JOURNAL OF COMBINATORIAL THEORY, Series B 47, 251–261 (1989)

Discrete Jordan Curve Theorems

ANDREW VINCE

Department of Mathematics,
University of Florida, Gainesville, Florida 32611

AND

C. H. C. LITTLE*

Department of Mathematics and Statistics,
Massey University, Palmerston North, New Zealand

Communicated by U. S. R. Murty

Received March 25, 1987

Discrete versions of the Jordan Curve Theorem are proved.

1. INTRODUCTION

There has been recent interest in combinatorial versions of classical theorems in topology. In particular, Stahl [5] and Little [3] have proved discrete versions of the Jordan Curve Theorem. The classical theorem states that a simple closed curve γ separates the 2-sphere into two connected components of which γ is their common boundary. The statements and proofs of the combinatorial versions in [3, 5] are given in terms of permutation pairs and colored graphs (see Sect. 4). In this paper short proofs of three graph theoretic versions of the Jordan Curve Theorem are given.

A graph G may have multiple edges but no loops. It is understood that each vertex in a cycle has degree 2. A cycle γ in a graph G will be said to have the *First Jordan Curve Property* (JCP1) if there exist connected proper subgraphs I and O of G such that $I \cap O = \gamma$ and $I \cup O = G_0$, where G_0 is the connected component of G containing γ. In particular, any path from a vertex of I to a vertex of O contains a vertex of γ. A family C of

* This paper was written while the second author was on leave at the University of Waterloo.

0095-8956/89 $3.00

cycles of a graph G is called a *double cover* if every edge in G is contained in exactly two cycles of C. The 1-skeleton triangulation of a closed surface is an example, where C is the set of triangles. More generally, any 2-cell embedding of a graph on a surface is an example, if the set C of boundaries of 2-cells contains only cycles.

For a set C of cycles in G let $K(C)$ be the subspace of the cycle space spanned by C and define $k = k(C) = |C| - \dim K(C)$. Let l denote the number of connected components of G. Define the Euler characteristic of G with respect to a double cover C by $\chi(G, C) = |V| - |E| + |C|$, where V and E are the vertex and edge sets, respectively. For a triangulation or a graph embedding, this is the usual definition. The following discrete version of the Jordan Curve Theorem is proved in Section 3. Two other discrete versions of the Jordan Curve Theorem appear as Theorem 2 in Section 3 and Theorem 3 in Section 4.

THEOREM 1. *If $\chi(G, C) = k + l$ for some double cover C, then every cycle not in C has the First Jordan Curve Property.*

Note that a 2-connected planar graph G, where C is the set of boundaries of regions, satisfies the hypotheses of the theorem with $k = l = 1$ and $\chi(G, C) = 2$.

2. DOUBLE COVERS

If H is a subgraph of a graph G with double cover C, let C_H denote the subset of C contained in H. The pair (G, C) is called *irreducible* if G has no proper subgraph H such that C_H is a double cover of H. In general, a subgraph H of G where (H, C_H) is irreducible is called an *irreducible component* of G. It is clear that G is the union of its irreducible components and that the intersection of any two irreducible components is either empty or consists of isolated vertices.

In the cycle space of a graph sums are modulo 2 (symmetric difference), and if $\sum c_i$ is a sum of cycles, it is always assumed, without loss of generality, that there are no repetitions among the c_i.

LEMMA 1. *If C is a double cover of a graph G, then*

(i) *$k(C)$ equals the number of irreducible components of (G, C).*

(ii) *$\chi(G, C) \leqslant k + l$ with equality if and only if C spans the cycle space of G.*

Proof. (i) Let (G_i, C_i), $1 \leqslant i \leqslant m$, be the irreducible components of (G, C) and let B be a subset of C consisting of $|C_i| - 1$ cycles from each set

C_i. If B is dependent, then some subset D of B forms a double cover of the union of the cycles in D, contradicting the irreducibility of C.

(ii) Let $\mathscr{C}(G)$ denote the cycle space of G. Then

$$\begin{aligned}\chi(G, C) &= |V| - |E| + |C| = |V| - |E| + k + \dim K(C)\\ &\leqslant |V| - |E| + k + \dim \mathscr{C}(G)\\ &= |V| - |E| + k + |E| - |V| + l = k + l\end{aligned}$$

and equality is achieved if and only if $\dim K(C) = \dim \mathscr{C}(G)$. ∎

LEMMA 2. *If C is an irreducible double cover of G, then*

(i) $k = l = 1$,

(ii) $\chi(G, C) \leqslant 2$ *with equality if and only if C spans the cycle space of G.*

Proof. If G is not connected, then for any component H of G the set of cycles C_H forms a double cover of H, contradicting the irreducibility of C. That $k = 1$ follows directly from (i) of Lemma 1 and part (ii) follows from part (ii) of Lemma 1 and part (i) of Lemma 2. ∎

In the case of equality in the second part of Lemma 1 we have:

LEMMA 3. *Let C be a double cover of a graph G. If $\chi(G, C) = k + l$ then for every block B of G*

(i) *(B, C_B) is irreducible,*

(ii) $\chi(B, C_B) = 2$.

Proof. Let $G_1, G_2, ..., G_m$ be the irreducible components of G and $C_i := C_{G_i}$ the irreducible double cover of G_i for each i. We claim that the G_i are exactly the blocks of G. From the definition of Euler characteristic it follows by induction that

$$\chi(G, C) \leqslant \sum \chi(G_i, C_i) - m + l,$$

with equality if and only if each G_i is a block of G. (1)

Also by Lemma 1

$$k(C) = m. \tag{2}$$

From (1), (2), and Lemma 2 it follows that

$$\chi(G, C) \leqslant \sum \chi(G_i, C_i) - m + l \leqslant 2m - m + l = m + l = k + l$$

with equality if and only if each G_i is a block of G. ∎

The dual can be defined for a graph G with respect to a double cover C. Let G^* be the graph with vertex set C such that for each edge $e \in E(G)$ there is an edge $e^* \in E(G^*)$ joining the two cycles in C containing e. For any vertex $v \in V(G)$ let $E(v)$ be the set of edges incident to v. Then the set $\{e^* \mid e \in E(v)\}$ forms a set $C^*(v)$ of disjoint cycles in G^*. Now $C^* = \bigcup \{C^*(v) \mid v \in V(G)\}$ is a double cover of G^*. The *dual* of (G, C) is denoted $(G, C)^* = (G^*, C^*)$. Thus if C is the set of boundaries of regions formed by a graph G embedded on a surface, then G^* is the ordinary dual. Lemmas 4 and 5 below follow from the definitions.

LEMMA 4. *If $G_1, ..., G_m$ are the irreducible components of G with respect to a double cover C, then G_i^* is connected for each i and $G^* = \bigcup G_i^*$, where the union is disjoint.*

LEMMA 5. *We have $(G, C)^{**} = (G, C)$ if and only if $|C^*(v)| = 1$ for all $v \in V(G)$. In this case $\chi(G^*, C^*) = \chi(G, C)$.*

For any cycle γ in G, let $E^*(\gamma) = \{e^* \mid e \in E(\gamma)\}$.

LEMMA 6. *Let G be a graph with double cover C and γ any cycle in G. Then $G^* - E^*(\gamma)$ has at most one more connected component than G^*.*

Proof. Let H^* be any connected component of $G^* - E^*(\gamma)$. Let D be the subset of C corresponding to $V(H^*)$ and H the subgraph of G that is the union of the cycles in D. Consider $\sigma = \sum_{c \in D} c$. By the construction $\sigma \subset \gamma$ and every vertex of σ has degree greater than 1. Therefore, either $\sigma = \emptyset$ or $\sigma = \gamma$. In either case H contains γ. Now consider any irreducible component G_0 of G. Then, by Lemma 4, if γ does not lie entirely in G_0, then $G_0^* - E^*(\gamma)$ remains connected. If γ does lie entirely in G_0, then $G_0^* - E^*(\gamma)$ has at most two connected components because each edge of γ is covered only twice by C. ∎

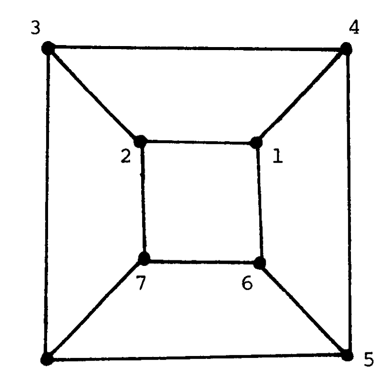

FIGURE 1

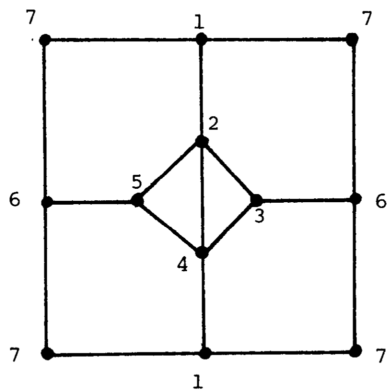

FIGURE 2

A curve γ in G is said to have the *Second Jordan Curve Property* (JCP2) if $G^* - E^*(\gamma)$ has one more connected component than G^*. If C is an irreducible double cover of G, then by Lemmas 4 and 6, γ satisfies JCP2 if and only if G^* has exactly one connected component and $G^* - E^*(\gamma)$ has exactly two. For a graph embedded on a surface, this definition corresponds to the intuitive notion of a separating curve.

JCP1 and JCP2 are not equivalent. In Fig. 1 the graph is the 1-skeleton of the cube with the understanding that the vertices labeled 1 are identified, and the cycles in C are the boundaries of the six faces. Then the cycle (2, 3, 4, 5, 6, 7, 2) has JCP2 but not JCP1. In the other direction the graph in Fig. 2 is understood to be embedded in the torus with opposite sides of the square identified. The cycles in C are the boundaries of the faces: (1, 2, 5, 6, 7), (1, 2, 3, 6, 7), (1, 4, 5, 6, 7), (1, 4, 3, 6, 7), (2, 3, 4), and (2, 4, 5). Then the cycle (1, 2, 3, 4, 1) has JCP1 but not JCP2.

3. DISCRETE JORDAN CURVE THEOREM

This section contains the proof of Theorem 1 and a second discrete Jordan Curve Theorem, Theorem 2, based on the dual graph.

Assume that C is an irreducible double cover of G and that $\chi(G, C) = k + l$. By Lemma 2, C spans the cycle space of G. Hence there is a subset D of C such that $\sum_{c \in D} c = \gamma$. Also $\sum_{c \in C-D} c = \gamma$. Let I and O be the union of the cycles in D and in $C - D$, respectively. Then I and O are exactly the "inside" and "outside" subgraphs of γ in the definition of JCP1.

Proof of Theorem 1. The theorem is first proved in the case that (G, C) is irreducible. Since every edge in G lies on some cycle of C, $I \cup O = G$. Clearly I and O are proper subgraphs of G. We next show that I and O are connected. Clearly $\gamma \subset I \cap O$. By way of contradiction, assume that there is a component I' of I that is disjoint from γ. Then $(I', D_{I'})$ contradicts the irreducibility of (G, C). The same argument shows that O is connected. It remains only to show that $I \cap O = \gamma$. Every edge in I, except those on γ, lies on exactly two cycles of D and therefore on no cycle of $C - D$. Hence for edge sets $E(I) \cap E(O) = E(\gamma)$. Finally assume, by way of contradiction, that there is a vertex v, not on γ, such that $v \in I \cap O$. Because I and O are connected and $\gamma \subset I \cap O$, for a suitable such v there must exist a cycle

