt and Gelbrich call two digit tiles T and T' *isomorphic* if there is an affine bijection ϕ from T to T' that preserves pieces at all levels. For a precise definition of "preserving the pieces" see [**BGe**].

THEOREM 2.8. *(Bandt and Gelbrich* [**BGe**]*, Gelbrich* [**Ge2**]*)*

1. *For any $N \geq 2$ there are finitely many isomorphism classes of digit tiles in $\mathbb{R}^2$ with N pieces and homeomorphic to a disk.*
2. *There are finitely many isomorphism classes of digit tiles in $\mathbb{R}^d$ with 2 pieces and homeomorphic to a d-ball.*

The authors have determined, for example, that there are three types of disk-like digit tiles in $\mathbb{R}^2$ with two pieces and seven types with three pieces. In [**GH**] Gröchenig and Haas give, in the 2-dimensional case, a sufficient condition on the pair (A, D) for $T(A, D)$ to be connected. In any dimension, Hacon, Salanha and Veerman [**HSV**] prove that any digit tile with two pieces is connected.

3. Radix Representation

To justify the terminology "digit" tile, consider an expanding matrix A as a base for an A-invariant lattice L (so $A(L) \subset L$), and a digit set $D \subset L$ as a set of digits for L. Use the Minkowski sum notation $X + Y = \{x + y \,|x \in X, \, y \in Y\}$ and $A(D) = \{A(d) \,|\, d \in D\}$, and let

$$D_n = \sum_{i=0}^{n-1} A^i(D) \quad \text{and} \quad D_\infty = \bigcup_{i=1}^{\infty} D_n. \tag{3.1}$$

Then D_n is the subset of the lattice that can be expressed using at most n digits, and D_∞ is the set of lattice points that can be expressed using any finite sequence of digits. Let the initial approximation to the tile $T = T(A, D)$ be a single point: $T_0 = \{0\}$. In this case Eq. (2.2) becomes

$$T := T(A, D) = \lim_{n\to\infty} \sum_{i=1}^{n} A^{-i}(D), \qquad (3.2),$$

where the limit is with respect to the Hausdorff metric. Then $T(A, D)$, according to Eq. (3.2), is the set of points in $\mathbb{R}^d$ that can be expressed using digits only to the "right" of the of the decimal point. In particular, consider the 1-dimensional case where $A = (10)$, $L = \mathbb{Z}$, and $D = \{0, 1, \dots, 9\}$. Then D_n is the set of integers that can be represented in the ordinary base 10 system using at most n digits; D_∞ is the set of non-negative integers; and T is the closed interval $[0, 1]$. For the above reasons we refer to the pair (A, D) as a *radix system* (positional number system) for lattice L.

The representation of numbers using positional number systems has an extensive literature prior to the advent of fractals. Knuth's classic [**Kn**] contains early references dating back to Cauchy, who noted that negative digits make it unnecessary for a person to memorize the multiplication table past 5×5. Gilbert [**Gi1, Gi2, Gi3**] considered radix representation for the Gaussian integers $\mathbb{Z}[i] = \{a + b\,i \,|\, a, b \in \mathbb{Z}\}$ and for integers in other algebraic number fields. For example, every Gaussian integer has a unique base $\beta = -1 + i$ representation of the form $\sum_{i=0}^{n} d_i \beta^i$, where $d_i \in D = \{0, 1\}$. This is analogous to a binary system for the Gaussian integers. Of course, this is just the radix system (A, D) for the lattice $\mathbb{Z}^2 = \mathbb{Z}[i]$, where A is the linear map given by $Ax = \beta x$.

Two obvious questions concerning radix representation are as follows.

QUESTION 3.1. Given an expanding integer matrix A and digit set D, when is it the case that every lattice point x has a unique representation of the form

$$x = \sum_{i=0}^{n-1} A^i(e_i), \; e_i \in D. \tag{3.3}$$

QUESTION 3.2. Given an expanding integer matrix A, does there exist a digit set D such that every lattice point x can be uniquely represented in the radix form (3.3).

Question 3.1 will be addressed as part of Theorem 4.2 in §4. The answer to Question 3.2 is "no, but almost." We mention two particular results. Here I is the identity matrix.

THEOREM 3.3. *(Vince* [**V1**]*) If* $\det(I - A) = \pm 1$*, then there is no digit set* D *such that every lattice point* x *has a unique representation of the form (3.3).*

Examples of such matrices include

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & a \\ 0 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -a \\ 1 & a \end{pmatrix}.$$

Recall that for any matrix A there exist orthogonal matrices U and V such that $U^T A V$ is diagonal [**GvL**]. The diagonal entries are called the *singular values of* A. Let C denote the canonical fundamental domain of the the origin with respect to the cubic lattice (the closure of C is a unit cube centered at the origin).

THEOREM 3.4. *(Vince* [**V1**]*) Let* A *be a* d*-dimensional matrix and* L *an* A*-invariant lattice. If the singular values of* A *are greater than* $3\sqrt{d}$ *and* $D = A(C) \cap L$*, then every lattice point* x *has a unique representation of the form (3.3). In the 1 and 2-dimensional cases, the bound* $3\sqrt{d}$ *can be improved to 2.*

The following previously known result follows directly from the two theorems above.

COROLLARY 3.5. For any Gaussian integer $\beta \in \mathbb{Z}[i]$, except 0, ± 1, $\pm i$, 2 and $1 \pm i$, there is a digit set D such that every Gaussian integer has a unique radix representation of the form $\sum_{i=0}^{n-1} e_i \, \beta^i, \quad e_i \in D$. No such digit set exists for $\beta = 2$ and $\beta = 1 \pm i$.

It follows from Eq. (3.2) that

$$T = \lim_{n\to\infty} A^{-n}(D_n);$$

so it should not be surprising that properties of the set D_∞ on a large scale are directly related to properties of the digit tile T on a small scale. The following theorem is an example. A set $S \in \mathbb{R}^d$ is *uniformly discrete* if there is a bound $r > 0$ such that distinct $x, y \in S$ satisfy $|x - y| \geq r$. Note that, in this theorem, it is not assumed that D is a digit set.

THEOREM 3.6. *(Lagarias and Wang* [**LW2**]*) Assume that A is a real expanding matrix with $|\det A| = m \in \mathbb{Z}$ and D a subset of $\mathbb{R}^d$ with $|D| = m$ and $0 \in D$. Then the following statements are equivalent.*

1. *$T := T(A, D)$ is the closure of its interior and $m(\partial T) = 0$.*
2. *All m^n elements of D_n are distinct and D_∞ is a uniformly discrete set.*

We now consider in more detail a radix system, called the generalized balanced ternary, which has applications to image processing. Consider a monic polynomial $f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$. In the quotient ring $\Lambda_f = \mathbb{Z}[x]/(f)$ let $\alpha = x+(f)$. Then Λ_f has the structure of a $\mathbb{Z}$-module with basis $(1, \alpha, \alpha^2, \dots, \alpha^{d-1})$. In other words Λ_f is a lattice which can be realized (in many ways) in $\mathbb{R}^d$ by embedding the d basis elements as d linearly independent vectors in $\mathbb{R}^d$.

If $f(x)$ is irreducible over $\mathbb{Z}$ then, as rings, $\Lambda_f = \mathbb{Z}[x]/(f) \cong \mathbb{Z}[\alpha]$ where α is any root of $f(x)$ in an appropriate extension field of the rationals. For example, if $f(x) = x^2 + 1$ then the lattice Λ_f is the ring of Gaussian integers $\mathbb{Z}[i]$ with basis $(1, i)$ and can be realized as the square lattice in the complex plane.

Consider the special case $f(x) = 1 + x + x^2 + \cdots + x^d$. Let $\omega = x + (f)$. In the ring $\Lambda_{\mathbf{d}} = \mathbb{Z}[x]/(f)$ we have $1 + \omega + \cdots + \omega^d = 0$ and $\omega^{d+1} = 1$. For the sake of symmetry we take as a generating set for the lattice $\Lambda_{\mathbf{d}}$ the set $(1, \omega, \omega^2, \dots, \omega^d)$ although it is linearly dependent. Embed the lattice $\Lambda_{\mathbf{d}}$ in d-dimensional Euclidean space by defining an inner product on pairs of basis elements $(1, \omega, \omega^2, \dots, \omega^d)$ by

$$(\omega^i, \omega^j) = \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{d} & \text{if } i \neq j. \end{cases}$$

In dimension $d = 1$ this is the integer lattice; for $d = 2$ it is the hexagonal lattice; and for $d = 3$ it is the lattice that consists of the centers of the tiling of space by truncated octahedra. In general it is the dual of the classical d-dimensional root lattice $\mathbf{A}_d$; so the weight lattice $A_d^* = \Lambda_{\mathbf{d}}$ [**CS**]. Now let $\beta = 2 - \omega$ and define a linear expanding map

$$A_\beta : \Lambda_{\mathbf{d}} \to \Lambda_{\mathbf{d}}$$

by

$$A_\beta(\mathbf{x}) = \beta\mathbf{x}.$$

Although not well-defined, a matrix for A_β with respect to the generating set $(1, \omega, \omega^2, \dots, \omega^d)$ is

$$A_\beta = \begin{pmatrix} 2 & 0 & 0 & \dots & 0 & -1 \\ -1 & 2 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2. \end{pmatrix}$$

Let

$$D_\beta = \{\epsilon_0 + \epsilon_1\omega + \epsilon_2\omega^2 + \cdots + \epsilon_d\omega^d \,:\, \epsilon_i \in \{0,1\}\},$$

where not all $\epsilon_i = 0$. Note that $|D_\beta| = 2^{d+1} - 1$. It can be shown that D_β is a digit set for A_β with respect to the lattice $\Lambda_{\mathbf{d}}$. Hence (A_β, D_β) is a radix system for lattice $\Lambda_{\mathbf{d}}$, called the *generalized balanced ternary* (GBT). Moreover, if follows from a variant of Theorem 4.9 that every lattice point in $\Lambda_{\mathbf{d}}$ has a unique representation in the GBT.

In dimension 1

$$A_\beta = (3) \qquad\qquad D = \{-1, 0, 1\}.$$

This is a base 3 system classically called the *balanced ternary.* Every integer (positive or negative) can be uniquely expressed base three using the three digits (trits). For $d = 2$ the digit set D_β is the subset of the hexagonal lattice consisting of the origin and all 6^{th} roots of unity. The corresponding tile $T(A_\beta, D_\beta)$ is shown in Figure 1. The GBT radix system has been suggested for spatial addressing of images as a viable alternative to a rectangular grid - for both geometric reasons (the round shape of the pixels) and algebraic reasons (the efficient algorithmic properties of the radix system [**GL, KVW, vR**]).

4. Self-Replicating Tiling

Let A be an expanding integer matrix, D a digit set, and assume that the pair (A, D) is basic. Section 2 concerns the self-affine tile $T(A, D)$. The term "tile" was used, rather than "set", because, given any self-affine tile T, there always exists a tiling of $\mathbb{R}^d$ by translates of T. To see this, iterate functional equation (2.5) to obtain

$$A^n(T) = \bigcup_{d \in D_n} (T + d).$$

Since (A, D) is pure, 0 lies in the interior of $A^n(T)$. Since A is an expansion, any ball centered at the origin lies in $A^n(T)$ for some n. In the notation of (3.1), the sets $D_1 \subset D_2 \subset \dots$ are nested because $0 \in D$ and hence

$$\mathcal{T}_\infty := \{T + d \mid d \in D_\infty\} \tag{4.1}$$

is a tiling of $\mathbb{R}^d$.

In fact, $\mathcal{T}_\infty$ is a special type of tiling by digit tiles, called a self-replicating tiling. A tiling $\mathcal{T}$ of $\mathbb{R}^d$ by copies of a single tile is called *self-replicating* if, for some linear expansion A, the expanded tile $A(T)$ is, for each $T \in \mathcal{T}$, tiled by elements of $\mathcal{T}$. Note that the self-replicating property is a global property of the tiling, not a property of the tile. This self-replicating property was investigated by Thurston [**Th**] for more general tilings to be discussed in §10. The tiling by twin dragons in Figure 6 is self-replicating; the image of each dragon under the mapping A is the union of two horizontally adjacent dragons. The following proposition is an easy consequence of Eq. (2.5).

