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Elementary Divisors of Graphs and Matroids

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New integer invariants of a graph G, introduced by U. Oberst, are obtained as the elementary divisors of the Laplacian matrix of G. The theory of elementary divisors is developed in the context of regular matroids. It is shown that the elementary divisors of a graph are actually invariants of its underlying matroid. Regular matroids, in turn, are related to lattices in euclidean space, and this leads to methods for computing the elementary divisors. Several properties of the elementary divisors of graphs are proved and the problem of how well these invariants distinguish between graphs is addressed.

1. Introduction

This paper concerns recent invariants of graphs, called elementary divisors, which arose in the work of U. Oberst on the algebraic topology of 1-complexes [3]. Oberst applied these invariants to finding necessary and sufficient conditions on an Abelian group A for the group of 1-chains over A to be the direct sum of the cycle and coboundary groups over A. Such a direct sum decomposition is well known when A is the field of real numbers. The intention of this paper is to place the elementary divisors into a matroid framework and to indicate that they, like the spectrum, may prove interesting from a combinatorial point of view.

In [3] the elementary divisors are defined in terms of the homology and cohomology of 1-complexes, but the following definition is equivalent. Let $A = A(G)$ be the adjacency matrix of a graph G, and $B(G)$ the diagonal matrix, where the diagonal entries are the degrees of the corresponding vertices. Then $L(G) = B(G) - A(G)$, called the *Laplacian* of G, can be put, uniquely, into Smith normal form

$$\begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix},$$

using only *integral* row and column operations, where D_r is an $r \times r$ diagonal matrix with positive integer entries such that each entry divides the next $d_1 \mid d_2 \mid \cdots \mid d_r$ and r is the rank of $L(G)$ [1]. The integers greater than 1 among $\{d_1, d_2, \ldots, d_r\}$ are called the *elementary divisors* of G. This multiset will be denoted $D(G)$, with values possibly occurring with multiplicity greater than 1. The elementary divisors of all connected graphs with 6 vertices and 10 edges are given in Figure 1. The graphs G_1 and G_2 in Figure 2 have elementary divisors $\{2, 6\}$ and $\{11\}$, respectively, which shows that the elementary divisors can distinguish pairs of graphs with the same degree sequence and that are topological equivalent. The elementary divisors depend only on the graph G and not on the particular matrix $L(G)$. This is a consequence of the fact [1] that two integral matrices L and L' have the same elementary divisors iff L and L' are equivalent. Equivalent means that $L' = PLQ$, where P and Q are invertible (over Z) integral matrices, i.e. L' can be obtained from L by a sequence of integral row and column operations. If $L = L(G)$ and $L' = L'(G)$ arise from the same graph G, by possibly different labelings of the vertices, then $L' = PLP^{\mathrm{T}}$ for some permutation matrix P and its transpose P^{T}. Therefore L and L' are equivalent.

In Section 2 the elementary divisors are generalized to unimodular matroids so that the elementary divisors of a graph are actually invariants of its underlying matroid. It is

