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Note

Dyck's Map $(3, 7)_8$ Is a Counterexample to a Clique Covering Conjecture

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Let $c(G)$ denote the minimum number of cliques necessary to cover all edges of a graph G. A counterexample is provided to a conjecture communicated by P. Erdős. If $c(G-e) < c(G)$ for every edge e, then G contains no triangles.

Let $c(G)$ denote the minimum number of cliques necessary to cover all edges of a graph G. If G contains no triangle, then the cliques are the edges of G. In this case removing any edge e must reduce the value of $c(G)$, that is,

$$c(G-e) < c(G). \tag{1}$$

P. Erdős [4] communicated the conjecture that this is the only situation in which (1) holds for all edges e in G.

Conjecture. If (1) holds for every edge e of a graph G, then G contains no triangle.

A counterexample is given in this note. Unfortunately the origin of the conjecture is not known to us. The symbol $\{p, q\}$ denotes the regular tessellation of a simply connected surface into p-gons, q incident at each

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vertex. If $1/p + 1/q < 1/2$ then the tesselation consists of infinitely many regular p-gons filling the hyperbolic plane. A *Petrie path* of $\{p, q\}$ is a "zigzag" path in which every two consecutive edges, but not three, belong to a face. The symbol $\{p, q\}_r$ denotes the map obtained from $\{p, q\}$ by identifying each pair of vertices that are separated by a Petrie path of length r. It is well known that the automorphism group of the map $\{p, q\}_r$ acts flag transitively, in particular, transitively on vertices, edges, and faces. Lists of finite maps $\{p, q\}_r$ are included in [1, 2]. In particular, $\{3, 7\}_8$ is a map on the orientable surface of genus 3 and has 56 faces, 84 edges, and 24 vertices. This particular map was studied extensively by W. Dyck [3] in 1880 in connection with Riemann surfaces and led to a good deal of interest in maps in general. The underlying graph G_0 of $\{3, 7\}_8$ provides a counterexample to the conjecture above. This particular graph seemed a likely candidate as a counterexample for the following reasons. Because of its symmetry, inequality (1) need only be checked for a single edge. The cliques of G_0 are simply the triangles in the triangulation $\{3, 7\}_8$. Let C be a clique covering, i.e., a set of triangles that covers the edges of G_0. Two triangles are said to be *adjacent* if they share a common edge. Three triangles are said to be *in a row* if one triangle is adjacent to both of the other triangles. The proof below is based on the fact that C must contain three triangles in a row. Smaller maps like the icosahedral map have clique covers without three triangles in a row.

THEOREM. *For every edge e of G_0 we have $c(G_0 - e) < c(G_0)$.*

Proof. It is not hard to verify that each K_3 in G_0 is a face of $\{3, 7\}_8$. Thus G_0 can have no K_4, and therefore the cliques of G_0 are exactly the boundaries of triangular faces of $\{3,7\}_8$. An adjacency table for these triangles is given below. We claim that in any covering of the edges by a set C of triangles, there must be three triangles in a row. If this is so with triangle w adjacent to triangles x and y, then in a minimum clique covering C of G_0, $C - w$ will cover $G_0 - e$ where x, y, z are the neighbors of w and $e = w \cap z$ (see Fig. 1). This implies that $c(G_0 - e) < c(G_0)$. Since the automorphism of G_0 acts transitively on edges, this is true for all edges e.

To verify the above claim let C be a clique covering and refer to Table I. Since each vertex degree is seven, there must exist incident to each vertex two adjacent triangles in C. By symmetry it can then be assumed, without loss of generality, that triangles 1 and 2 are both in C. By way of contradiction assume that G_0 has no three triangles in a row in C. This forces triangles 3, 8, 10, 7 out of C. For example, if 10 were in C then, according to Table I, triangle 2 is adjacent to both 1 and 10 and hence 1, 2, and 10 would be three triangles in C in a row. In any clique covering C of G_0 no two adjacent triangles lie in the complement of C which forces 4, 26, 6, 15,

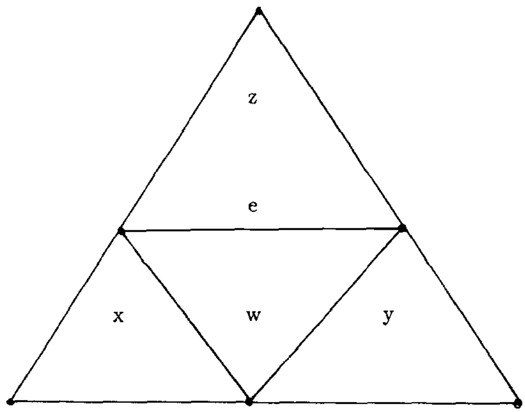

FIGURE 1

TABLE I

Adjacency List for the Graph G_0

1:	3	8	2	2:	1	10	7
3:	4	9	1	4:	5	23	3
5:	6	24	4	6:	7	25	5
7:	2	26	6	8:	15	20	1
9:	3	13	16	10:	14	11	2
11:	10	12	19	12:	13	11	18
13:	17	12	9	14:	21	37	10
15:	22	31	8	16:	9	34	22
17:	30	44	13	18:	12	53	54
19:	11	55	39	20:	8	33	21
21:	20	32	14	22:	16	43	15
23:	4	38	30	24:	5	41	29
25:	6	49	28	26:	7	39	27
27:	26	31	47	28:	25	37	46
29:	24	36	45	30:	23	35	17
31:	15	27	42	32:	21	50	51
33:	38	20	48	34:	16	54	41
35:	30	56	40	36:	29	49	39
37:	28	53	14	38:	33	23	45
39:	36	26	19	40:	35	25	47
41:	34	24	46	42:	31	49	48
43:	22	51	52	44:	52	55	17
45:	29	50	38	46:	28	51	41
47:	52	40	27	48:	42	56	33
49:	42	36	54	50:	55	32	45
51:	32	43	46	52:	43	44	47
53:	18	37	56	54:	49	34	18
55:	19	44	50	56:	53	35	48

20, 14, 11 in C; 5, 21 out of C; 24, 32 in C. Now either face 12 is in C or it is not. If 12 is in C then 19 is out; 55, 39 in; 36, 50 out; 29, 45 in; 45 out. Now 45 both in and out is a contradiction. On the other hand, if 12 is not in C then 13 in; 16 out; 34, 22 in; 41, 43 out; 46, 51 in; 32 out. Now 32 both in and out is also a contradiction. ∎

It is not known whether G_0 is the smallest counterexample to the conjecture.

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