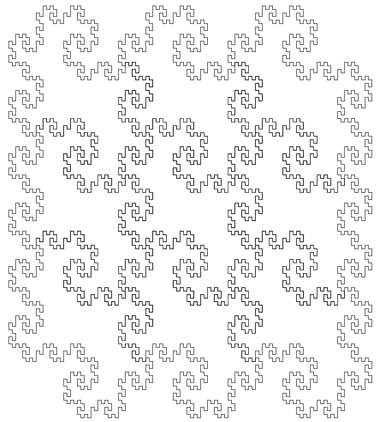


FIGURE 6. Lattice tiling by twin dragons.

PROPOSITION 4.1. Given any basic digit tile T, the corresponding tiling $\mathcal{T}_\infty$ is self-replicating.

According to Proposition 4.1, for any basic digit tile T there is a self-replicating tiling by translates of T. The proposition does not imply, however, that this tiling is by translation by the integer lattice, as is the case in Figure 6. A tiling $\mathcal{T}$ is a *lattice tiling* of $\mathbb{R}^d$ if $\mathcal{T}$ is a tiling by translation by a lattice, i.e., $\mathcal{T} = \{T+x \mid x \in L\}$ for some lattice L. Consider Example 4.1; the corresponding tiling $\mathcal{T}_\infty$ shown in Figure 7 is not a lattice tiling.

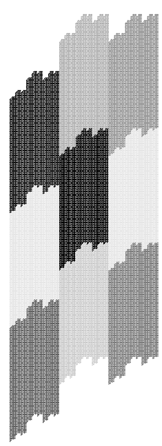


FIGURE 7. Not a lattice tiling.

EXAMPLE 4.1.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$
$$D = \{(-1,-3),(-1,-1),(-1,1),(0,-2),(0,0),(0,2),(1,-2),(1,0),(1,2)\}$$

The following theorem [**V4**] is central and gives ten equivalent conditions for the existence of a self-replicating lattice tiling. All terms not yet defined will be discussed after the statement of the theorem.

THEOREM 4.2. *Let $T = T(A,D)$ be a basic digit tile. Let $T_n = F^{(n)}(T_0)$ be the approximating tiles, where T_0 is the unit d-cube centered at the origin with edges parallel to the axes. The following statements are equivalent. Limits are with respect to the Hausdorff metric.*

1. $\mathcal{T} := \{T + x \,|\, x \in \mathbb{Z}^d\}$ *is a tiling of* $\mathbb{R}^d$.
2. $\mathcal{T} := \{T + x \,|\, x \in \mathbb{Z}^d\}$ *is a self-replicating tiling of* $\mathbb{R}^d$.
3. $m(T) = 1$.
4. *The characteristic function $\chi_T(x)$ is a scaling function of a multiresolution analysis.*
5. $\lim_{n\to\infty} \partial T_n = \partial T$.
6. $\lim_{n\to\infty} \partial T_n$ *is not space filling.*
7. $D_\infty = \mathbb{Z}^d$.
8. *Every lattice point has a unique finite address.*
9. *Every lattice point in the ball $\mathcal{B}(A,D)$ has a finite address.*
10. $\lambda(A,D) < |\det A|$.

Condition (3) states that the Lebesgue measure of T is 1. This is clearly necessary if statement (1) is to hold. The converse appears in [**GH**]. It is known that $m(T)$ is always an integer [**LW3**], but is not always 1. For the tile of Example 4.1 the measure is 2. Figure 8, showing the first few approximations to this tile, may provide insight into why the tile is "stretched."

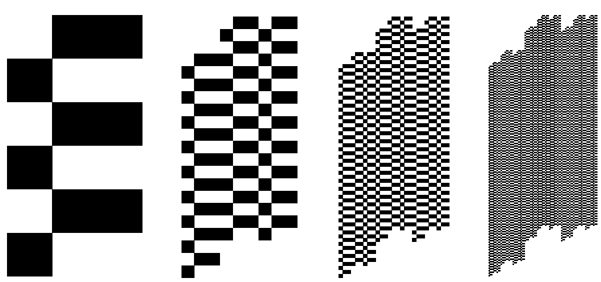


FIGURE 8. Digit tile with Lebesgue measure 2.

The equivalence of conditions (1) and (4) is due to Gröchenig and Madych [**GM**]. An important application of digit tiling is to wavelets, the construction of orthonormal wavelet bases in $\mathbb{R}^d$. The multiresolution analysis machinery produces an orthonormal wavelet bases of $L^2(\mathbb{R}^d)$. We refer the reader to [**GM, Str**] and any number of introductory texts, for example [**Ch**], rather than elaborating on wavelets in this paper.

Conditions (5) and (6) concern the boundary of the approximating tiles; proof of their equivalence to the other conditions appears in [**V4**]. Condition (5) states that the boundaries of the approximating tiles approach the boundary of the limit tile in the Hausdorff topology. It is easy to see that this is not the case for the tile in Figure 8. Condition (6) states that, if the conditions of Theorem 4.2 fail, then the behavior of the boundary is indeed pathological; the limit of the boundaries of the approximates is space filling - contains some open set. In the case of Example 4.1, the limit is the whole tile T.

Conditions (7) and (8) relate to Question 3.1 in §3. They state that every lattice point x has a unique base A representation with digits D. In other words $x = \sum_{i=0}^{n-1} A^i(e_i)$, $e_i \in D$. The proof of the equivalence of conditions (7), (8) and (9) to the other conditions in [**V1**] relies on the concept of A-adic integer, analogous to the classical p-adic integer, p a prime (see [**Se**] for background on the p-adic integers). The set of *A-adic integers* is the completion of $\mathbb{Z}^d$ with respect to the metric induced by the norm

$$|x| = \frac{1}{|\det A|^\nu},$$

where ν is the greatest integer such that $x \in A^\nu(\mathbb{Z}^d)$. Analogous to the p-adic case, there is a canonical representation of each A-adic number in the form

$$x = \sum_{i=0}^{\infty} A^i(e_i), \qquad e_i \in D.$$

Define the *address* of such an A-adic as

$$\dots e_3 e_2 e_1 e_0.$$

It can be shown that, given a digit set D for A, each point in $\mathbb{Z}^d$ has an address that eventually repeats, in the same sense as an ordinary repeating decimal. A lattice point is said to have a *finite address* if $e_n = 0$ for all n sufficiently large. In fact, there is an easy algorithm to obtain the address of any lattice point x.

Algorithm $(x_0 = x)$

$$\begin{aligned} e_n &\equiv x_n \mod A(\mathbb{Z}^d) \\ x_{n+1} &= A^{-1}(x_n - e_n) \end{aligned}$$

Moreover there is a computable bound on the number of iterations of this algorithm sufficient to determine whether or not the lattice point has a finite address.

Example 4.3. In 1-dimension the 3-adic address, i.e., $A = (3)$, of the integer 2 with respect to digit set $\{-1, 0, 4\}$ is $(-1)(4)(-1)$:

$$\begin{aligned} x_0 &= 2 & e_0 &= -1 \\ x_1 &= 1 & e_1 &= 4 \\ x_2 &= -1 & e_2 &= -1 \\ x_3 &= 0 & e_3 &= 0. \end{aligned}$$

Note that condition (8) together with formula (3.2) imply that every point in $\mathbb{R}^d$, except those on the overlap of two tiles in $\mathcal{T}$, can be uniquely expressed in the form

$$e_n \dots e_1 e_0 . e_{-1} e_{-2} \cdots := \sum_{i=-\infty}^{n} A^i(e_i), \; e_i \in D.$$

The representation of points on the overlap of two tiles is not unique; for example for $A = (10)$ and $D = \{0, 1, \dots, 9\}$, we have $.999 \cdots = 1$.

Conditions (9) and (10) are algorithmic. They provide efficient methods to check that all conditions in Theorem 4.2 hold. The number $\lambda(A, D)$ in condition (10) is the largest eigenvalue of certain easily computable matrix. A definition and discussion of this matrix appears in §5. Condition (9) states that there is ball $\mathcal{B}(A, D)$ centered at the origin, whose radius $r(A, D)$ depends only on (A, D), such that, if every lattice point in $\mathcal{B}(A, D)$ has a finite address, then all lattice points do. In the case that A is a similarity with expansion factor c, an explicit value of the radius is easy to express:

$$r(A, D) = \frac{\max\{|d| \; : \; d \in D\}}{c - 1}.$$

Applying the formula for $r(A, D)$ to Example 4.4, the only lattice point in $\mathcal{B}(A, D)$ is the origin, which obviously has a finite address. By condition (9) in Theorem 4.2, the corresponding lattice tiling is a self-replicating lattice tiling; it is shown in Figure 9.

Example 4.4.

$$\begin{aligned} A &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \\ D &= \{(0,0), (1,0), (0,1), (-1,0), (0,-1)\} \\ r(A, D) &= \frac{1}{\sqrt{5} - 1} = .8090 \dots \end{aligned}$$

It should be remarked that either condition (7) or (8) automatically implies that (A, D) is basic [**V1**]. In Example 4.5 there are 21 points in $\mathcal{B}(A, D)$ to check using condition (9), including the point $(-1, 0)$. The algorithm gives the repeating address $(1,0), (0,0), (1,0), (0,0), \dots$ for the point $(-1, 0)$, not a finite address. The problem in this case is that (A, D) is not basic; it is not pure. As pointed out in §2, there is a related basic pair (A', D') such that $T(A, D)$ and $T(A', D')$ are the same up to translation. Then $T(A', D')$ does satisfy the conditions of Theorem 4.2. The corresponding tiling is the twin dragon tiling in Figure 6, which is indeed a self-replicating lattice tiling.

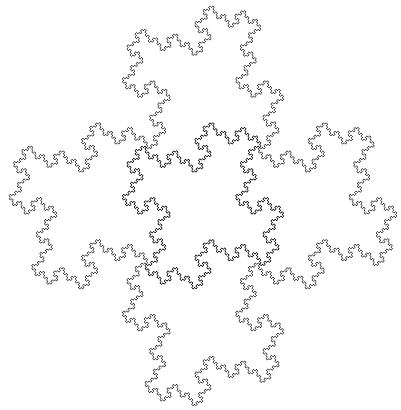


FIGURE 9. Self-replicating tiling.

EXAMPLE 4.5.

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\
D &= \{(0,0),(1,0)\} \\
r(A,D) &= \frac{1}{\sqrt{2}-1} = 1+\sqrt{2} = 2.4142\ldots
\end{aligned}
$$

5. Dimension of the Boundary

For some well studied tiles, like the Twin Dragon in Figure 3, the Hausdorff dimension of the boundary is known and has been computed by various means. More recently Duvall and Keesling [**DK**] determined the Hausdorff dimension of the boundary of a particular tile, the Lévy Dragon. In [**Kees**] Keesling showed that the Hausdorff dimension of the boundary of any self-similar tile in $\mathbb{R}^d$ is less than d, but that this dimension could be arbitrarily close to d. This section outlines a method due to Duvall, Keesling and Vince [**DKV**] for determining the Hausdorff dimension of the boundary of any self-similar digit tile. After our results were obtained we came across unpublished preprints by Veerman [**Ve**] and by Strichartz and Wang [**SW**] which contain similar results obtained by different methods. The only condition that is needed on the digit tile T for our formula in Theorem 5.1 is that one of the equivalent conditions given in Theorem 4.2 holds for T. This is not unexpected in light of conditions (5) and (6) of that theorem. The method given below either determines precisely the Hausdorff dimension of the boundary of T or it determines that condition (10) of Theorem 4.2 fails. The problem of determining an exact formula for a self-affine (not necessarily self-similar) digit tile remains open.

Recall the definition of Hausdorff dimension; an introductory treatment can be found, for example, in [**F1**]. An ϵ-cover of a set $X \subset \mathbb{R}^d$ is a collection of sets of diameter at most ϵ such that X is contained in their union. Let $|U|$ denote the diameter of the set U, and let s be a non-negative number. For any $\epsilon > 0$ define

$$H^s_\epsilon(X) = \inf \{\sum_{i=1}^{\infty} |U_i|^s \,:\, \{U_i\} \text{ is an } \epsilon\text{-cover of } X\}.$$

As ϵ decreases, the collection of possible covers is reduced; hence $H^s_\epsilon(X)$ decreases. Define the *s-dimensional Hausdorff measure* of X by

$$H^s(X) = \lim_{\epsilon \to 0} H^s_\epsilon(X).$$

It is easy to show that there is a critical value of s at which this limit jumps from ∞ to 0. Define the *Hausdorff dimension* by

$$\dim_H(X) = \inf \{s \,:\, H^s(X) = 0\} = \sup \{s \,:\, H^s(X) = \infty\}.$$

To state the main result, the contact matrix, first defined by Gröchenig and Haas [**GH**], is introduced. Given an expanding integer matrix A and digit set D for the integer lattice in $\mathbb{R}^d$, a set $N = N(A, D)$ of integer lattice points, called the *neighborhood* for (A, D), is used to index the rows and columns of the contact matrix.