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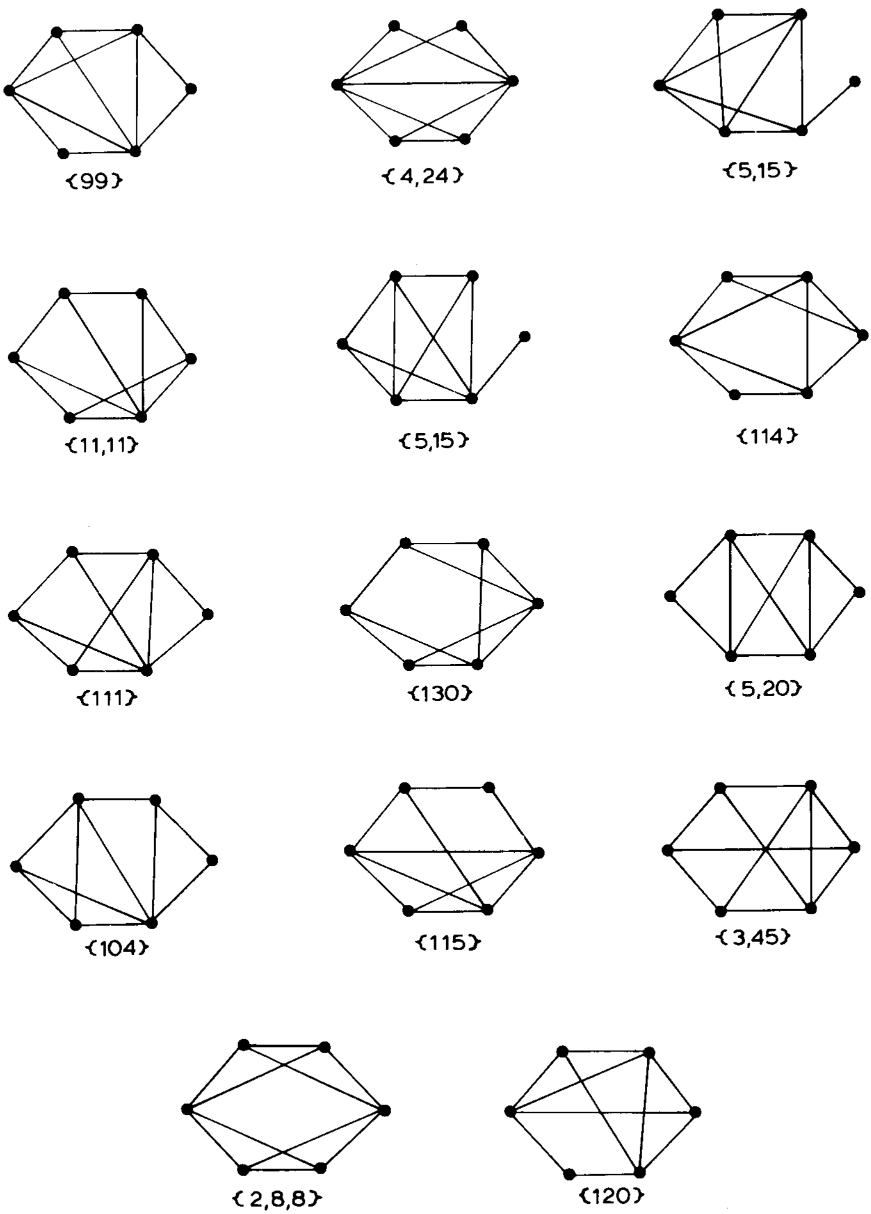

FIGURE 1. Elementary divisors of connected (6, 10)-graphs.

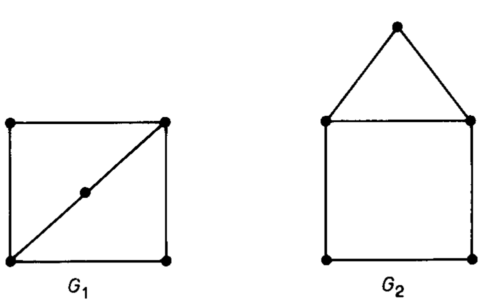

FIGURE 2. $D(G_1) = \{2, 6\}$ and $D(G_2) = \{11\}$.

shown that a matroid and is its dual have the same elementary divisors. Matroids, in turn, are related to lattices in euclidean space which leads, in Section 3, to methods for computing the elementary divisors of a graph. Several properties of the elementary divisors of graphs are proved in Section 3 and the problem of how well these invariants distinguish between graphs is addressed.

2. Unimodular and Lattice Matroids

All matroids will be finite. For basic definitions see [4, 6, 7]. A matroid that can be co-ordinatized over every field is called *unimodular* or, *regular*. In particular, the cycle matroid of a graph is unimodular [6]. With respect to a fixed basis for an n-dimensional vector space V, a co-ordinatization of a rank r matroid in V can be represented as an $r \times n$ matrix. An $r \times n$ integer matrix with $r \leq n$ is called *unimodular* if every $r \times r$ submatrix has determinant 0 or ± 1. Likewise, the matrix is called *totally unimodular* if every square submatrix has determinant 0 or ± 1. It is well known [4] that a matroid is unimodular iff it has a totally unimodular co-ordinatization over the rationals Q (equivalently over the integers Z). In this paper all co-ordinatization will be over Z.

Define the *elementary divisors* of a unimodular matroid M to be the set of elementary divisors of AA^{T}, where A is *any* unimodular co-ordinatization of M. To see that the elementary divisors are invariants of the matroid, let A and B be co-ordinatizations of isomorphic unimodular matroids. It is well known [7] that $B = PADQ$, where P is an $r \times r$ unimodular matrix, D is a diagonal matrix with ± 1 entries on the diagonal and Q is a permutation matrix. Then $BB^{\mathrm{T}} = PADQQ^{\mathrm{T}}D^{\mathrm{T}}A^{\mathrm{T}}P^{\mathrm{T}} = PAA^{\mathrm{T}}P^{\mathrm{T}}$. