MAP DUALITY AND GENERALIZATIONS

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Abstract. A map is an embedding of a graph into a surface so that each face is simply connected. Geometric duality, whereby vertices and faces are reversed, is a classic construction for maps. A generalization of map duality is given and discussed both graph and group theoretically.

1. Introduction. A *map* is an embedding of a graph into a surface so that each face is simply connected. The geometric dual of a map, whereby vertices and faces are reversed, is a classic construction. The dual of the cube (as a map on the sphere) is the octahedron and the dual of the icosahedron is the dodecahedron. Maps have been generalized to hypergraph embeddings (hypermaps) [12] and to cell complexes in dimensions greater than 2 [3][10]. Maps and their generalizations have been investigated combinatorially and algebraically, as well as topologically, by way of rotation systems [4], permutations schemes [9], edge colored graphs [8][10], and map groups [6][10]. Several authors [5][7][8][13] have used these methods to extend the notion of duality. Recently Jones and Thornton [7] and James [5] investigate a group of operations on maps and hypermaps in terms of the automorphism group of a certain finitely presented group. These operations include the classical dual of a map.

This paper also concerns a generalization of map duality. Preliminary results on maps and their generalizations appear in Section 2. In particular, a graph theoretic generalization of map, called an n-graph, will be the central concept. In Section 3, a simple geometric duality construction is given that includes, as special cases, the duality of Jones and Thornton and of James. This duality is an equivalence

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX relation on n-graphs that preserves connectedness, the number of vertices and the automorphism group (Theorem 1). Duality is especially interesting for symmetric n-graphs, those with the greatest degree of symmetry. In fact, a pair of symmetric n-graphs are dual if and only if they have the same number of vertices and isomorphic automorphism groups (Theorem 4). The minimum symmetric cover of an arbirary n-graph is defined in Section 5. Although a pair of dual n-graphs need not be symmetric, it is shown that duality can always be lifted to the minimum symmetric cover (Theorem 5). The duality construction is discussed in group theoretic terms in Section 4. There is a bijection between n-graphs and permutation representaions of a certain group W. A pair of n-graphs are dual if and only if the corresponding permutation representations have the same image (Theorems 2 and 3). Jones and Thornton [7] and James [5] define certain map operations in terms of automorphisms of W, and, in Section 6, these operations are related to the notion of duality in this paper. Section 7 closes with two open problems on self-dual maps.

2. Graph encoded maps. Let $[n] = \{0, 1, \ldots, n-1\}$. An *n-graph* G is a finite graph, with loops and multiple edges allowed, regular of degree n (loops add only one to the degree), together with an edge coloring $f : E(G) \to [n]$ such that colors on incident edges are distinct. Two vertices that are joined by an edge colored i are called *i-adjacent*. In [8] and [10], it is shown how any map K on a surface can be uniquely represented by a 3-graph $G(K)$. An example is shown in Fig. 1. The map K on the sphere, having two vertices, two edges and two faces, appears in Fig. 1a. The 3-graph $G(K)$ that encodes K appears in Fig. 1b. This example is extremely simple so that it can easily be used to illustrate the concepts in this section. Slightly more complicated are the 3-graph in Fig. 8a and the 3-graph in Fig. 7a that enclodes the pyramid with square base (both maps on the sphere). A map can be regarded as a 2-dimensional cell complex. More generally, any polytope or cell complex K whose topological realization is an n-dimensional manifold (or even pseudomanifold) can be uniquely represented by an $(n+1)$-graph $G(K)$. Loops in an n-graph correspond to the boundary of the manifold.

An *i-residue* of an n-graph G is any connected component of G obtained by removing all edges colored i. Call two residues *incident* if the their intersection is non-empty. Let $G(K)$ be the $(n+1)$-graph that represents a cell complex K whose underlying space is an n-manifold. In [10] a bijection is provided between the faces of K and the residues of $G(K)$. Under this bijection i-faces of K correspond to i-residues of $G(K)$, and two faces are incident if and only the corresponding residues are incident. In the example of Fig. 1, the 0-residue ($\{1, 2\}$-colored cycle) marked v' in $G(K)$ corresponds to the vertex v in K, and the 1-residue marked e' corresponds to the edge e. The fact that v' and e' have non-empty intersection in $G(K)$ corresponds to the fact that vertex v and edge e are incident in K. Altogether in this example, $G(K)$ has two 0-residues, two 1-residues and two 2-residues, corresponding to the two vertices, two edges and two faces of K, respectively.

FIG. 1. A MAP K AND THE 3-GRAPH $G(K)$ THAT ENCODES IT.

The faces of a complex K are partially ordered by inclusion. Analogously, call an n-graph G *ordered* if there is a partial order $<$ on the residues such that for any pair x and y of residues, $x < y$ or $y < x$ if and only if x and y are incident residues. The 3-graph $G(K)$ in Fig. 1b is ordered, the ordering on the set of residues being inherited from the partial order on the set of faces in the corresponding map K in Fig. 1a. Some checking shows that the 3-graph in Fig. 2, having 12 residues, is not ordered. Therefore this 3-graph is not associated with any map. In fact, a connected 3-graph G is ordered if and only if $G = G(K)$ for some map K on a surface [10]. In light of this fact, the terms "map" and "ordered 3-graph" will be used interchangably.

FIG. 2. NON-ORDERED 3-GRAPH.

A necessary and sufficient condition for an n-graph to be ordered can be given in terms of the diagram. For an n-graph G, the *diagram* $D(G)$ is a complete graph with node set $[n]$. Each edge $\{i, j\}$ in the diagram is labeled with an integer p_{ij} equal to one half the least common multiple of the lengths of the $\{i, j\}$-colored cycles in G. By convention, the edge is omitted when $p_{ij} = 2$, and the edge label is omitted when $p_{ij} = 3$. The diagrams of the 3-graphs in Fig. 1b, 2, 7a and 8a are shown in Fig. 3a,b,c,d, respectively. For example, the diagram of the graph G in Fig. 1b has three nodes because G is a 3-graph. The nodes are not labeled in this case because there is no ambiguity. In this diagram the edge joining nodes 0 and 1, 1 and 2, and 2 and 0, are each labeled 2 because the (12), (02) and (01)-colored cycles in G all have length $4 = 2 \cdot 2$. By convention, an edge is omitted from the diagram when the label is 2, so this diagram consists simply of three disjoint nodes. In general, note that there is no edge joining nodes i and j in the the diagram of G if and only if all $\{i, j\}$-colored cycles in G have length 2 or 4. A diagram is called *linear* if $p_{i,j} = 2$ for all i, j with $|i - j| \geq 2$ i.e., if the diagram has the form in Fig. 3f. (The labeling of the nodes $0, 1, \ldots, n-1$ is understood to be from left to right.) The diagram in Fig. 3a, c and d are linear while the diagram in Fig. 3b is not linear. In [10] it is proved that an n-graph G is ordered if and only if $D(G)$ is linear.

Fig. 3. Diagrams.

A map on a surface, being ordered, has a linear diagram with 3 nodes. A map consisting of p-gons, q of them incident at each vertex, is traditionally called *type* $\{p, q\}$. In the diagram of a map of type $\{p, q\}$, the two edges are labeled p and q. This is because the (0,1)-colored cycles of $G(K)$ have length $2p$, corresponding to the fact that each face of K is incident with p edges and p vertices of K, and the (1,2)-colored cycles of $G(K)$ have length $2q$, corresponding to the fact that each vertex of K is incident with q edges and q faces of K. For example, the cube is of type $\{4, 3\}$, with three 4-gons incident at each vertex, and its diagram is given in Fig. 3e. (By convention, the 3 on the second edge is deleted.)

Although not every 3-graph corresponds to a map, every 3-graph does correspond to a hypermap [10]. A *hypergraph* $H = (X, E)$ consists of a vertex set X and a family E of subsets of X called edges, whose union is X. If all the edges have cardinality 2, then H is a graph. Define a *hypermap* $\overline{H}$ (hypergraph embedding) as a two colored map K on a surface. More precisely there is a coloring function $K_2 \to \{1, 2\}$ from the set K_2 of 2-faces of the map, such that any pair of 2-faces that share an edge are assigned distinct colors. The underlying hypergraph of $\overline{H}$ is $H = (X, E)$, where X is the set of vertices of K and a subset $e \subset X$ is an edge in E if e is the complete set of vertices of a 2-face in K colored 1. In the case of a graph, all the faces colored 1 are 2-gons. To obtain the 3-graph $G(H)$ corresponding to a hypermap H, "blow up" each vertex of H into a cycle as in Fig. 4.

FIG. 4. THE 3-GRAPH OF A HYPERMAP.

3. Dual n-graphs. Although most of the examples in this section will be maps or hypermaps, there is no added complexity in considering n-graphs, and, in fact, the treatment is simplified by doing so. Let G be an n-graph and let $\tau = i_1 i_2 \dots i_m$ be a word in the elements of $[n]$. A *path of type* τ in G is a path whose edges are labeled successively $i_1, i_2, \dots, i_m$. Call τ an *involution* for G if every path of type $\tau^2 = i_1 i_2 \dots i_m i_1 i_2 \dots i_m$ is closed. Let $T = \{\tau_i \,|\, i \in [n]\}$ be a set of n involutions. Construct another graph $T(G)$ as follows. The vertices of $T(G)$ are the vertices of G. Two vertices u and v are i-adjacent in $T(G)$ if u and v are joined by a path of type τ_i in G.

Lemma 1. *If G is an n-graph and T a set of involutions of G, then $T(G)$ is an n-graph.*

Proof. It must be shown that $T(G)$ is regular and properly colored. Clearly each vertex of $T(G)$ is incident with at least one edge of each color. If a vertex x is incident with two edges colored i, then there would be two vertices u and v such that the unique paths of type τ_i beginning at u and v, respectively, end at x. Since τ_i is an involution this would imply that the unique path of type τ_i beginning at x ends at both u and v, clearly impossible unless $u = v$. $\square$

Call two n-graphs G and G^* *dual* if there exist sets T and T^* of involutions such that $G^* = T(G)$ and $G = T^*(G^*)$. It is routine to verify that duality is an equivalence relation on n-graphs and $G \sim G^*$ will denote that G and G^* are dual.

Lemma 2. *If $G \sim G^*$, then G is connected if and only if G^* is connected.*

Proof. Assume that G is connected. To show that G^* is connected, let u and v be vertices of G^*. Since G is an n-graph, u and v are joined by a path, say of type $i_1 i_2 \dots i_m$ in G. Denote the vertices of this path by $u = u_0, u_1, \dots, u_m = v$. By the definition of duality, u_{j-1} and $u_j, 1 \leq j \leq m$, are joined by a path of type $\tau^*_{i_j} \in T^*$. Therefore u and v are joined by a path in G^*. $\square$

An *isomorphism* $f : V(G) \to V(G')$ between n-graphs is a graph isomorphism that, in addition, preserves colors. That is, u and v are i-adjacent in G if and only if $f(u)$ and $f(v)$ are i-adjacent in G'. An *automorphism* of an n-graph G is an isomorphism from G onto itself. Let $\Gamma(G)$ denote the automorphism group of G. An n-graph is called *symmetric* if $\Gamma(G)$ is transitive on vertices. It is straightforward to show that the automorphism group of a symmetric n-graph must, in fact, act sharply transitive on vertices. The 3-graphs corresponding to (the boundary complexes of) the Platonic solids, for example, are symmetric. For maps and cell complexes, symmetry corresponds exactly to flag transitivity, that is, transitivity on ordered tuples of faces $(f_0, f_1, \dots)$, where f_i is an i-dimensional face and f_i is contained in f_{i+1} for all i. A flag in the cell complex corresponds to a vertex in the corresponding n-graph. In the literature, a flag transitive map is often called a *regular* map. Hereafter $\approx$ will denote group isomorphism.

Theorem 1. *If $G \sim G^*$ then $\Gamma(G^*) \approx \Gamma(G)$. Also G is symmetric if and only if G^* is symmetric.*

Proof. Let $f : V(G) \to V(G)$ be an automorphism of G. We will show that f also induces an automorphism of $T(G)$. Assume that u and v are i-adjacent in $T(G)$. Since there is a path of type τ_i from u to v and f is an automorphism of G, there is a path of type τ_i from $f(u)$ to $f(v)$. Hence $f(u)$ and $f(v)$ are i-adjacent. The converse is similarly proved. The second statement of the theorem now follows from the first. $\square$

Example 1. Let σ be a permutation of $[n]$ and let $T = T_\sigma = \{\tau_i \mid i \in [n]\}$, where $\tau_i = \sigma(i)$. (Each τ_i is a string of length one in this case.) Then $T(G)$ is obtained from G by merely permuting the edge colors. Let $T^* = T_{\sigma^{-1}}$ where σ^{-1} is the inverse permutation. Clearly G and $G^* = T(G)$ are dual because $T^*(G^*) = G$. In this case, call G^* a *permutation dual* of G There are $n!$ possible permutation duals for any n-graph, corresponding to the $n!$ possible permutations of $[n]$. When $n = 3$, six duals of a map are obtained, which were first mentioned by Wilson [13]. In the case where G is a map and σ is the transposition $(0\,2)$, the dual $T_\sigma(G)$ corresponds to the standard geometric dual of the map (interchanging the role of 0 and 2-dimensional faces). In general the *order reversing* permutation $\sigma(i) = n-1-i,\ i = 0,\dots,n-1$, corresponds to the standard geometric dual of a higher dimensional polytope or complex. The six permutation duals of the cube are drawn as maps or hypermaps (not as 3-graphs) in Fig. 5.

FIG. 5. PERMUTATION DUALS OF THE CUBE.

Example 2. Assume that j and k are non-adjacent nodes ($p_{jk} = 2$) in the diagram of an n-graph G. Then, by the remarks above concerning the diagram, every path of type $(jk)^2$ is closed in G. Take $T = \{\tau_i \mid i \in [n]\}$ where

$$\tau_i = \begin{cases} jk & \text{if } i = j \\ i & \text{otherwise.} \end{cases}$$

Let $T^* = T$. Again G and $G^* = T(G)$ are dual because $T^*(G^*) = G$. Consider, for example, a map on a surface and take $j = 0$, $k = 2$. The construction of the dual is shown in Fig. 6a. The two duals of the cube corresponding to $j = 0$, $k = 2$ and $j = 2$, $k = 0$ are shown as maps on a torus (not as 3-graphs) in Fig. 6b and 6c. (Like labeled edges are to be identified.) The case $j = 0$, $k = 2$ is exactly the map constructed by Coxeter and Moser [1], whose faces are the Petrie paths in the original map. It is possible to generate up to 18 duals of a map by applying Examples 1 and 2. (In certain symmetric cases, some of these duals may be identical.) Up to six of these are ordered, and hence are themselves maps. For the cube these six are the cube, octahedron, and the maps in Fig. 6. Those in Fig. 6d and 6e are the ordinary duals to those in Fig. 6b and 6c.

Fig. 6. Dual maps of the cube

Example 3. Given integers $j, k \in [n]$, let $T = \{\tau_i \,|\, i \in [n]\}$ where

$$\tau_i = \begin{cases} jkj & \text{if } i = k \\ i & \text{otherwise.} \end{cases}$$

Clearly any path of type τ_i^2, $i \in [n]$ is closed. Again choose $T^* = T$ to show that G and $G^* = T(G)$ are dual. The example, shown in Fig. 7, is such a 3-graph and its dual, where $T = \{101, 1, 2\}$. Fig. 7a is the 3-graph of a pyramid with square base. Fig. 7b is the 3-graph of a hypermap on a surface of genus 3. Note that in Fig. 7b, there are 30 vertices; there are four places in the figure where 0-colored edges cross, that are *not* vertices.

Fig. 7. A dual of a pyramid.

Example 4. This example is presented mainly because it will be of interest at the end of Section 6. Let G be an ordered n-graph, $n > 3$. Consider the cases of even and odd n separately. For $n = 2k+1$ let $T = \{\tau_i \,|\, i \in [n]\}$ where

$$\tau_i = \begin{cases} 0k \quad \text{or} \quad k(n-1) \quad \text{or} \quad 0k(n-1) & \text{if } i = k \\ i & \text{otherwise.} \end{cases}$$

For $n = 2k$ let $T = \{\tau_i \,|\, i \in [n]\}$ where either

$$\tau_i = \begin{cases} 0(k-1) & \text{if } i = k-1 \\ i & \text{otherwise} \end{cases}$$

or

$$\tau_i = \begin{cases} k(n-1) & \text{if } i = k \\ i & \text{otherwise} \end{cases}$$

or

$$\tau_i = \begin{cases} 0(k-1) & \text{if } i = k-1 \\ k(n-1) & \text{if } i = k \\ i & \text{otherwise.} \end{cases}$$

In each case it is straightforward to show that T is a set of involutions, since the length of an $\{i,j\}$-colored cycle in an ordered n-graph is 2 or 4 if i and j are not consecutive. We leave it as an exercise to verify that G and $T(G)$ are dual. In this example each dual is also ordered.

Example 5. For a more sporadic example, consider the 3-graph G in Fig. 8a that encodes a map on the sphere. Then $T = \{0, 1210, 2\}$ is a set of involutions for G. To show that G and $G^* = T(G)$ are dual, choose $T^* = \{0, 121, 2\}$. The dual G^*, which encodes a map on a projective plane, is shown in Figure 8b. In this figure, note that the intersection of the 0-colored edges is *not* a vertex in the 3-graph.

FIG. 8. A DUAL OF A MAP ON THE SPHERE.

4. Permutation representations. Traditionally maps have been discussed in terms of certain groups defined in terms of generators and relations [1]. This algebraic approach will be generalized to n-graphs in Sections 4 and 6, and will be applied to duality. Let

$$W = < r_0, r_1, \ldots, r_{n-1} \,|\, r_0^2 = r_1^2 = \cdots = r_{n-1}^2 = 1 >,$$

and let S_V denote the symmetric group of permutations of the set V. A homomorphism $\alpha : W \to S_V$ is called a *permutation representation* of W. Two such representations $\alpha : W \to S_V$ and $\beta : W \to S_V$ are called *equivalent* if there exists a permutation $\sigma \in S_V$ such that $\beta(w) = \sigma^{-1} \cdot \alpha(w) \cdot \sigma$ for all $w \in W$. In other words the following diagram commutes, where the vertical arrow is conjugation by σ.

$$\begin{array}{ccc} W & \xrightarrow{\alpha} & S_V \\ & & \downarrow \\ & & S_V \end{array}$$

Given an n-graph G with vertex set V, let ρ_i, $i \in [n]$ denote the involution in S_V that interchanges each pair of i-adjacent vertices. For example, in the 3-graph of Fig. 1bc, the three involutions are $\rho_0 = (12)(34)(56)(78)$, $\rho_1 = (13)(24)(57)(68)$ and $\rho_2 = (17)(28)(46)(35)$. In general, the set $\{\rho_i \,|\, i \in [n]\}$ of involutions induce a permutation representation $\alpha : W \to S_V$ defined on generators by $\alpha(r_i) = \rho_i$. Call two permutation representations α and α^* *dual* if they have the same image in S_V i.e., $\alpha^*(W) = \alpha(W)$. This algebraic definition of duality and the geometric definition in Section 3 are equivalent.

Theorem 2. *There is a bijection between n-graphs (up to isomorphism) and permutation representations (up to equivalence). Under this bijection, dual n-graphs correspond to dual permutation representations.*

Proof. The permutation representation α corresponding to the given n-graph G is defined above. That this correspondence is a bijection is proved in [10].

Concerning the statement about duallity in the theorem, let G and G^* be n-graphs and α and α^* be the corresponding permutation representations. First assume that α and α^* are dual and hence have the same image Σ in S_V. To show that G and G^* are dual n-graphs, let N and N^* be the kernels of α and α^*, resp., so that there are canonical homomorphisms h and h^* and isomorphisms γ and γ^* so that the two triangles in the diagram commute.

$$\begin{tikzcd} & W/N \arrow[d, "\gamma"] \\ W \arrow[r, "\alpha"] & \Sigma \\ & W/N^* \arrow[u, "\gamma^*"'] \end{tikzcd}$$

If w denotes a word in the generators of W, let $\overline{w}$ denote the coset it represents in W/N (or W/N^*). For $i \in [n]$, let $\overline{r_{j_1} r_{j_2} \dots r_{j_m}}$ be the image of $\overline{r}_i$ under the isomorphism $\gamma^{-1} \circ \gamma^* : W/N^* \to W/N$, so that $\gamma(\overline{r_{j_1} r_{j_2} \dots r_{j_m}}) = \gamma^*(\overline{r}_i)$. Define a set $T = \{\tau_i \,|\, i \in [n]\}$ by $\tau_i = j_1 j_2 \dots j_m$. We claim that $G^* = T(G)$. To see this, note that u is joined to v by a path of type τ_i in G if and only if $v = (\alpha(r_{j_1}) \cdot$

$\alpha(r_{j_2}) \cdot \ldots \cdot \alpha(r_{j_m}))u = \alpha(r_{j_1} r_{j_2} \ldots r_{j_m})u = \gamma(\overline{r_{j_1} r_{j_2} \ldots r_{j_m}})u = \gamma^*(\overline{r}_i)u = \alpha^*(r_i)u$ if and only if u is i-adjacent to v in G^*. A set T^* such that $G = T^*(G^*)$ can be constructed in a similar manner, thus showing that G and G^* are dual.

Conversely assume that G and $G^* = T(G)$ are dual n-graphs on the same vertex set V, for some set $T = \{\tau_i \,|\, i \in [n]\}$ of involutions. By definition, if $\tau_i = j_1 j_2 \ldots j_m$, then $\alpha^*(r_i) = \alpha(r_{j_1}) \cdot \alpha(r_{j_2}) \cdot \ldots \cdot \alpha(r_{j_m})$. Since the r_i generate W, $\alpha^*(W) \subseteq \alpha(W)$. Containment in the other direction is similarly proved. $\square$

If our intention is to investigate only ordered n-graphs (maps on surfaces in the case $n = 3$), then Theorem 2 should be altered slightly. Theorem 3 below indicates that it is appropriate to consider permutation representations of

$$W' =< r_0, r_1, \ldots, r_{n-1} \,|\, r_i^2 = 1,\, i \in [n],\, (r_i r_j)^2 = 1,\, |i - j| \geq 2 > .$$

This group is obtained from W by adding relations that state that each product $r_i r_j$ has period 2 for non-consecutive i and j.

Theorem 3. *There is a bijection between ordered n-graphs (up to isomorphism) and permutation representations $\alpha : W' \to S_V$ (up to equivalence). Under this bijection, dual n-graphs correspond to dual permutation representations.*

Proof. Let G be an n-graph and $\alpha : W \to S_V$ the corresponding permutation representation given by Theorem 2. If N is any normal subgroup of W contained in the kernel α, then there is an induced permutation representation $\overline{\alpha} : W/N \to S_V$ defined by $\overline{\alpha}(\overline{w}) = \alpha(w)$, so that the following diagram commutes,

$$\begin{array}{ccc} W & \xrightarrow{\alpha} & S_V \\ {\scriptstyle h}\big\downarrow & & \\ W/N & & \end{array}$$

where h denotes the canonical homomorphism $h : W \to W/N$. The assignment $\alpha \mapsto \overline{\alpha}$ induces a bijection between permutation representations of W whose kernel contains N and permutation representations of W/N. Take N to b the normal closure of $\{(r_i r_j)^2 \,:\, |i - j| \geq 2\}$ in W. In this case, G is an ordered n-graph if and only if all (i, j)-colored cycles with $|i - j| \geq 2$ have length 2 or 4 if and only if N lies in the kernel of α. Thus, the permutation representations of W whose kernel contains N are exactly the permutation representations of W corresponding to ordered n-graphs. Now we have the desired bijection between ordered n-graphs and permutation representations $\overline{\alpha} : W' = W/N \to S_V$.

The second statement follows from Theorem 2, and the fact that $\alpha : W \to S_V$ and $\beta : W \to S_V$ have the same image if and only if the induced permutation representations $\overline{\alpha} : W' \to S_V$ and $\overline{\beta} : W' \to S_V$ have the same image. $\square$

5. Symmetric duals and the minimum symmetric cover. Two results concerning symmetric n-graphs are proved in this section. The first is the converse of Theorem 1. The second states that duality of arbitrary n-graphs lifts to the minimum symmetric cover.

The concept of the group of an n-graph is required for the proofs. Again let $W =< r_0, r_1, \dots, r_{n-1} \,|\, r_0^2 = r_1^2 = \cdots = r_{n-1}^2 = 1 >$. For an n-graph G, let $\alpha : W \to S_V$ be the permutation representation defined on generators by $\alpha(r_i) = \rho_i$. For a fixed vertex v of G, let $H(G) = \{w \in W \,|\, \alpha(w)(v) = (v)\}$. Up to conjuagacy in W, the group $H(G)$ is independent of the choice of v and is called the *group of G*. Moreover, it is known [10] that $H(G)$ is the unique subgroup of W, up to conjugacy, such that G is isomorphic to the Schreier coset graph of W with respect to $H(G)$. Recall that the vertices of the Schreier coset graph are the right cosets of $H(G)$ in W, and two vertices in the Schreier coset graph are i-adjacent if the one of the corresponding cosets is obtained from the other by multiplication by r_i on the right.

It is known [11] that an n-graph G is symmetric if and only if $H(G)$ is normal in W, in which case $\Gamma(G) \approx W/H(G)$. Hence, in the case of a symmetric n-graph G, there exists a set X of elements in W such that the group $H(G)$ is the normal closure of X in W. In other words, $H(G)$ is the smallest normal subgroup of W containing X. Each element of X can be thought of as a word in the letters $r_0, r_1, \dots, r_{n-1}$ and $W/H(G)$ is isomorphic to the group W modulo the set of relations $\{w = 1 \,|\, w \in X\}$. These relations are referred to as the *defining relations* of the symmetric n-graph G.

Moreover, the generators r_i in the presentation of W have a geometric interpretation as reflections, and the $r_i r_j$ as rotations. Defining relations for the 3-graph in Fig. 1b can be given by $(r_1r_2)^2 = (r_0r_2)^2 = (r_0r_1)^2 = 1$. These relations indicate that a rotation of period 2 about any vertex, edge or face, resp., is an automorphism of the map in Fig. 1a. In the more complicated 3-graph in Fig. 6b, defining relations for this symmetric map on the torus can be given by $(r_0r_1)^6 = (r_1r_2)^3 = (r_0r_1r_2)^4 = 1$. The relations $(r_0r_1)^6 = 1$ and $(r_1r_2)^3 = 1$ indicate that rotations of period 6 about face centers and rotations of period 3 about vertices are map automorphisms. The relation $(r_0r_1r_2)^4 = 1$ indicates that a translation one step along a Petrie path is a map automorphism.

Theorem 4. *Let G and G^* be symmetric n-graphs. If $\Gamma(G) \approx \Gamma(G^*)$, then $G \sim G^*$.*

Proof. Let $H = H(G)$ and $H^* = H(G^*)$ be the groups of G and G^*, resp. Then, by the remarks above, H and H^* are normal in W, and $W/H^* \approx \Gamma(G^*) \approx \Gamma(G) \approx W/H$. Denote the resulting isomorphism by $\phi : W/H^* \to W/H$. Regard G as the Schreier coset graph of W with respect to H; similarly regard G^* as the Schreier coset graph of W with respect to H^*. By abuse of language, no distinction will be made between a vertex of the graph and the corresponding coset in the quotient. Then ϕ induces a bijection between $V(G^*)$ and $V(G)$. The notation v^* and v will be used for a vertex and its image, resp., under this bijection. For $i \in [n]$ let $Hr_{j_1}r_{j_2}\dots r_{j_m} = \phi(H^*r_i)$; let $\tau_i = j_1j_2\dots j_m$; and let $T = \{\tau_i \,|\, i \in [n]\}$. Note that τ_i is an involution in G because $Hr_{j_1}r_{j_2}\dots r_{j_m}r_{j_1}r_{j_2}\dots r_{j_m} = [\phi(H^*r_i)]^2 = \phi(H^*r_i^2) = H$. Now vertices u^* and v^* are i-adjacent in G^* if and only if $u^* = v^*r_i = v^* \cdot H^*r_i$ if and only if $\phi(u^*) = \phi(v^*) \cdot Hr_{j_1}r_{j_2}\dots r_{j_m}$ if and only if $u = v \cdot r_{j_1}r_{j_2}\dots r_{j_m}$ if and only if u and v are joined by path of type τ_i in G. Therefore $G^* = T(G)$. That $G = T^*(G^*)$, for an appropriate T^*, is proved similarly by cosidering the inverse isomorhism $\phi^{-1} : W/G \to W/H^*$. $\square$

Let G and G' be n-graphs. A surjective function $f : V(G') \to V(G)$ is called a *covering* if f preserves i-adjacency for all $i \in [n]$. In terms of maps, a covering corresponds to a topological covering of the respective surfaces, possibly ramified at vertices of the embedded graph or at face centers [10]. A covering will be denoted simply $f : G' \to G$. Two coverings $f_1 : G'_1 \to G$ and $f_2 : G'_2 \to G$ are called *equivalent* if there exists an isomorphism $\theta : G'_1 \to G'_2$ such that $f_1 = f_2 \circ \theta$. A covering $f : \widehat{G} \to G$ of G by a symmetric n-graph $\widehat{G}$ is called a *minimum symmetric cover* if f satisfies the following universal property: for any covering $g : G' \to G$ of G by a symmetric n-graph G', there exists a covering $\overline{g}$ such that the following diagram commutes.

$$\begin{array}{ccc} G' & \xrightarrow{\overline{g}} & \widehat{G} \\ & & \big\downarrow f \\ & & G \end{array}$$

Lemma 3. *For an n-graph G, the minimum symmetric covering of G exists and is unique up to equivalence.*

Proof. The uniqueness follows in the usual way from the universal property. To prove existence, let $H(G)$ be the group of G (unique up to conjugacy) and let

$$N(G) = \cap_{w \in W} w^{-1} H(G) w.$$

So $N(G)$ is the largest subgroup of $H(G)$ that is normal in W in the sense that it contains every subgroup of $H(G)$ that is normal in W. Further let $\widehat{G}$ be the n-graph with group $N(G)$ i.e., $\widehat{G}$ is the Schreier coset graph of W with respect to $N(G)$. Since $N(G) \leq H(G)$, there is a canonical covering $f : \widehat{G} \to G$, defined in terms of the Schreier coset graph by $f(N(G)w) = H(G)w$ for all $w \in W$. Since $N(G) \triangleleft W$, it follows, from the remarks at the beginning of this section, that $\widehat{G}$ is symmetric. Furthermore, if G is finite, then $\widehat{G}$ is also finite [11].

To show that $\widehat{G}$ satisfies the universal property, again represent G, G' and $\widehat{G}$ as Schreier coset graphs. In [10] it is shown that any covering, in this case g, can be given, up to equivalence, by $g(H(G')w) = H(G)w$, where $H(G')$ and $H(G)$ are groups of G' and G, resp., and $H(G') \leq H(G)$. Since G' is symmetric, $H(G')$ is normal in W. Since $H(\widehat{G}) = N(G)$ is the largest subgroup of $H(G)$ that is normal in W, and $H(G')$ is also a subgroup of $H(G)$ normal in W, then $H(G') \leq H(\widehat{G})$. Now if $\overline{g}$ is defined by $\overline{g}(H(G')w) = H(\widehat{G})w$, the diagram commutes. $\square$

Theorem 5. *If $G \sim G^*$ and $\widehat{G}$ and $\widehat{G}^*$ are the minimum symmetric covers of G and G^*, resp., then $\widehat{G} \sim \widehat{G}^*$.*

Proof. By Theorem 4 it is sufficient to prove that $\Gamma(\widehat{G}) \approx \Gamma(\widehat{G}^*)$. Let α and α^* be permutation representations of G and G^*, resp., and let $N = N(G)$ and $N^* = N(G^*)$. Now it is sufficient to define a chain of isomorphisms

$$\Gamma(\widehat{G}^*) \approx W/N* \approx W/\ker\alpha^* \approx \text{im } \alpha^* \approx \text{im } \alpha \approx W/\ker\alpha \approx W/N \approx \Gamma(\widehat{G}).$$

The middle isomorphism, im $\alpha^* \approx$ im α, follows from Theorem 2 and the duality of G and G^*. The last isomorphism follows from the remarks at the beginning of this section and the facts that $\widehat{G}$ is the Schreier coset graph of W with respect to N and that $\widehat{G}$ is symmetric. Similarly for the first isomorphism. Finally, if H is the group of G, then, using the Schreier coset graph to represent G, the permuatation representation α is defined on the generators r_i by $(\alpha r_i)(Ha) = Har_i$ for all i. Therefore $(\alpha w)(Ha) = Haw$ for all $w \in W$, and hence $\ker\alpha = \{w \in W \mid Haw = Ha$ for all a $\in$ W $\} = N$. Similarly $\ker\alpha^* = N^*$, and the remaining isomorphisms follow. □

6. Automorphism induced duality. This section gives a group theoretic construction of dual n-graphs and, in particular, discusses two examples that appear in the literature. Let G be an n-graph and $\alpha : W \to S_V$ be the corresponding permutation representation of W given in Theorem 2. Further, let N be any given normal subgroup contained in the kernel of α, and let $Aut(W/N)$ denote the automorphism group of W/N. Each $\phi \in Aut(W/N)$ induces an n-graph G_ϕ, dual to G, as follows. Consider the homomorhisms

$$W \xrightarrow{\ h\ } W/N \xrightarrow{\ \phi\ } W/N \xrightarrow{\ \overline{\alpha}\ } S_V,$$

where h is the canonical homomorhism, and $\overline{\alpha}$ is the induced homomorphism defined by $\overline{\alpha}(\overline{w}) = \alpha(w)$. Thus ϕ induces a dual permutation representation $\alpha_\phi : W \to S_V$ defined by $\alpha_\phi = \overline{\alpha} \circ \phi \circ h$. Consequently, by Theorem 2, ϕ induces an associated n-graph G_ϕ, dual to G. Call any such dual n-graph *Aut(W/N)-induced.* The remainder of this section is concerned with two examples.