The neighborhood $N(A, D)$ is defined as follows. Let $\{e_1, \dots, e_d\}$ denote the canonical basis of $\mathbb{R}^d$ and let $N_0 = \{0\} \cup \{\pm e_1, \dots, \pm e_d\}$. Then $N(A, D)$ is the unique smallest finite set $N \subset \mathbb{Z}^d$ such that $N_0 \subseteq N$ and $D + N \subseteq A(N) + D$. The neighborhood can easily be computed using the following algorithm, and it is easy to show that the algorithm terminates after a finite number of steps. Because D is a set of coset representatives of $\mathbb{Z}^d/A(\mathbb{Z}^d)$, for any lattice point y the equation $Ax + d = y$ has a unique solution pair (x, d), where $x \in \mathbb{Z}^d$ and $d \in D$.

Algorithm

$N = N_0$

Repeat until the two sets are equal:

$N \leftarrow N \cup \{x \in \mathbb{Z}^d \,|\, Ax + d = y$ for some $d \in D$ and $y \in D + N\}$.

For each $x \in N$ and $d \in D$, let x_d denote the unique lattice point such that $d + x \in Ax_d + D$. By the definition of N we have $x_d \in N$. Let C' be the $k \times k$ matrix whose rows and columns are indexed by the elements in N and whose entries are as follows. For $x, y \in N$

$$c_{xy} = |\{d \in D \,|\, x_d = y\}|.$$

By convention let the first index of C' correspond to the element $0 \in N$. Note that $c_{00} = |D|$ and $c_{0y} = 0$ for $y \neq 0$. Thus the first row of C' consists of all zeros except for one entry. Let C denote the $(k-1) \times (k-1)$ matrix obtained from C' by removing the first row and column. Call C the *contact matrix* for the pair (A, D). (In [**GH**] it is actually C' that is referred to as the contact matrix.)

According to the Perron-Frobenius Theorem for non-negative matrices, C has a real eigenvalue λ such that, for any other eigenvalue μ, we have $\lambda \geq |\mu|$. In other words, the spectral radius of C is an eigenvalue.

THEOREM 5.1. *(Duval, Keesling and Vince* [**DKV**]*) Let $T = T(A, D)$ be a self-similar digit tile where A has expansion factor c and the contact matrix has* *largest eigenvalue* $\lambda := \lambda(A, D)$*. Under any of the conditions in Theorem 4.2 we have*

$$\dim_H(\partial T) = \frac{\log \lambda}{\log c}.$$

Examples. *Twin dragon.* The dimension of the boundary of the Twin Dragon (Example 2.1 and Figure 3) has been calculated by various means. Using our method the neighborhood is the following set of lattice points:

$$N = \{(0,0),(0,1),(1,0),(1,-1),(0,-1),(-1,0),(-1,1)\}$$

Ordering the elements of $N \setminus \{0\}$ as above (clockwise around a hexagon) the contact matrix C, computed using the definition, is the following integer matrix with cyclical structure.

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial is easy to compute because of the near diagonal structure of the matrix:

$$\det(C - \lambda I) = \lambda^4(1-\lambda)^2 - 4 = (\lambda+1)(\lambda^2 - 2\lambda + 2)(\lambda^3 - \lambda^2 - 2).$$

So the largest eigenvalue of C is the real root of $\lambda^3 - \lambda^2 - 2$. Hence the Hausdorff dimension of the twin dragon is

$$\dim_H \partial T = \frac{\log \lambda}{\log \sqrt{2}} \simeq 1.523627$$

Gasket. For the Gasket (Example 2.2 and Figure 4), the neighborhood N is again in a hexagonal pattern:

$$N = \{(0,0),(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1)\}$$

The contact matrix is a cyclic matrix with three ones in each row:

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Hence the Perron-Frobenius eigenvector, the unique eigenvector with positive entries, is the all ones vector. The corresponding eigenvalue is $\lambda = 3$.

$$\dim_H \partial K = \frac{\log 3}{\log 2} = 1.5849625\ldots$$

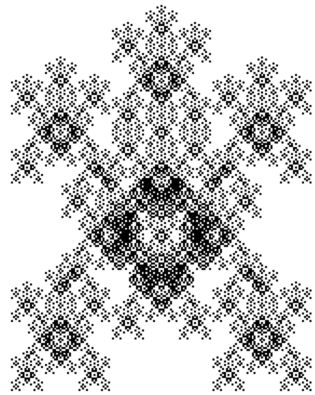


FIGURE 10. Lander.

Lander. The lander is the digit tile $T(A, D)$ where $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and $D = \{(0,0), (1,1), (1,-1), (2,0), (-1,-2), (3,-2), (-1,2), (3,2), (1,3)\}$. The dimension of the boundary of the Lander in Figure 10 is somewhat greater than for the other examples.

$$\dim_H \partial T \simeq 1.913624$$

Sketch of the proof of Theorem 5.1. Let T_0 be the unit cube centered at the origin with edges parallel to the axes and let $T_n = F^{(n)}(T_0)$ denote the n^{th} approximation to the tile $T := T(A, D)$ as given in Eq. (2.3). Then T_n is the non-overlapping union of copies of cubes of edge length $1/c^n$. For each lattice point, consider the unit cube centered at that point. Hence the neighborhood $N := N(A, D)$ can also be regarded as the non-overlapping union of cubes. Let N_n denote the neighborhood N contracted by a factor of $1/c^n$. Then it can be shown by induction that the sum of the elements in the n^{th} power C^n of the contact matrix C is approximately equal to the number α_n of small cubes q in T_n such that the neighborhood, centered at q, lies both inside and outside of T_n. In other words, α_n counts the number of small cubes in T_n "close" to ∂T_n. When we use the term "approximately" here we mean that there are upper and lower bounds of one quantity by a constant multiple of the other quantity, where the constants do not depend on n.

What simplifies the calculation of the Hausdorff dimension of ∂T is that, for the boundary of a self-similar digit tile, the Hausdorff dimension coincides with the box-counting dimension. This is a consequence of a result of Falconer [**F2**] on sub-self-similar sets. Consider the collection of cubes in the ϵ-coordinate mesh of $\mathbb{R}^d$. For a given set $X \in \mathbb{R}^d$ let $\beta_\epsilon(X)$ denote the number of such cubes that intersect X. The *box-counting dimension* is defined by

$$dim_B(X) = \lim_{\epsilon \to 0} \frac{\log \beta_\epsilon(X)}{-\log \epsilon}.$$

Letting $\epsilon = 1/c^n$, it can be proved, in the case of our digit tile T, that the number of small cubes that intersect ∂T is approximately equal to the number of small cubes α_n in T_n that are "close" to ∂T_n. Therefore

$$dim_H(\partial T) = dim_B(\partial T) = \lim_{n\to\infty} \frac{\log \alpha_n}{n \log c} = \lim_{n\to\infty} \frac{\log |C^n|}{n \log c},$$

where $|C|$ denotes the sum of all the entries in a matrix C. What completes the theorem is the fact that the largest eigenvalue of any nonnegative matrix C is given by the formula $\lambda(A, D) = \lim_{n\to\infty} |C^n|^{1/n}$. $\square$

6. Construction of the Boundary

The main result of this section is an explicit correspondence between two known methods for constructing digit tiles in the plane. The IFS method produces the tile itself; the recurrent set method, due to Dekking [**De1, De2**], produces the boundary of the tile. The proof of the theorem in this section appears in [**V4**]. Another connection between the IFS and recurrent set method appears in Bedford [**Be1, Be2**] in the context of constructing Markov partition boundaries for hyperbolic toral endomorphisms. Kenyon [**Ke3**] uses the recurrent set method in a setting discussed in §10.

The IFS "data" from which a digit tile $T = T(A, D)$ is constructed is simply the expanding matrix A and the digit set D. The pair (A, D) will be referred to as *tile data* if

1. A is an expanding 2×2 integer matrix and
2. D is a set of coset representatives of $\mathbb{Z}^2/A(\mathbb{Z}^2)$.

We use an integer matrix to keep the exposition simple. As explained in §2, all results are easily extended to the case of a tile based on a general lattice.

The "data" for the recurrent set method is a free group endomorphism ([**Lo**] is an introductory text on combinatorial group theory). Let $G := G\langle a, b\rangle$ be the free group on two generators a and b. Thus G consists of all words in the letters $\{a, b, a^{-1}, b^{-1}\}$, including the empty word e. The operation is concatenation, and the only relations are $aa^{-1} = e = a^{-1}a$ and $bb^{-1} = e = bb^{-1}$. Consider an endomorphism $\sigma : G \to G$. Note that σ is determined by its action on a and b. Define a matrix

$$A_\sigma = \begin{pmatrix} m_{aa} & m_{ab} \\ m_{ba} & m_{bb} \end{pmatrix},$$

where $m_{\alpha\beta}$ is the number of occurrences of α in $\sigma(\beta)$, counting α^{-1} as occurring -1 time. Here α and β are each either a or b. This process is called abelianization.

EXAMPLE 6.1. Twin dragon.

$$\begin{aligned} \sigma(a) &= ab \\ \sigma(b) &= a^{-1}b \end{aligned}$$

$$A_\sigma = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

EXAMPLE 6.2. Gasket.

$$\begin{array}{rcl} \sigma(a) & = & a^{-1}b^{-1}abaa \\ \sigma(b) & = & ba^{-1}ba \end{array}$$

$$A_\sigma = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Denote by $f : G \to \mathbb{R}^2$ the homomorphism determined by $f(a) = (1,0)$ and $f(b) = (0,1)$. Let $w = \alpha_1\alpha_2 \dots \alpha_n$ be any word in which each α_i is an element of $\{a, b, a^{-1}, b^{-1}\}$, and consider the sequence of points $x_i \in \mathbb{R}^2$, $i = 0, 1, \dots, n$, given by $x_0 = (0,0)$ and $x_i = f(\alpha_1\alpha_2 \dots \alpha_i) = f(\alpha_1) + f(\alpha_2) + \cdots + f(\alpha_i)$, $i \geq 1$. Join the points $x_0, x_1, \dots, x_n$ sequentially by line segments to obtain a polygonal path $p(w)$ and let

$$K_n := K_n(\sigma) = A_\sigma^{-n} p(\sigma^n(aba^{-1}b^{-1})). \tag{6.1}$$

Basically the path is obtained by traveling one unit left or right for an occurrence of a or a^{-1}, resp., in the string and one unit up or down for an occurrence of b or b^{-1}, resp.; then the path is contracted by A^{-n}. It is known [**De1**] that, if A_σ is expanding, then the sequence $\{K_n\}$ converges with respect to the Hausdorff metric to a closed curve

$$K := K(\sigma) = \lim_{n\to\infty} K_n.$$

Some line segments may be traversed by K_n more than one time. We impose the convention that each traversal of a line segment in one direction cancels a traversal of that line segment in the opposite direction. Thus K_n can consist of several closed curves, and hence K_n, and also K, may be disconnected. It can happen that the winding number of K_n about a point is more than 1. In this case there is no well defined region enclosed by K_n. The following result makes this situation easy to detect [**V4**].

LEMMA 6.3. *If the winding number of K_1 about every point of $\mathbb{R}^2 \setminus K_1$ is either 0 or 1, then the same is true of K_n, $n > 1$.*

The endomorphism $\sigma : G \to G$ will be referred to as *boundary data* if

1. A_σ is expanding, and
2. the winding number of K_1 about every point of $\mathbb{R}^2 \setminus K_1$ is either 0 or 1.

From Eq. (6.1) the path $A(K_1(\sigma))$ has sides that are parallel to the axes and joins integer lattice points. Let D_σ be the set of lattice points that are the lower left corners of unit squares that lie inside $A(K_1(\sigma))$.