$$\tau\colon v = v_0, v_1, ..., v_m, ..., v_n = v, \ 0 < m < n,$$

such that $(v_{i-1}, v_i) \in E(I)$ for $i = 1, ..., m$ and $(v_{i-1}, v_i) \in E(O)$ for $i = m+1, ..., n$. Since C spans the cycle space of G, there is a subset F of C such that $\sum_{c \in F} c = \tau$. Then there must exist a sequence $F' = (c_1, ..., c_k)$ of cycles in F with the following properties: (1) $(v, v_1) \in c_1$ and $(v, v_{n-1}) \in c_k$; (2) $v \in c_i$ for $1 \leqslant i \leqslant k$; and (3) there are edges $(v, u_i) \in c_i \cap c_{i+1}$ for $i = 1, 2, ..., k-1$. Because $\sum_{c \in D} c = \gamma = \sum_{c \in C - D} c$ and no edge incident to v lies in γ, either $F' \subset D$ or $F' \subset C - D$. But this implies that edges (v, v_1) and (v_{n-1}, v) both lie in I or both lie in O. This contradicts $E(I) \cap E(O) = E(\gamma)$.

Now consider the general case where (G, C) may be reducible. Let B be the block in which γ is contained. Lemma 3 states that the result above can be applied to B. Let I_B and O_B be the two subgraphs of G guaranteed by the theorem. Consider the connected components of the graph obtained from G by removing B. Let I' and O' be the union of those components that have a vertex in common with I_B and O_B, resp. If a component has a vertex in common with both, i.e., with γ, then it is placed arbitrarily in one of I' or O', but not both. Then $I = I_B \cup I'$ and $O = O_B \cup O'$ are the subgraphs required in Theorem 1. ∎

The following discrete Jordan Curve Theorem has the advantage that the converse also holds. It has the disadvantage that JCP2, in contrast to JCP1, depends on the double cover C (via the construction of the dual). Comments on the converse of Theorem 1 are made in Section 5.

THEOREM 2. *Let G be a graph with double cover C. Then every cycle in G has the Second Jordan Curve Property if and only if $\chi(G, C) = k + l$.*

Proof. Assume $\chi(G, C) = k + l$ and that γ is a cycle in G. Then γ must lie in some block B of G. By Lemma 3, (B, C_B) is irreducible and $\chi(B, C_B) = 2$. By Lemma 4, B^* is one of the connected components of G^*. Let D be the subset of C and I and O the graphs defined at the beginning of this section. Then $B^* - E^*(\gamma) = I^* \cup O^*$ and $I^* \cap O^* = \varnothing$. Therefore $B^* - E^*(\gamma)$ has one more connected component than B^*; and hence the same is true for $G^* - E^*(\gamma)$ and G^*.

Conversely assume that $\chi(G, C) < k + l$, and let γ be a cycle not in the span of C. Assume, by way of contradiction, that $G^* - E^*(\gamma)$ has more components than G^*. Then for some connected component G_0^* of G^* there are two connected components, H^* and another, of $G_0^* - E^*(\gamma)$. Let D and $\sigma = \sum_{c \in D} c$ be as defined in the proof of Lemma 6. As in the proof of that lemma, either $\sigma = \gamma$ or $\sigma = \varnothing$. But if $\sigma = \varnothing$, then H^* is the only connected component of $G_0^* - E^*(\gamma)$, a contradiction. Hence $\gamma = \sigma = \sum_{c \in D} c$, contradicting the assumption that γ is not in the span of C. ∎

4. Orientable Graphs

Let G be a graph with double cover C. For a cycle γ let $E'(\gamma)$ be the set of edges incident to, but not on γ. Define an equivalence relation on $E'(\gamma)$ as follows: $e_1 \sim e_2$ if edges e_1 and e_2 both belong to the same cycle in C. The equivalence relation is then the transitive closure of $\sim$. It is not difficult to show that if $E'(\gamma)$ is not empty and $|C^*(v)| = 1$ for all $v \in V(G)$, then there are either 1 or 2 equivalence classes. Call (G, C) *orientable* if (1) $|C^*(v)| = 1$ for all $v \in V(G)$ and (2) there are two equivalence classes for every cycle γ not in C. In this case call the equivalence classes $L' = L'(\gamma)$ and $R' = R'(\gamma)$. If $\gamma \in C$ define $L' = L'(\gamma) = E'(\gamma)$ and $R' = R'(\gamma) = \varnothing$. The terminology is motivated by the fact that for an embedding of a graph on a surface property (1) holds and property (2) holds exactly if the surface is orientable. On this surface the "left" edges L' lie on the "opposite side" of γ from the "right" edges R'. By a path from edge e to edge e' is meant a path from a vertex of e to a vertex of e'. Let $L = L(\gamma)$ and $R = R(\gamma)$ be the sets of vertices reachable from edges of L' and R', resp., by paths not containing a vertex in γ. Clearly $L \cup R = V(G_0) - V(\gamma)$, where G_0 is the connected component of G containing γ. A cycle γ is said to have the *Third Jordan Curve Property* (JCP3) if $L \cap R = \varnothing$. In particular, if γ has JCP3, then every path from L' to R' crosses γ.

Lemma 7. *Let γ be a cycle in an orientable graph. If γ has JCP2 then γ has JCP3. Conversely if $L(\gamma) \neq \varnothing$, $R(\gamma) \neq \varnothing$, and γ has JCP3 then γ has JCP2.*

Proof. Without loss of generality we may assume that G is connected. If G is connected and (G, C) is orientable, then C is an irreducible cover. Otherwise any vertex in the intersection of two irreducible components would violate property (1) in the definition of orientability. For a vertex $v^* \in V(G^*)$ let $c(v^*)$ denote the corresponding cycle in C. Now assume that $G^* - E^*(\gamma)$ has two components I^* and O^*. Then $I = \bigcup_{v^* \in I^*} c(v^*)$ and $O = \bigcup_{v^* \in O^*} c(v^*)$ are subgraphs of G such that $I \cup O = G$ and

$E(I) \cap E(O) = E(\gamma)$. Also $L' \subset I$ and $R' \subset O$ (or vice versa). To show that $L \cap R = \emptyset$ it is sufficient to show that I and O have no vertices in common except those on γ. But if I and O have such a vertex in common, property (1) in the definition of orientability of G implies that I and O also have an edge, not in γ, in common. This is a contradiction.

Conversely assume that $G^* - E^*(\gamma)$ is connected. Then there exists a path $\not{p}^*$ from v_1^* to v_2^* in $G^* - E^*(\gamma)$ such that $c(v_1^*)$ contains a vertex in L and $c(v_2^*)$ contains a vertex in R. Consider the subgraph $H = \bigcup_{v^* \in \not{p}^*} c(v^*)$ of G. Let $\not{p} = v_1 v_2 \cdots v_n$ be a shortest path in H from a vertex in L to a vertex in R. To show that $L \cap R \neq \emptyset$, it is sufficient to show that $\not{p}$ does not cross γ. Assume, by way of contradiction, that v_{i+1} is the first vertex of $\not{p}$ that crosses γ. Then v_i lies in either L or R. If v_i lies in R, then the path $v_1 v_2 \cdots v_i$ contradicts the minimality of $\not{p}$. If v_i lies in L, assume that subpath $v_i v_{i+1} \cdots v_{j-1}$ lies on γ and that v_j does not lie on γ. By the definition of L' edge $v_{j-1} v_j$ lies in L' and v_j lies in L. Then path $v_j v_{j+1} \cdots v_n$ contradicts the minimality of $\not{p}$. ∎

THEOREM 3. *Let G be an orientable graph with double cover C. Then every cycle in G has the Third Jordan Curve Property if and only if $\chi(G, C) = 2l$.*

Proof. As in the proof of Lemma 7, for (G, C) orientable, the connected components are irreducible. Therefore $k = l$ by Lemmas 1 and 2. If $\chi(G, C) = 2l = k + l$ then by Theorem 2 and Lemma 7 every cycle has JCP3. For the converse assume $\chi(G, C) < 2l$. It is sufficient to find a cycle γ, not in $K(C)$, and such that $L(\gamma)$ and $R(\gamma)$ are non-empty. Then, exactly as in the proof of Theorem 3, γ will not have JCP2 and, by Lemma 7, will not have JCP3. Thus let γ be any cycle not in $K(C)$. To complete the proof we need only show that $L \neq \emptyset$ and $R \neq \emptyset$. Assume, by way of contradiction, that $L = \emptyset$. Let D be the set of all cycles in C containing edges in L'. Note that L' contains only chords of γ, because $L = \emptyset$, and so the cycles in D contain only edges and chords of γ. Hence $\sum_{c \in D} c = \gamma$, contradicting the assumption that $\gamma \notin K(C)$. ∎

Examples: permutation pairs and 3-graphs. These examples are of interest in relating the results of this paper to those of [3, 5]. Both examples are generalizations of graph embeddings. The 3-graph is slightly more general, allowing for non-orientable, as well as orientable, embeddings. Permutation pairs and 3-graphs are defined below and the correspondence between them is explained. The versions of the Jordan Curve Theorem in [3, 5], based on these concepts, are cases of Theorem 3 in this section.

Let $A = \{v, e, f\}$. A *3-graph* G is a connected, regular graph G of degree 3 together with a coloring $E(G) \to A$ such that incident edges have different colors. These graphs and their generalizations have been investigated by

Ferri, Gagliardi, Lins, Vince [1, 2, 6, 7], and others. Let G_a, $a \in A$, denote the subgraph of G obtained by deleting all edges of color a. The components of G_a are two-colored cycles of G, called *faces of type a*. A face F_a of type a is *incident* to a face F_b of type b if $F_a \cap F_b \neq \emptyset$. In the case that the faces of type e are all 4-gons, the faces of type v, e, and f can be realized as the vertices, edges, and faces, resp., of a graph embedding, in such a way that the incidence relation above corresponds to the ordinary notion of incidence. Moreover the original graph G is the dual graph of the barycentric subdivision of this embedding [6].

A *permutation pair* consists of a pair of permutations (π_v, π_f) of a finite set S. The concept was introduced by Stahl [4], and a similar representation of maps by permutations is due to Tutte [8]. For a permutation pair (π_v, π_f) define $\pi_e = \pi_v \pi_f$. For $a \in A = \{v, e, f\}$ define a *face of type a* as a cycle in the permutation π_a. A face F_a of type a is called *incident* to a face F_b of type b if $F_a \cap F_b \neq \emptyset$. In the case that π_e is an involution (order 2), the faces of type v, e, and f can be realized as the vertices, edges, and faces, resp., of a graph embedding on an orientable surface [4].

There is a bijection between the set of bipartite 3-graphs and the set of permutation pairs. Given a 3-graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$, let σ_v be the order 2 permutation of V that maps each vertex to the vertex joined to it by an edge of color v; similarly for σ_e and σ_f. Then the corresponding permutation pair is (π_v, π_f), where π_v and π_f are permutations of V_1 given by $\pi_v = \sigma_e \sigma_f$ and $\pi_f = \sigma_v \sigma_e$.

Conversely, given a permutation pair (π_v, π_f), the corresponding 3-graph has vertex set $S \times \{1, -1\}$ where for each $i \in V_1$, the vertex $(i, 1)$ is joined to vertices $(\pi_f i, -1)$, $(i, -1)$, and $(\pi_v^{-1} i, -1)$ by edges colored v, e, and f, resp. [3]. The significance of G being bipartitte is the following. When G represents a graph embedding, this embedding is orientable if and only if G is bipartite [6].

Stahl [5] gives a version of the Jordan Curve Theorem in terms of permutation pairs and Little [3] interprets this result in terms of 3-graphs via the correspondence in the paragraph above. Both relate to the results of this paper as follows. Let G be a 3-graph. Consider the set of cycles $C_2 = G_v \cup G_e \cup G_f$, i.e., the 2-colored cycles. Since G is connected, C_2 is clearly an irreducible double cover of G. It is not hard to show that (G, C_2) is orientable if $\chi(G, C_2) = 2$. Then Theorem 3 implies that G has the JCP3 if and only if $\chi(G, C_2) = 2$. This is analogous to the theorem of [3, 5]. It would be interesting to know if the two theorems are actually equivalent.

5. An Open Problem

The converse of Theorem 1 in general is false. Figure 3 shows a connected graph where $\chi(G, C)=2\neq 3=k+1$, but every cycle not in C has the First Jordan Curve Property. The double cover of G in this counter-example is $C=\{1\ 2\ 4), (2\ 3\ 4), (1\ 2\ 3\ 4), (3\ 5\ 6), (5\ 6\ 1), (3\ 5\ 1\ 6)\}$. Note that C is a reducible double cover; there are two irreducible components. The converse of Theorem 1 in the irreducible case is open.

Conjecture 1. If $\chi(G, C)\neq 2$ for an irreducible double cover C of G, then there exists a cycle, not in C, that does not have the First Jordan Curve Property.

The following conjecture is related to Conjecture 1.

Conjecture 2. If $\chi(G, C)\neq 2$ for an irreducible cover C, then there exists a cycle γ, not in C, such that $G-\gamma$ is connected.

The connection between Conjecture 1 and Conjecture 2 is as follows. If "cycle" in Conjecture 2 is strengthened to "chordless cycle," then Conjecture 2 implies Conjecture 1. To see this, assume $\chi(G, C)\neq 2$ and let γ be a chordless cycle such that $G-\gamma$ is connected. Assume, by way of contradiction, that γ has JCP1 with separating sets I and O. Let $I'=I-\gamma$ and $O'=O-\gamma$. Then $I'\cup O'=G-\gamma$ and $I'\cap O'=\varnothing$, contradicting the connectedness of $G-\gamma$. The condition that γ is chordless is needed to ensure that neither I' nor O' is empty.

The following strong version of Conjecture 1 seems plausible, but is false: If $\chi(G, C)\neq 2$ for an irreducible double cover C, then the First Jordan

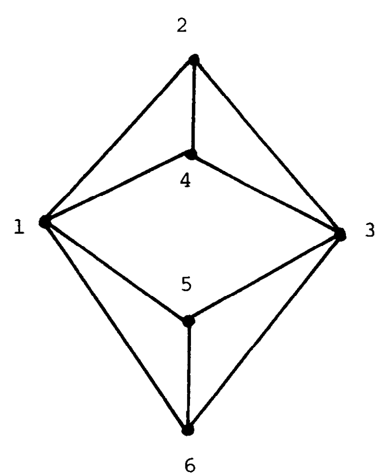

Figure 3

Curve Property fails for every cycle not in $K(C)$. The 3-graph in Fig. 2 again serves as a counterexample. Here $\chi(G, C) = 0$, but in Section 2 a curve γ, not in $K(C)$, was given that does have the Jordan curve property.

References

1. M. Ferri and C. Gagliardi, Crystallization moves, *Pacific J. Math.* **100** (1982), 85–103.
2. S. Lins, Graph-encoded maps, *J. Combin. Theory Ser. B* **32** (1982), 171–181.
3. C. Little, Cubic combinatorial maps, *J. Combin. Theory Ser. B* **44** (1988), 44–63.
4. S. Stahl, Permutation-partition pairs: A combinatorial generalization of graph embeddings, *Trans. Amer. Math. Soc.* **259** (1980), 129–145.
5. S. Stahl, A combinatorial analog of the Jordan Curve Theorem, *J. Combin. Theory Ser. B* **35** (1983), 28–38.
6. A. Vince, Combinatorial maps, *J. Combin. Theory Ser. B* **34** (1983), 1–21.
7. A. Vince, *N*-graphs, submitted for publication.
8. W. T. Tutte, Combinatorial oriented maps, *Canad. J. Math.* **31** (1979), 986–1004.