Example 1. $Aut(W)$-induced duality. This is the case when N is trivial and $\alpha_\phi = \alpha \circ \phi$. $Aut(W)$ is exactly the set of hypermap operations that are the subject of the paper [5] for the case $n = 3$. In that paper James proves that $G_\phi = G$ for each inner automorphism ϕ of W. So $Out(W) = Aut(W)/Inn(W)$ is the relevant group of operations, and, furthermore, $Out(W)$ is isomorphic to $PGL(2, Z)$. James further remarks that if $\phi \in Aut(W)$ then each $\phi(r_i)$ is conjugate to some r_j. Therefore $Aut(W)$ is generated by a set of automorphisms each of which either permutes $\{r_0, r_1, \ldots, r_{n-1}\}$ or is of the form

$$\phi(r_i) = \begin{cases} r_j r_k r_j & \text{if } i = k \\ r_i & \text{otherwise} \end{cases}$$

for some $j, k \in [n]$. This implies that any $Aut(W)$-induced dual of G is generated by some combination of the constructions of Examples 1 and 3 of Section 3.

Example 2. $Aut(W')$-induced duality. The group W' is as defined in Section 4. The following lemma and its corollaries will be useful before considering $Aut(W')$-induced duality.

Lemma 4. *Let α and β be permutation representions of an arbitrary group W, and let N lie in the kernel of both α and β. If ϕ is an automorphism of W such that $\beta = \alpha \circ \phi$, then there exists an automorphism ϕ' so that the following diagram commutes.*

Proof. Define $\phi'(\overline{w}) = \overline{\phi(w)}$ for all $w \in W$. The lemma is then a routine exercise. $\square$

Corollary 1 below is a direct consequence of Lemma 4, and Corollary 2 follows from Corollary 1 by letting N be the normal closure of $\{(r_i r_j)^2 \, : \, |i - j| \geq 2\}$.

Corollary 1. *Suppose G and G^* are $Aut(W)$-induced dual n-graphs and that $N \triangleleft W$ lies in the kernel of both the permutation representations of G and G^*. Then G and G^* are also $Aut(W/N)$-induced duals.*

Corollary 2. *Suppose G and G^* are ordered n-graphs and that G^* is an $Aut(W)$-induced dual of G. Then G^* is also an $Aut(W')$-induced dual of G.*

According to Corollary 2, for a given ordered n-graph, the $Aut(W')$-induced duals already include the $Aut(W)$-induced ordered duals. Therefore, for ordered n-graphs, $Aut(W')$ is the pertinent group of operations, and these are exactly the map operations considered by Jones and Thornton [7]. In that paper, analogous to the results of James, they show that each inner automorphism of W' acts trivially on a given n-graph G; so again $Out(W') = Aut(W')/Inn(W')$ is the relevant group of operations. Furthermore, when $n = 3$, this group is isomorphic to S_3, the symmetric group on 3 elements, and when $n > 3$, to D_4, the dihedral group of 8 elements [5]. The six possible duals of a map (ordered 3-graph) are all generated by the operations in Examples 2 of Section 3 and the order reversing permutation dual in Example 1. For the cube, the six duals are the cube itself, the octahedron and the four maps in Fig. 6. For $n > 3$, it can be shown that the eight possible $Aut(W')$-induced duals of a given map (corresponding to the elements of D_4) are generated by our constructions in Example 4 of Section 3, together with the order reversing permutation dual in Example 1.

7. Self dual maps. Let G be an n-graph and T a set of involutions such that G and $G^* = T(G)$ are dual. It may occur that G and G^* are isomorphic. In the classical case, where G is a map and $T = \{2, 1, 0\}$, if G and $G^* = T(G)$ are isomorphic, then the map G is usually called *self dual*. This definition implies a homeomorphism of the surface taking the embedded graph onto its geometric dual. Two questions are posed concerning self dual maps. In the 3-graphs corresponding to a pair of self dual maps, the (0,1)-colored cycles (2-residues) in the 3-graph correspond to the (2,1)-colored cycles (0-residues) in its dual. Hence a self dual map must be of type $\{q, q\}$ for some integer q.

Question 1. *Is it true that if G is a symmetric map of type $\{q, q\}$, then G is self dual?*

We believe the answer to Question 2 is no, but have no counterexample. Here $\cong$ denotes isomorphism of n-graphs.

Question 2. *With notation as in Theorem 5, does $\widehat{G} \cong \widehat{G}^*$ necessarily imply $G \cong G^*$?.*

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