THEOREM 6.4. *(Vince [**V4**]) The mapping $\Theta : \sigma \mapsto (A_\sigma, D_\sigma)$ induces a bijection from the collection of all boundary data to the collection of all tile data such that*

$$\partial T_n(A_\sigma, D_\sigma) = K_n(\sigma).$$

Moreover, if any of the conditions in Theorem 4.2 hold, then

$$\partial T(A_\sigma, D_\sigma) = K(\sigma).$$

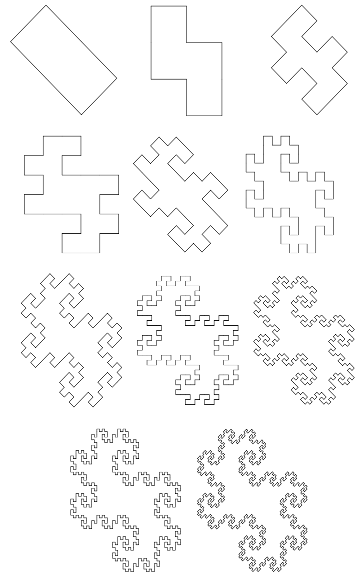


FIGURE 11. Approximations to the boundary of the twin dragon.

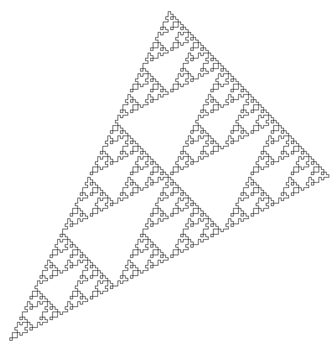


FIGURE 12. Boundary of the gasket.

The bijection is algorithmic and was used to draw Figure 11, which gives the first approximations to the boundary of the twin dragon corresponding to the approximations in Figure 3 drawn by the IFS method. The endomorphism is that of Example 6.1. Figure 12 shows the boundary of the topologically more complicated gasket originally pictured in Figure 4. The endomorphism is that of Example 6.2.

7. Lattice Tiling Problem and Aperiodic Proto-tile Problem

One part of Hilbert's 18^{th} problem asks whether there exists a polyhedron, copies of which tile space, but which is not the fundamental region of a group of isometries. In other words, the symmetry group of the tiling is not transitive on tiles. Examples were discovered early on, a polyhedron in 3 dimensions by Reinhardt [**Re**] in 1928 and a convex pentagon in 2 dimensions by Kershner [**Ker**] in 1968. All known examples, however, are periodic. A tiling of $\mathbb{R}^d$ is *periodic* if its symmetry group contains translations in d linearly independent directions.

A strong version of Hilbert's question is whether there exists a single tile which admits only nonperiodic tilings. A *nonperiodic* tiling is one that admits no translations. The Penrose tiles comprise a set of two tiles, copies of which tile the plane in uncountably many ways, but no such tiling is periodic. A set of *proto-tiles*, copies of which tile $\mathbb{R}^d$ but only nonperiodically, is called *aperiodic*. The Schmitt-Conway-Danzer (SCD) tile [**Da, Sch**], for example, is a single, convex, aperiodic tile in $\mathbb{R}^3$ (under the restriction that mirror image copies of the proto-tile are not allowed and screw symmetry does not count as a periodic symmetry). The SCD tile provides a solution to the above question, but the following questions remain open.

Question 7.1. Does there exists a single aperiodic proto-tile in $\mathbb{R}^2$?

Question 7.2. Does there exist an aperiodic proto-tile that tiles R^d by translation?

The answer to Question 7.2 in dimension 1 is no [**LW1**]. Venkov [**Ven**] answered Question 7.2 in 1954 in any dimension for the case of a convex proto-tile, a result independently rediscovered by McMullen [**McM**]. Their result: if a convex T tiles $\mathbb{R}^d$ by translation, then there is a lattice tiling of $\mathbb{R}^d$ by copies of T. The same result is true in dimension 2 for polyominoes[1] [**BN, KV, WvL**]. However, the Venkov-McMullen result is not true for non-convex tiles in general. The 1-dimensional tile $[0,1]\cup[2,3]$ allows a a tiling, but no lattice tiling of $\mathbb{R}$. Szabó [**Sz**] constructs a 3-dimensional, centrally symmetric, star polyhedron whose translates tile $\mathbb{R}^3$, but admits no lattice tiling of $\mathbb{R}^3$. A lattice tiling is periodic, but a periodic tiling is not necessarily a lattice tiling. So Question 7.2 remains unresolved in the non-convex, non-polyomino case.

A natural place to seek an example that might affirmatively answer Question 7.2 is among the digit tiles. Any digit tile T admits a tiling by translation as given by Eq. (4.1) in §4. However this tiling is sometimes not periodic, as in Example 4.1 and Figure 7. The tile in Figure 7, however, does admit a lattice tiling - by translation by the lattice generated by vectors $(1,0)$ and $(0,2)$. Gröchenig and Haas [**GH**] conjectured that every digit tile admits a lattice tiling. What makes the conjecture difficult is the existence of tiles, as in Example 4.1, that do not satisfy the conditions of Theorem 4.2. The lattice tiling conjecture was recently verified by Lagarias and Wang; so it is not possible to find an aperiodic digit tile. Note that the tiling guaranteed by their theorem is not necessarily self-replicating in the sense of §4.

Theorem 7.3. *(Lagarias and Wang* [**LW4**]*) Every digit tile T admits a lattice tiling of $\mathbb{R}^d$ for some lattice $L\subseteq\mathbb{Z}^d$.*

For remarks on Question 7.1 see Penrose's paper [**P2**]. Although there is no known single aperiodic proto-tile in $\mathbb{R}^2$, the analogous problem for coverings of $\mathbb{R}^2$ is solved. Moreover, the result has received considerable attention recently because of its implications for the structure of real quasicrystals. Consider the marked regular decagon on the left in Figure 13. This proto-tile is used to cover the plane with overlap allowed, but only according to the following overlap rule: two decagons may overlap only if shaded regions overlap and the overlap area is greater than or equal to the area of the overlap hexagon in the center illustration in

[1] A *polyomino* is a rookwise connected tile formed by joining unit squares at their edges.

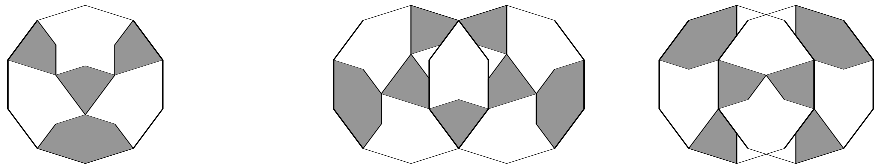


FIGURE 13. Overlapping marked decagons.

Figure 13. The figure shows the two possible sizes of the overlap. Gummelt [**Gu**] proved that every covering by marked decagons that satisfies the overlap rule is nonperiodic. Moverover, by dissecting each decagon into Penrose acute and obtuse triangles (Figure 15), such decagon coverings can be put into correspondence with the Penrose tilings.

Jeong and Steinhardt [**JS**] subsequently proved that both the Penrose matching rules for Penrose tilings and the overlap rule for decagon coverings can be replaced by a condition on the density of certain clusters. More precisely, the Penrose tilings are the tilings by Penrose rhombs for which the density of certain clusters of tiles (clusters whose union is essentially the Gummelt decagon) is maximum. This result led Jeong and Steinhardt to hypothesize that quasicrystals are formed from a single type of atomic cluster that can share atoms with neighboring clusters and that quasicrystals maximize cluster density. Evidence for such a model recently came from electron microscopy [**St**]. Electron micrographs of $Al_{72}Ni_{20}Co_8$ show striking similarities to the decagon coverings in Gummelt's paper.

8. Crystallographic digit tiling

A *crystallographic group* Γ is a discrete, cocompact group of isometries of Euclidean space. *Discrete* means that any ball contains at most finitely many points in the Γ-orbit of any point. *Cocompact* means that the quotient space $\mathbb{R}^d/\Gamma$ is compact. A lattice group, the group of translations by the points of a lattice, is a special case of a crystallographic group. A fundamental theorem of Bieberbach states that it Γ is a d-dimensional crystallographic group, then Γ contains a *translation subgroup*, a subgroup generated by translations in d independent directions.

Under any of the conditions of Theorem 4.2 a self-replicating digit tiling is a lattice tiling. This means that

$$\mathcal{T} = \{\gamma(T) \mid \gamma \in L\},$$

where L is a lattice group. But a lattice group L is only one of 17 crystallographic groups in the plane and only one of 230 crystallographic groups in 3-space. This section briefly describes a generalization, due to Gelbrich [**Ge1**], from lattice tiling to crystallographic tiling. A *crystallographic tiling* is of the form

$$\mathcal{T} = \{\gamma(T) \mid \gamma \in \Gamma\},$$

where Γ is a crystallographic group.[2]

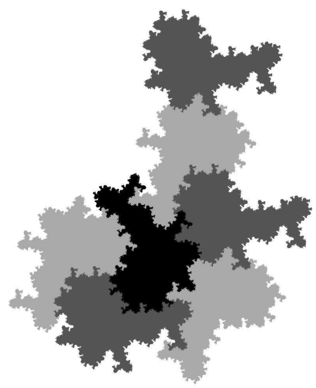
The basic construction of digit tiles given in §2 is based on a lattice L. The linear expansion A maps L into itself; so ALA^{-1} is the subgroup of translations by

[2]The term "crystallographic" is often used interchangeably with the term "periodic." A crystallographic tiling is periodic by Bieberbach's theorem, but a periodic tiling is not necessarily crystallographic. The symmetry group of a periodic tiling may not act transitively on the tiles.

points of the sublattice $A(L)$. A set of coset representatives of L/ALA^{-1} consists of translations by a digit set D. To generalize, let Γ be any crystallographic group; let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a linear expanding map such that $A\Gamma A^{-1} \subset \Gamma$ and let $D = \{d_1, \ldots, d_N\}$ be a set of right coset representatives of $\Gamma/A\Gamma A^{-1}$. Then the contractions

$$f_i(x) = A^{-1} \circ d_i(x)$$

provide an iterated function system with a unique attractor, say $T := T(\Gamma, A, D)$. The analogue of Theorem 2.5 holds: $T(\Gamma, A, D)$ is a compact set that is the closure of its interior. Call $T(\Gamma, A, D)$ a *crystallographic digit tile.*



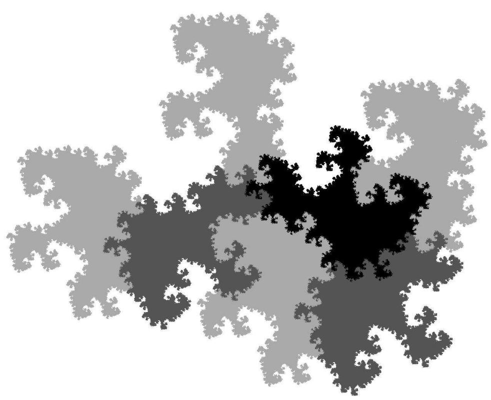


FIGURE 14. Crystallographic digit tilings: sea horse and coral reef.

Using the same reasoning as for ordinary digit tiles, every crystallographic digit tile in $\mathbb{R}^d$ admits a tiling of $\mathbb{R}^d$ that is self-replicating in the sense of §4. Some crystallographic tilings, courtesy of Gelbrich and Giesche [**GeG**], are shown in Figure 14 and are reminiscent of fractalized Escher prints. Analogous to (4.1) it can be shown that every self-replicating crystallographic tiling is of the form

$$\mathcal{T} = \{\gamma(T) \mid \gamma \in \Gamma_0\},$$

where Γ_0 is a subset (not necessarily a subgroup) of Γ.

The analogous result to Theorem 7.3, that every crystallographic digit tile admits a crystallographic tiling seems likely, but is open as far as we know. The issue is, given a crystallographic tile T, whether there exists a tiling $\{\gamma(T) : \gamma \in \Gamma_0\}$ where Γ_0 is a crystallographic group. Generalizing the results of §4, §5 and §6 to crystallographic tiles would also be of interest.

9. Hierarchical tiling

All tilings in §1 - §8 are by copies of a single tile. We now turn to tilings by copies of tiles taken from a finite set of *proto-tiles.* Many of the concepts that occur in the remainder of this paper are valid in a general context; so we introduce the notions of hierarchy and hierarchical tiling and frame the theory in this setting. Hierarchy is the basic notion; the tilings will be produced automatically from the hierarchy.

9.1. Hierarchy. Let $\mathcal{P} = (P_0, P_1, P_2, \dots)$ be a sequence of finite proto-tile sets. Define *inradius* (P_n) to be the largest r such that each proto-tile in P_n contains a ball of radius r. Call $\mathcal{P}$ a *hierarchy* if the following three conditions are satisfied.

1. $\lim_{n\to\infty}$ inradius $(P_n) = \infty$.
2. Each tile in P_{n+1} has a unique subdivision into the non-overlapping union of isometric copies of tiles in P_n.

The *subdivision rule* in condition (2) must be unique in the sense that each tile in P_{n+1} can be subdivided into the non-overlapping union of isometric copies of tiles in P_n in a unique way. (If there is ambiguity, for example if a proto-tile has nontrivial symmetry, then it is common to color some points in the tiles so that colors must match. In the IFS approach discussed in §10 this coloring is unnecessary.) Let $\mathcal{S}$ be an non-overlapping set of tiles in $\mathbb{R}^d$ taken from P_n. Using the subdivision rule there is a unique set $\mathcal{S}_{(1)}$ of tiles from P_{n-1} obtained by subdividing each tile in $\mathcal{S}$ according to the subdivision rule. Repeat to obtain from $\mathcal{S}$ the k^{th} *subdivision* $\mathcal{S}_{(k)}$, $k \le n$, by tiles in P_{n-k}.

3. For any given m, each tile in P_m appears in the $(n-m)^{th}$ subdivision of each tile in P_n for all n sufficiently large.

The square hierarchy example in Figure 15 shows the first three proto-tile sets and the first and second subdivisions. (Each proto-tile set consists of a single tile.) The second hierarchy in Figure 13 is by acute and obtuse Penrose triangles. Each proto-tile set consists of two tiles. The second subdivision is shown. (It can also be considered as the fourth subdivisions in the finer hierarchy shown in [**GS**, p. 540].) To insure uniqueness of the subdivision rule, the vertices of the triangles should be appropriately colored, as is usually done for the Penrose tiles. In both of these examples the proto-tile sets $P_0, P_1, \dots$ have the same cardinality. Moreover, corresponding tiles in P_n and P_{n+1} are similar, the ratio being 2 in the case of the squares and the golden ratio τ in the case of the Penrose tiles. (These are examples of what are commonly called *local inflation rules*). In general, this does not have to be the case for a hierarchy.

9.2. Hierarchical tiling. A tiling by copies of tiles taken from a proto-tile set P will be called a *P-tiling.* A *patch* of a tiling is a subset of tiles whose union is a topological ball. The definition of hierarchy concerns the proto-tile sets, not

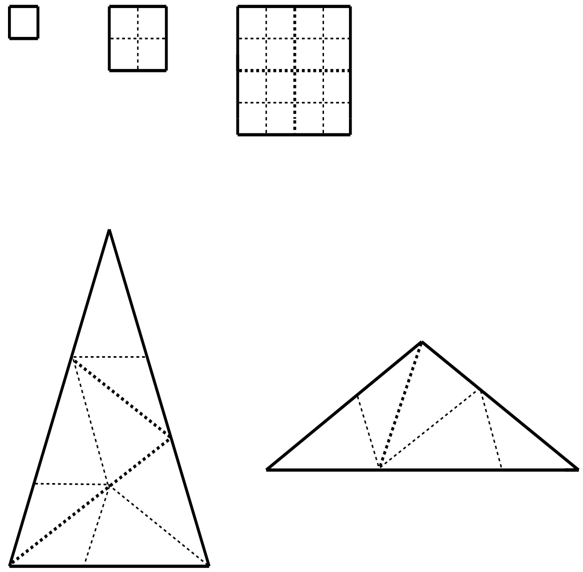


FIGURE 15. Square and Penrose hierarchies.

tilings by these proto-tiles. Now define a tiling $\mathcal{T}$ to be *hierarchical* if there exists a hierarchy $P_0, P_1, P_2, \ldots$ and a sequence $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ of tilings with $\mathcal{T}_0 = \mathcal{T}$ such that

1. $\mathcal{T}_n$ is a P_n-tiling for all n.
2. $\mathcal{T}_n$ is the subdivision of $\mathcal{T}_{n+1}$ for each n.
3. Each patch in $\mathcal{T}$ appears in the n^{th} subdivision of some tile in P_n, for n sufficiently large, n depending only on the size of the patch.

The last condition is to eliminate from consideration tilings such as the following. Combine the square tiling of the left half-plane and the square tiling of the right half-plane offset slightly along a vertical "fault" where the two half-planes meet.

If $\mathcal{T}$ is a tiling with hierarchy $\mathcal{P}$, then $\mathcal{T}$ will be referred to as a $\mathcal{P}$-tiling. We also use the terminology $\mathcal{P}$ *admits* the tiling $\mathcal{T}$. Note that if $\mathcal{P} = (P_0, P_1, P_2, \ldots)$ is a hierarchy then so is any infinite subsequence $\mathcal{P}' = (P_{i0}, P_{i1}, P_{i2} \ldots)$ with the obvious subdivision rule coming from the subdivision rule for $\mathcal{P}$. Moreover, if $P_0 = P_{i0}$, then a tiling $\mathcal{T}$ is a $\mathcal{P}$-tiling if and only if $\mathcal{T}$ is a $\mathcal{P}'$-tiling. Such hierarchies $\mathcal{P}$ and $\mathcal{P}'$ will be considered *equivalent*.

If, for every $\mathcal{P}$-tiling $\mathcal{T}$, the sequence $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ is uniquely determined, then we say that $\mathcal{P}$ *forces uniqueness*. The hierarchy of squares in Figure 15 does not force uniqueness; for the tiling $\mathcal{T}$ of the plane by squares, there are infinitely many ways to choose the sequence of tilings $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$. The Penrose hierarchy does force uniqueness on any Penrose tiling of the plane by thick and thin triangles. In other words, the subdivision rule for the Penrose hierarchy is locally invertible; the subdivision rule for the square hierarchy is not. If the hierarchy for a tiling forces uniqueness, then the tiling is commonly said to satisfy the *unique composition property* or the *local inflation/deflation property*.

A tiling $\mathcal{T}$ is of *finite type* if, for any positive number r, there are at most finitely many patches, up to congruence, within a ball of radius r. A tiling $\mathcal{T}$ has the *local isomorphism property* if, for any patch $\mathcal{Q}$ of $\mathcal{T}$, there is a number R such that any ball of radius R contains, up to congruence, a copy of $\mathcal{Q}$. Two tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to be *locally isomorphic* if every patch of $\mathcal{T}_1$ can be found in $\mathcal{T}_2$ and vice versa. Local isomorphism is an equivalence relation.[3] Nonperiodic tilings that are both of finite type and satisfy the local isomorphism property have been referred to as quasiperiodic. Since the term quasiperiodic has multiple definitions in the literature we will not use it.[4]

THEOREM 9.1. *Let $\mathcal{P}$ be a hierarchy.*

1. *Every $\mathcal{P}$-tiling is of finite type.*
2. *Every $\mathcal{P}$-tiling has the local isomorphism property, and any two $\mathcal{P}$-tilings are locally isomorphic.*
3. *If $\mathcal{P}$ forces uniqueness, then every $\mathcal{P}$-tiling is nonperiodic.*

PROOF. Let $\mathcal{P} = (P_0, P_1, P_2, \dots)$ be the hierarchy and $(\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \dots)$ the corresponding sequence of tilings with $\mathcal{T}_0 = \mathcal{T}$. By condition (3) in the definition of hierarchical tiling, any ball of radius r in a $\mathcal{P}$-tiling $\mathcal{T}$ is contained in the k^{th} subdivision $p_{(k)}$ of some tile $p \in P_k$, where k depends only on r. Since there are at most finitely many configurations within the k^{th} subdivision of the tiles of P_k, the finite type property is verified.

To verify the local isomorphism property, let $\mathcal{Q}$ be a patch in $\mathcal{T}$. Again, $\mathcal{Q}$ appears in the k^{th} subdivision $p_{(k)}$ of some proto-tile $p \in P_k$. But by condition (3) in the definition of hierarchy, the proto-tile p, in turn, appears in the subdivision of each tile in P_n for n sufficiently large. Finally, since the tiles are compact, there is a number R, depending only on n, such that any ball of radius R contains some tile in $\mathcal{T}_n$. Therefore any ball of radius R contains $\mathcal{Q}$. The same reasoning shows that any two $\mathcal{P}$-tilings are locally isomorphic.

Assume that $\mathcal{P}$ forces uniqueness, and assume, by way of contradiction, that $\mathcal{T}$ admits a translational symmetry. This induces a translational symmetry of $\mathcal{T}_1$; otherwise uniqueness of $\mathcal{T}_1$ is violated. Repeating this argument implies that, for each n, there is a translational symmetry of $\mathcal{T}_n$. But this is impossible because inradius$\,(P_n) \to \infty$ by condition (1) in the definition of hierarchy. $\square$

Note that, for the set P of Penrose tiles, the standard matching rules guarantee that every P-tiling is a $\mathcal{P}$-tiling. Since the Penrose hierarchy forces uniqueness, it follows from Theorem 9.1 that no P-tiling is periodic. In this case we call P an *aperiodic set*; no tiling by copies of tiles in P is periodic.

So far it has not been assumed that a given hierarchy $\mathcal{P} = (P_0, P_1, \dots)$ admits even a single tiling. The existence of $\mathcal{P}$-tilings is now addressed. If $p_n \in P_n$ and $p_{n+1} \in P_{n+1}$ then, in accordance with the subdivision rule, p_n can possibly appear

[3]Concerning terminology in the literature, two tiling in the same local isomorphism class are sometimes called *locally indistinguishable*, and a tiling with the local isomorphism property is sometimes called *repetitive*. Another equivalence relation among tilings, *mutual local derivability*, will not come into play in this paper. We use the terms "finite type" and "local isomorphism" with respect to congruence. Analogous versions with respect to translations are also often used.

[4]This paper does not discuss the well known projection method for constructing "quasiperiodic" point sets. It is interesting to note, however, that there exists such sets for which the window system can be interpreted as a self-similar tile with fractal boundary; see [**LGJJ**].

several times in p_{n+1} (or not at all). Let $S(p_n, p_{n+1})$ be a set of symbols denoting the positions of p_n in the subdivision of p_{n+1}. If p_n does not appear in p_{n+1}, then $S(p_n, p_{n+1})$ is empty. Consider any sequence $C = (c_0, c_1, \dots)$ where each $c_n \in S(p_n, p_{n+1})$ for some $p_n \in P_n$, $p_{n+1} \in P_{n+1}$ and, if $c_{n-1} \in S(q_{n-1}, q_n)$ and $c_n \in S(p_n, p_{n+1})$ then $q_n = p_n$. Construct a tiling from C as follows. Start with $\mathcal{Q}_0 := p_0$; $\mathcal{Q}_0$ is embedded in the subdivision $\mathcal{Q}_1$ of tile p_1 in position c_0; p_1, in turn, is embedded in the subdivision $\mathcal{Q}_2$ of p_2 in position c_1. Continue in this way to obtain a nested sequence $\mathcal{Q}_0 \hookrightarrow \mathcal{Q}_1 \hookrightarrow \mathcal{Q}_2 \hookrightarrow \dots$ of patches. The union $\bigcup_n \mathcal{Q}_n$ is a *partial* tiling. We use the term "partial" because the union may not be all $\mathbb{R}^d$. Call two such sequences C and C' *equivalent* if there is an integer k such that the sequences C and C' agree after the first k terms. Because of the uniqueness of subdivision, equivalent sequences yield the same partial tiling up to congruence. Call an equivalence class of sequences a *code* for the tiling it produces. So there is a well-defined mapping from the set of codes onto the set of partial $\mathcal{P}$-tilings. (The mapping may not be one-to-one; the square tiling of the plane, for instance, has infinitely many codes.)

If, in condition (3) in the definition of hierarchy, it is required that each tile in P_m appears in the interior (not intersecting the boundary) of each tile in P_n, then we call the hierarchy *interior*. The following result is surely known; in particular it has long been known for the Penrose hierarchy [**GS**].

THEOREM 9.2. 1. *If a hierarchy $\mathcal{P}$ is interior, then $\mathcal{P}$ admits (full) tilings.*
2. *If $\mathcal{P}$ forces uniqueness, then there is a bijection between the set of codes and the set of partial tilings (up to isometry). In particular $\mathcal{P}$ admits uncountably many partial tilings (uncountably many full tilings if $\mathcal{P}$ is interior).*

PROOF. Concerning (1), the property of being interior insures that for some code the union $\bigcup_n \mathcal{Q}_n$ described above covers all $\mathbb{R}^d$, hence producing a full tiling.

Concerning (2), given a $\mathcal{P}$-tiling $\mathcal{T}$, any code $C(\mathcal{T}) = (c_0, c_1, \dots)$ for $\mathcal{T}$ is obtained as follows. Choose an arbitrary tile $T_0 \in \mathcal{T}$, where T_0 has proto-tile type p_0, Then T_0 is contained at position c_0 in a unique tile T_1 of proto-tile type p_1 at the next level. In general T_n of type p_n is contained at position c_n in a unique tile of type p_{n+1}. Moreover if C and C' are both codes for $\mathcal{T}$ then they must be equivalent because, any two initial tiles in $\mathcal{T}$ are contained in the same single tile at a sufficiently high level.

Because of condition (1) in the definition of hierarchy, there are at least two choices for the next embedding at infinitely many stages. So there are uncountably many codes, hence uncountably many tilings. □

The code for the Penrose tiling by acute and obtuse triangles can be denoted by binary digits 0 or 1 in such a way that each partial tiling is given by a unique binary sequence which contains no subsequence 11. (This code is with respect to the finer hierarchy mentioned in reference to Figure 13.) Every such binary sequence, except $(000\dots)$, $(10001000\dots)$ and $(00100010001\dots)$ yields a tiling of $\mathbb{R}^2$. The exceptions yield partial tilings which can easily be extended to full tilings. Hence by Theorem 9.2 there is a bijection between the set of codes and set of Penrose tilings. The Penrose tiling with code $(000\dots)$, called the *cartwheel*, has been singled out in the literature. For example it is shown in [**GS**] that, except for seven exceptions, every tile in the cartwheel tiling lies in a patch of tiles whose symmetry group is the dihedral group D_5. A special case of Theorem 10.1 in the next section implies the surprising property that the cartwheel is the unique Penrose tiling $\mathcal{T}$ for which an expansion by the golden ratio sends each tile in $\mathcal{T}$ to the union of tiles in $\mathcal{T}$.

10. Self-affine and Self-similar tiling

The basic concept in §9 is a hierarchy $\mathcal{P}$. From a given hierarchy, tilings are produced according to Theorem 9.2, infinitely many in the case that $\mathcal{P}$ forces uniqueness. This section concerns two special types of hierarchies, self-affine and self-similar, and their associated tilings. After defining self-affine and self-similar hierarchy (§10.1), a few important results concerning the associated tilings are presented (§10.2). An alternative approach based on graph iterated function systems is given in §10.3. Examples appear in §10.4.

10.1. Definitions. Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a linear expanding map. Let $P = \{T_1, T_2, \dots, T_N\}$ be a finite set of proto-tiles, and let

$$P_n = \{A^n(p) \,|\, p \in P\}. \tag{10.1}$$

The subdivision rule for the first level of a hierarchy $\mathcal{P} = (P_0, P_1, \dots)$ is given explicitly as follows for each $i = 1, 2 \dots, N$:

$$A(T_i) = \bigcup g_{ij}^k(T_j), \tag{10.2}$$

where the union is non-overlapping with indices $j = 1, 2, \dots, N$ and "multiplicities" $k = 1, 2, \dots, k(i,j)$, and each g_{ij}^k is an isometry. The functional equation (10.2) states that each large tile $A(T_i)$ is the non-overlapping union of copies of the small tiles $T_1, \dots, T_N$. In this union, each tile of type T_j can appear one or more times $(k(i,j) \geq 1)$ or not at all $(k(i,j) = 0)$.

To define the subdivision rule on P_n for $n > 1$, make the following assumption:

$$A \circ g_{ij}^k \circ A^{-1} \quad \text{is an isometry for all} \quad i, j, k. \tag{10.3}$$

Assumption (10.3) allows Eq. (10.2) to be iterated to obtain a subdivision rule at every level. The matrix $M = (k(i,j))$ of multiplicities from (10.2) is called the *substitution matrix* for the subdivision. Thus $k(i,j)$ is the number of times T_j appears in T_i. Condition (3) in the definition of hierarchy in §9.1 is equivalent to some power of M being strictly positive, i.e., M is what is called a *primitive matrix*. If this is the case $\mathcal{P}$ satisfies all three conditions in the definition of hierarchy.

Assumption (10.3) holds if either

1. g_{ij}^k is a translation for each i, j, k, or
2. A is a similarity.

In case (1) the hierarchy $\mathcal{P}$ will be called *self-affine* and in case (2) *self-similar*. If both (1) and (2) hold we call the hierarchy *translationally self-similar*. Let P_n be as in Eq. 10.1 and let $P'_n = \{A'^n(p) \,|\, p \in P\}$, where $A' = \phi \circ A$ for some isometry ϕ. Note that $\mathcal{P} = (P_0, P_1, \dots)$ and $\mathcal{P}' = (P'_0, P'_1, \dots)$ are the same hierarchy. In particular, in the self-similar case it can be assumed that $A(x) = cx$ where $c > 1$. In either case, the remarks in §9.2 imply that replacing A by $\phi \circ A^s$, where ϕ is an isometry and s any positive integer results in an equivalent hierarchy as defined in §9.2.

A $\mathcal{P}$-tiling will be called *self-affine* if $\mathcal{P}$ is a self-affine hierarchy and *self-similar* if $\mathcal{P}$ is a self-similar hierarchy. It is unfortunate that the term "self-similar" has slightly different definitions in various publications on the subject. The definition of self-similar in [**So2**], for example, assumes that both conditions (1) and (2) hold, translationally self-similar in our terminology. A self-similar tiling in [**Ke3, So1, Th**] has an additional property we will call special. A self-affine or self-similar tiling $\mathcal{T}$ is *special* if the image $A'(T)$ is, for any $T \in \mathcal{T}$, the union of tiles in $\mathcal{T}$. Here A' can be any linear map of the form $\phi \circ A^s$, which, as discussed in the paragraph above, results in a hierarchy equivalent to the original hierarchy. This definition of special is a direct generalization from §4 of the term self-replicating; in that case $s = 1$.

10.2. Some results. In this section several miscellaneous results on self-affine and self-similar tilings are presented. Let $\mathcal{P}$ be either a self-affine or a self-similar hierarchy and denote by $\Omega_{\mathcal{P}}$ the set of all $\mathcal{P}$-tilings. The *subdivision operator* $\sigma : \Omega_{\mathcal{P}} \to \Omega_{\mathcal{P}}$ is defined as follows. Using the notation $A(\mathcal{T}) = \{A(T) \mid T \in \mathcal{T}\}$ define

$$\sigma(\mathcal{T}) = A(\mathcal{T})_{(1)},$$

the first subdivision of the inflated tiling $A(\mathcal{T})$. According to the next result, the special self-affine and self-similar tilings are the ones with a repeating code.

THEOREM 10.1. *The following statements are equivalent for a self-affine or self-similar tiling $\mathcal{T}$.*

1. *$\mathcal{T}$ is a fixed point of the subdivision operator σ^s for some positive integer s.*
2. *There is a repeating code for $\mathcal{T}$ of the form $C(\mathcal{T}) = (c_1, c_2, \dots, c_s, c_1, c_2, \dots, c_s, \dots)$.*
3. *The tiling $\mathcal{T}$ is special.*

PROOF. (1) $\Longleftrightarrow$ (2) First, $\sigma^s(\mathcal{T}) = \mathcal{T}$ if and only if the two tilings have a same code (up to equivalence), say $(c_1, c_2, \dots)$. But if $C(\mathcal{T}) = (c_1, c_2, \dots)$, then, by the definition of the subdivision operator, $C(\sigma^s(\mathcal{T})) = (c'_1, c'_2, \dots, c'_s, c_1, c_2, \dots)$ for some symbols $c'_1, c'_2, \dots, c'_s$. Hence, by the definition of equivalent codes, $\sigma^s(\mathcal{T}) = \mathcal{T}$ if and only if $c_{k+s} = c_k$ for k sufficiently large. This is the case if and only if $c(T)$ repeats with period s.

(1) $\Longleftrightarrow$ (3) The tiling $\mathcal{T}$ is a fixed point of the subdivision operator σ^s if and only if $A^s(\mathcal{T})_{(s)} = \phi(\mathcal{T})$ for some isometry ϕ. This is the case if and only if, for each tile $T \in \mathcal{T}$, we have $A^s(T) = \bigcup_{i=1}^K \phi(T_i)$ for some tiles $T_i \in \mathcal{T}$. This equation is equivalent to $(\phi^{-1} \circ A^s)(T) = \bigcup_{i=1}^K T_i$; in other words, $\mathcal{T}$ is special. □

COROLLARY 10.2. Every self-affine or self-similar hierarchy admits a special tiling.

PROOF. Property (3) in the definition of the hierarchy, i.e. that the substitution matrix is primitive, implies that the hierarchy admits a tiling whose code repeats. The result then follows from Theorem 10.1. □

The following result concerns the unique composition property defined in §9.2. The third part of Theorem 9.1 states, in particular, that a self-affine tiling with the unique composition property (local inflation/deflation) must be nonperiodic. A proof of the converse in the 1-dimensional case appeared in [**Mo**]. The converse is true in general.

THEOREM 10.3. *(Solomyak* [**So2**]*) If a self-affine tiling is nonperiodic then it has the unique composition property.*

The next result concerns tile frequencies. Recall, for example, that the frequencies of the two Penrose tiles in any Penrose tiling exist and the ratio of the two frequencies is the golden ratio [**GS**]. The existence of uniform frequencies of patches in cubes was established by Lunnon and Pleasants for substitution tilings by tiles that are polytopes [**LuP**]. In general, let $\mathcal{Q}$ be a patch in a tiling $\mathcal{T}$. Let $L_{\mathcal{Q}}(X)$ denote the number of translates of $\mathcal{Q}$ in a region $X \subset \mathbb{R}^d$. The frequency freq($\mathcal{Q}$) of the patch is defined as the following limit, if it exists,

$$\lim_{n\to\infty} \frac{L_{\mathcal{Q}}(X_n)}{Vol(X_n)},$$

where X_n is a region with d-dimension measure Vol(X_n) that tends to infinity in such a way that the boundary of X_n does not wriggle too much. A precise definition and the following statement appear in [**So1**].

THEOREM 10.4. *(Solomyak) If $\mathcal{T}$ is a self-affine tilling, then the frequencies of patches exist.*

For a nonempty patch $\mathcal{Q}$ in a translationally self-similar tiling $\mathcal{T}$, define the *locator set*

$$L_{\mathcal{Q}}(\mathcal{T}) = \{x \in \mathbb{R}^d \mid \text{there exists } \mathcal{Q}' \subset \mathcal{T} \text{ with } \mathcal{Q} = \mathcal{Q}' - x\}.$$

Voronoi tilings based on these locator sets can be constructed. Priebe [**Pri**] proves an interesting finiteness property concerning the number of these derived Voronoi tilings of $\mathcal{T}$.

There is a growing body of work on the dynamical systems arising from the action by translation on a certain space of tilings. Solomyak [**So1**] gives a comprehensive survey of results on the dynamics of self-affine tilings, including a proof of unique ergodicity. We refer the interested reader to the cited paper and the references therein.

Perhaps the best known property of translationally self-similar tilings concerns possible expansion constants. For a self-similar tiling of the plane $\mathbb{R}^2 \cong \mathbb{C}$ the map A can be represented as multiplication by an *expansion constant* $\lambda \in \mathbb{C}$. The next theorem was announced by Thurston with a proof of necessity. Kenyon gave a constructive proof of sufficiency and a generalization to self-affine tilings in $\mathbb{R}^d$ [**Ke1**].

THEOREM 10.5. *(Thurston* [**Th**], *Kenyon* [**Ke3**]*) A translationally self-similar tiling of the plane with expansion constant λ exists if and only λ is a complex Perron number, that is, an algebraic integer whose Galois conjugates, except $\overline{\lambda}$, are less than $|\lambda|$ in modulus.*

Concerning Theorem 10.5, it is not hard to show that, for a translationally self-similar tiling, $|\lambda|^2 = \lambda\overline{\lambda}$ is a real Perron number. In fact, this is essentially what is done in the proof of Proposition 10.1 later in this paper. The proof that $|\lambda|^2$ is a Perron number is based on the fact that the area of each proto-tile increases by a factor of $|\lambda|^2$ under the inflation by λ and this inflated area is an integer linear combination of the areas of the original proto-tiles. To show the stronger result that λ itself is a Perron number, Thurston considers certain distinguished points (capitals or control points) for each proto-tile, and a certain finite set of differences between control points in the tiling. Then λ inflates this set of differences so that the inflated differences are an integer linear combination of the original differences.

We conclude this section with a very brief comment on the diffraction spectrum of a self-similar tiling. One of the common definitions of quasicrystal is that of an atomic structure whose X-ray diffraction shows Bragg peaks - sharp spots in the diffraction pattern. For a discrete set Y of points in $\mathbb{R}^d$ (an atomic arrangement say), consider the distribution $f(x) = \sum_{y \in Y} \delta_y$, where δ_x is the Dirac delta. The X-ray diffraction of Y can be described using the Fourier transform $\widehat{\gamma}$ of a related distribution γ, called the autocorrelation. See [**Ba**] or [**Se**], for example, for definitions and background. Under mild conditions $\widehat{\gamma}$ can be decomposed into a discrete part (Bragg spectrum) and continuous part (diffuse spectrum). Concerning tilings, by choosing a distinguished point for each type of tile, the spectrum of a $\mathcal{P}$-tiling can be discussed. In several examples of self-similar tilings it was noticed that, for the existence of nontrivial Bragg spectrum, it is necessary that the Perron-Frobenius eigenvalue (the largest eigenvalue) of the substitution matrix be a Pisot number [**BT**]. A *Pisot number* is an algebraic integer $\beta > 1$ such that all its other Galois conjugates lie inside the unit circle. In the generality below, the result is due to Gähler and Klitzing [**GK**].

THEOREM 10.6. *(Gähler and Klitzing) If $c > 1$ is the expansion factor of a self-similar tiling with nontrivial Bragg spectrum, then c must be a Pisot number.*

That c is a Pisot number is equivalent to the Perron-Frobenius eigenvalue of the substitution matrix being a Pisot number. Gähler and Klitzing go on to give a nice description of the Bragg spectrum of a self-similar tiling, which leads to distinguishing three types of such tilings: quasiperiodic, limit-periodic and limit-quasiperiodic.

10.3. Graph iterated function systems. This section concerns a constructive approach to self-affine and self-similar tilings based on graph iterated function systems. Whereas the attractor to an IFS is a single compact set, the attractor of a graph IFS is a finite collection of compact sets. This generalization can be found in [**MW**] as well as in the literature on image compression. Bandt [**B1, B3**] applies the method to tilings.

Using the same notation as in §2 let $\mathcal{C} := \mathcal{C}(\mathbb{R}^d)$ denote the space of nonempty compact subsets of $\mathbb{R}^d$, complete with respect to the Hausdorff metric, and let $\mathcal{C}^N$ be the N-fold Cartesian product of copies of $\mathcal{C}$. A *graph iterated function system* (GIFS) is a directed graph G, possibly with loops and multiple edges in which the vertices of G are labeled by $\{1, 2, \ldots, N\}$ and each edge e is labeled with a contraction $f_e : \mathbb{R}^d \to \mathbb{R}^d$. It is also assumed that G is *strongly connected*, i.e., that there is a directed path from any vertex to any other. Let E_{ij} denote the set of edges from vertex i to vertex j. Define the function

$$F \,:\, \mathcal{C}^N \to \mathcal{C}^N$$

as follows. If $\mathbf{X} = (X_1, X_2, \ldots, X_N) \in \mathcal{C}^N$, then

$$F(\mathbf{X}) = (F_1(\mathbf{X}), F_2(\mathbf{X}), \ldots, F_N(\mathbf{X})),$$

where

$$F_i(\mathbf{X}) = \bigcup_{j=1}^{N} \bigcup_{e \in E_{ij}} f_e(X_j).$$

It can be shown that F is a contraction on $\mathcal{C}^N$, and consequently has a unique fixed point $\mathbf{T} = (T_1, T_2, \ldots, T_N)$.

Now consider the special case where each contraction is of the form

$$f_e(x) = A^{-1} \circ g_e,$$

where A is an expanding linear map and g_e is an isometry. The definition of fixed point implies

$$A(T_i) = \bigcup_{j=1}^{N} \bigcup_{e \in E_{ij}} g_e(T_j), \; i = 1, 2, \ldots, N, \tag{10.4}$$

which is precisely Eq. (10.2). So, if each g_e is a translation or A is a similarity, then call the GIFS *self-affine* or *self-similar*, respectively. In this case the sequence

$$\mathcal{P} = \{\mathbf{T}, A(\mathbf{T}), A^2(\mathbf{T}), \ldots\} \tag{10.5}$$

is a self-affine or self-similar hierarchy whose substitution rules are determined by (10.4) provided

1. T_i is the closure of its interior for each i, and
2. the unions in Eq. (10.4) are non-overlapping.

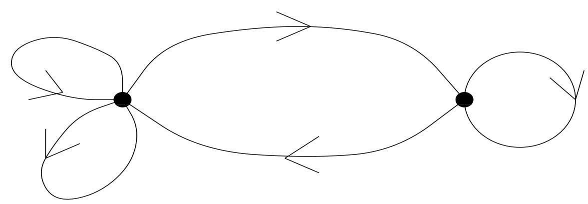


FIGURE 16. Graph iterated function system for the Penrose tiles.

From [**Kees**] it is known that $m(\partial T_i) = 0$. In the definition of GIFS, the condition that G be strongly connected is equivalent to condition (3) in the definition of hierarchy in §9.1. So, assuming conditions (1) and (2) given just above, the $\mathcal{P}$-tilings are self-affine or self-similar tilings and, conversely, every self-affine or self-similar tiling can be obtained by such a GIFS construction.

Figure 16 shows the self similar GIFS whose fixed point is the pair of Penrose tiles shown in Figure 15. The two loops directed from the left node correspond to the two similarities taking the acute Penrose triangle to two smaller similar copies in its first subdivision. The edge directed from the first to the second node corresponds to the similarity taking the obtuse Penrose triangle to a smaller similar copy in the first subdivision of the acute Penrose triangle. Likewise, the two edges directed from the right node correspond to similarities taking each of the two Penrose triangles to smaller similar copies in the first subdivision of the obtuse triangle.

Assuming condition (1) holds, it is not difficult to give a necessary and sufficient condition for condition (2). Note that, in the GIFS terminology, the $N \times N$ matrix $M = (|E_{ij}|)$ is the substitution matrix as defined in §10.1.

PROPOSITION 10.1. Assume that condition (1) holds for a self-affine or self-similar GIFS. Then condition (2) holds if and only if $|\det A|$ is the Perron-Frobenius eigenvalue (the largest real eigenvalue) of the substitution matrix M.

PROOF. Let x_i denote the Lebesgue measure of tile T_i. The unions in Eq. 10.4 are non-overlapping if and only if

$$|\det A|\, x_i = \sum_{j=1}^{N} |E_{ij}|\, x_j, \quad i = 1, 2, \ldots, N.$$

This means that $|\det A|$ is an eigenvalue of M. But for a non-negative matrix, the only eigenvalue with a positive eigenvector is the Perron-Frobenius eigenvalue. □

Given a directed edge path $p = e_1 e_2 \cdots e_n$ and a contraction $f : \mathbb{R}^d \to \mathbb{R}^d$ we introduce the notation

$$f_p = f_{e_1} \circ f_{e_2} \circ \cdots \circ f_{e_n}.$$

The following proposition follows directly from Eq. (2.3) and allows for an algorithm to produce approximations of each of the proto-tiles $T_1, T_2, \ldots, T_N$ in the self-affine or self-similar hierarchy.

PROPOSITION 10.2. Let $\mathbf{T} = (T_1, T_2, \ldots, T_N)$ be the fixed point of a GIFS G, and let $E_i^{(n)}$ denote the set of all finite, directed edge paths of length n in the graph G with initial vertex i. Then T_i is the limit with respect to the Hausdorff metric of the sets $\{f_p(0) \mid p \in E_i^{(n)}\}$ as $n \to \infty$.

According to the proposition above, the graph G can be regarded as a finite state machine. If the initial state is vertex i, then the tile T_i is the language accepted by the machine. (In fact, this is the point of view taken by Thurston [**Th**] in the Pisot tiling example in §10.4.) Recall that a *finite state machine* M over the alphabet F is a finite set S (the states of the machine), a map $t : F \times S \to S$ (the state transition map), together with a distinguished element $I \in S$ (the initial state), and a distinguished set $OK \subset S$ (the accepting states). A finite state machine can be represented as a directed graph in which each state is represented by a node and each transition $(f, s) \mapsto s'$ is represented by an arc from s to s' labeled f. A word w in the alphabet F is *accepted* by M if, when you start at I and go along the direction given by w, you end up in OK. An infinite word is accepted if each finite prefix is accepted. The GIFS graph G is made into finite state machine by declaring the vertices of G accepting states and adding "fail states" so that the transition map is defined for on all $F \times S$.

In a code $(c_0, c_1, \ldots)$ for a self-affine or self-similar tiling, the position c_n of a tile $A^n(T_j)$ in tile $A^{n+1}(T_i)$ is completely determined by f_e where e is the appropriate edge from vertex i to vertex j in the graph G. Therefore, a code for such a tiling corresponds to (the equivalence class of) an infinite directed path in G with a given terminal vertex. (Two edge paths with the same terminal vertex are equivalent if they coincide except possibly for the last finite number m of edges.) If the hierarchy forces uniqueness, then there is a bijection between such equivalences classes of directed paths and the (partial) tilings. In fact, the tilings can be given explicitly. In the self-affine case each contraction can be written in the form

$$f_e(x) = A^{-1}x + d_e, \tag{10.6}$$

where $d_e \in \mathbb{R}^d$. In the self-similar case each contraction is of the form

$$f_e(x) = c\, g_e(x), \tag{10.7}$$

where $c < 1$ and g_e is an isometry. If p is an infinite, directed edge path in the graph G with fixed terminal vertex, let $p(n)$ denote the finite, directed edge path with the same terminal vertex consisting of the last n edges in p. By carefully applying the definitions we obtain the following tilings.

PROPOSITION 10.3. Let $\mathbf{T} = (T_1, T_2, \ldots, T_N)$ be the proto-tiles of a self-affine or self-similar hierarchy corresponding to the graph iterated function system G.

1. If $\mathcal{T}$ is a self-affine tiling (contractions of the form 10.6) with code given by path $p = \cdots e_2 e_1 e_0$, then
$$\mathcal{T} = \bigcup \{ \sum_{j=0}^{n} (A^j (d_{e'_j} - d_{e_j}) + T_i) \},$$
where the union is over all n and all edge paths $q = e'_n \cdots e'_1 e'_0$ with the same initial vertex as $p(n)$, and i is the terminal vertex of q.
2. If $\mathcal{T}$ is a self-similar tiling (contractions of the form 10.7) with code given by path p, then
$$\mathcal{T} = \bigcup \{ (g_{p(n)}^{-1} \circ g_q)\ (T_i) \},$$
where the union is over all n and all edge paths q that have the same length and initial vertex as $p(n)$, and i is the terminal vertex of q.

10.4. Examples. Four types of examples of self-similar hierarchies are mentioned in this section. Recall that a self-similar hierarchy is completely determined by the first subdivision rule, that is, by Eq. (10.4) of the GIFS graph G.

Polygonal hierarchies. Numerous sporadic self-similar hierarchies using a single polygonal tile have been constructed [**GS**]. A simple example is the **L**-shaped triomino hierarchy with subdivision rule as given by the third diagram in Figure 2. This particular hierarchy forces uniqueness; so by the results of §9 there are uncountably many **L**-shaped triomino tilings, all nonperiodic, of finite type and locally isomorphic. This is called the *chair tiling*, and it has obvious analogues in higher dimensions.

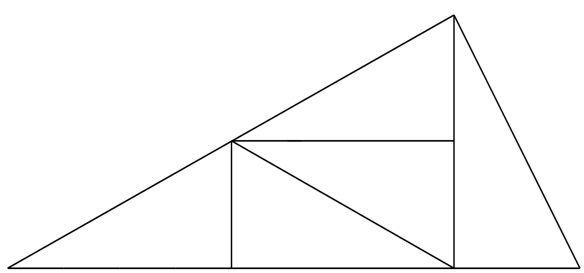


FIGURE 17. Subdivision rule for the pinwheel tiling.

The best known polygonal self-similar hierarchy is the Penrose hierarchy in Figure 13 - already discussed in §9. Another important hierarchy is the the pinwheel hierarchy [**R1**] based on $1, 2, \sqrt{5}$ right triangles, the subdivision rule shown in Figure 17. This hierarchy has the property that, up to congruence, there is one proto-tile, but in any of the uncountably many, nonperiodic pinwheel tilings, the tile appears in (countably) infinitely many orientations.

Hierarchies using a free group endomorphism. For certain special cases, Kenyon [**Ke3**] has extended the recurrent set method of §6.[5] Using essentially the same notation as in §6, let $G := G\langle a_1, \ldots, a_N \rangle$ denote the free group on N generators; let $\sigma : G \to G$ be an endomorphism. Using the notation $[a\ b] = aba^{-1}b^{-1}$ for the commutator, assume that each $\sigma([a_i\ a_j])$ is the product of conjugates of various $[a_{i_r}\ a_{j_r}]$. Kenyon finds a family of endomorphisms that satisfy this assumption. Take, for example, the case $N = 3$ and let

$$\begin{array}{rcl} \sigma(a) & = & b \\ \sigma(b) & = & c \\ \sigma(c) & = & c^q a^{-s} b^{-r} \end{array}$$

where $q, r \geq 0$, $s \geq 1$. Then there is a complex root λ of $x^3 - qx^2 + rx + s = 0$ such that, if $f : G \to \mathbb{C}$ is the homomorphism determined by $f(a) = 1, f(b) = \lambda, f(c) = \lambda^2$ and p denotes the corresponding polygonal path, then

$$\begin{array}{rcl} A_n & = & \lambda^{-n} p(\sigma^{(n)}([a\ b])) \\ B_n & = & \lambda^{-n} p(\sigma^{(n)}([b\ c])) \\ C_n & = & \lambda^{-n} p(\sigma^{(n)}([a\ c])) \end{array}$$

converge in the Hausdorff metric to closed curves A, B, C, respectively. Let T_a, T_b, T_c denote the enclosed compact tiles. Then λT_a is T_b; λT_b is the non-overlapping union of s translates of T_c and r translates of T_b; and λT_c is the non-overlapping union of q translates of T_b and s translates of T_a. This gives a subdivision rule for a translationally self-similar hierarchy. Some associated tilings are illustrated in [**Ke3, So1**]. Figure 18 is an example with six types of tiles, courtesy of R. Kenyon, whose expansion is a complex root of $x^4 + x + 1$. It is also a Pisot tiling as defined in the next paragraph.

Pisot tilings. Thurston [**Th**] considers radix representation of a real number on the line or complex number in the plane in the form $z = \sum_{i=i_0}^{n} a_i \beta^{-i}$, where β is a fixed real (complex) number and the a_i are chosen from a finite set D of algebraic integers in $\mathbb{Q}(\beta)$, and D contains 0. In general, D is not a digit set in the sense of §2. It is not difficult to choose D so that every number z has such a radix representation, but the representation is usually not unique.

The first step in constructing a self-similar hierarchy is to choose an ordering of $D : d_1 < d_2 < \cdots < d_N$. A *proper* representation of a number z is the one which is greatest in the corresponding lexicographic order. A representation of z is *weakly proper* if every finite initial segment of z can be extended to a proper representation. As a one dimensional example consider base $\tau = \frac{1+\sqrt{5}}{2}$ and $D = \{0, 1\}$ with $0 < 1$. Then $.101010\ldots$ is weakly proper, but not proper because $1 = .101010\ldots$. In this example the weakly proper representations are exactly those that contain no two consecutive 1's.

Thurston shows that if β is a complex (or real) Pisot number, an algebraic integer such that all its Galois conjugates except β and $\overline{\beta}$ lie inside the unit circle,

[5] Also related is the work of Garcia-Escudero and Kramer [**G-EK**] concerning an interpretation of certain 2-dimensional tilings using automorphisms of free groups.

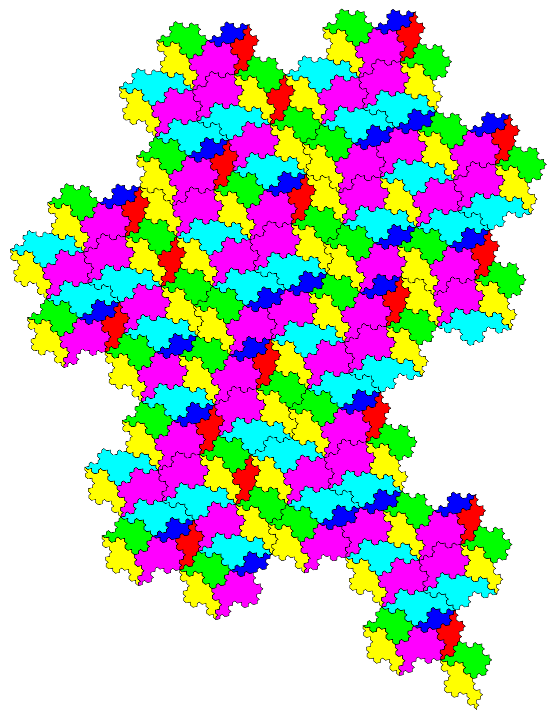


FIGURE 18. Pisot tiling.

then there exists a finite state machine $M(\beta, D)$, as defined in §10.3, which will recognize whether a sequence of elements from D gives a weakly proper representation for some number z. (In the one dimensional case, the finite state machine can be explicitly constructed from the *carry sequence*, which is the sequence of digits in the weakly proper representation of 1. If β is a Pisot number then the carry sequence is eventually periodic. In the example above $.101010\ldots$ is the carry sequence.)

A self-similar hierarchy can be constructed from the finite state machine. Turn the finite state machine into a graph iterated function system as follows. Given a pair (β, D), where β is a Pisot number, first remove all the FAIL states (the states that are not OK) from the associated finite state machine $M(\beta, D)$. Then relabel the edges as follows. On each edge e replace its label d_e by the contraction $f_e(z) = \beta^{-1}(z + d_e)$. This graph G, with say N nodes, determines a GIFS. The attractor of this GIFS is $(T_1, T_2, \ldots, T_N)$, where T_j can be described as follows. According to Proposition 10.2, the tile T_j consists of all points $z = \sum_{i=1}^{\infty} a_i \beta^{-i}$, where $a_i \in D$ for all i, and where the word $a_1 a_2 a_3 \ldots$ is accepted by the finite state machine $M(\beta, D)$ with vertex j as the initial state. In other words, T_j consists of all real (complex) numbers with decimal expansion only to the right of the decimal point and with weakly proper representation corresponding to a directed path in G starting at vertex j. Let E_{ji} denote the set of edges from vertex j to vertex i. Since multiplication by β is just a right shift of the decimal point we have the subdivision rule for each j:

$$\beta(T_j) = \bigcup_{i=1}^{N} \bigcup_{e \in E_{ji}} T_i + d_e.$$

The union is non-overlapping because of the uniqueness of weakly proper representation.

The Pisot tiling of R. Kenyon in Figure 18 has six types of tiles and uses radix β where β is a complex root of $x^4 + x + 1$ with modulus greater than 1.

Dual hierarchies. Given a self-affine or self-similar hierarchy $\mathcal{P}$ in terms of a GIFS graph G, the construction of a dual hierarchy $\mathcal{P}^*$ is outlined by Thurston [**Th**] and expanded on and generalized by Gelbrich [**Ge3**] and by Praggastis [**Pra**]. It also appears in a paper on the construction of sofic partitions of hyperbolic toral automorphisms by Kenyon and Vershik [**KeVe**]. We sketch the basic idea of the construction given in [**Ge3**].

Given a GIFS graph G define a dual graph G^* as follows. If G has vertex set $\{1, 2, \ldots, N\}$, let G^* have vertex set $\{1^*, 2^*, \ldots, N^*\}$. Each edge in G labeled with a contraction $f : \mathbb{R}^d \to \mathbb{R}^d$ is replaced by an oppositely directed edge in G^* labeled by its dual f^*, which is defined in the next paragraph.

A toral automorphism $\tilde{A} : \mathbb{R}^M \to \mathbb{R}^M$ is a linear map leaving some lattice L invariant and such that $|\det \tilde{A}| = 1$. If each eigenvalue of $\tilde{A}$ has modulus $\neq 1$, then $\mathbb{R}^M = E_s \oplus E_u$ such that $\tilde{A}_s = \tilde{A}|_{E_s}$ is a contraction and $\tilde{A}_u = \tilde{A}|_{E_u}$ is an expansion. It is known that, for a map $A : \mathbb{R}^d \to \mathbb{R}^d$ that is the expansion for certain self-similar or self-affine hierarchies, there exists a toral automorphism $\tilde{A} : \mathbb{R}^M \to \mathbb{R}^M$ that is a lifting of A. This means that there is an embedding

$$i : \mathbb{R}^d \hookrightarrow \mathbb{R}^M$$

such that $i(\mathbb{R}^d) = E_u$ and $\tilde{A} \circ i = i \circ A$. Let $A^* = \tilde{A}^{-1}|_{E_s}$ be the inverse of the lifting restricted to the complementary space.

More generally, for such a self-affine hierarchy (and sometimes for a self-similar hierarchy) an affine contraction $f : \mathbb{R}^d \to \mathbb{R}^d$ with linear part A can also be lifted to an affine map $\tilde{f} : \mathbb{R}^M \to \mathbb{R}^M$ given by $\tilde{f}(x) = \tilde{A}(x - b)$ where $\tilde{A}$ leaves E_u and E_s invariant and maps L bijectively onto itself. Let $f^* : E_s \to E_s$ be defined by

$$f^*(x) = A^*(x) + \mathrm{proj}_{E_s} b.$$

Now the dual graph G^*, and thus the dual hierarchy $\mathcal{P}^*$, is defined.

Some examples of this dual construction appear in [**So1, Th**]. Figure 19, courtesy of R. Kenyon, shows the 2-dimensional dual of a 1-dimensional Pisot tiling that uses the real root of $x^3 - x^2 - 1$ as base and $\{0, 1\}$ as digit set. The subdivision rule for the three types of tiles is of the form: $T_1 = f_1(T_2)$; $T_2 = f_2(T_3)$; $T_3 = f_1(T_1 \cup T_3)$.

Gelbrich [**Ge3**] computes the dual of the Penrose hierarchy and gives illustrations of some associated tilings. These tilings appeared previously in [**BGu**] and have the following appealing property. For the Penrose tiles (kite and dart, thick and thin rhombs, or acute and obtuse triangles), somewhat artificial matching rules guarantee that the tilings are self-similar and, consequently, that the proto-tile set is aperiodic. For the dual proto-tile set, the matching rules are a direct consequence of the fractal shape of the boundaries of the two proto-tiles. Every tiling by copies of the dual proto-tiles must be a self-similar tiling.

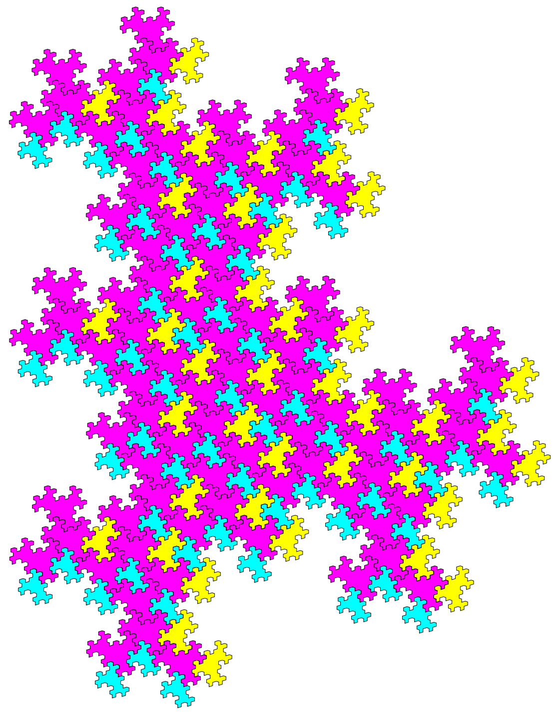


FIGURE 19. Dual of a Pisot tiling.

It is not always the case that (1) the dual tiles have non-empty interior and (2) the union in (10.4) for the dual is non-overlapping. But these two conditions turn out to be equivalent [**Ge4**].

11. Concluding Remark

A main question at this point is how, in general, to construct the self-affine and self-similar hierarchies - and hence tilings. Any such hierarchy is the attractor of a GIFS. So from the GIFS point of view the issue is how to choose the parameters (the linear map A and translations d_e in 10.6 or the expansion factor c and isometries g_e in 10.7) so that the tiles in the attractor of the GIFS have nonempty interior (condition 1 in §10.3). In the case of a single proto-tile this was done in §2 by choosing the set of translations d_e as a digit set D. In the absence of periodicity, however, there is no obvious analogue of the quotient $D = L/A(L)$ of a lattice by the sublattice. There are known sufficient conditions to insure nonempty interior, including the "open set condition" [**F1**] and an equivalent algebraic condition due to Bandt and Graf [**BGr**], but these are usually not readily applicable in practice. A reasonable approach to the problem appears open at this time.

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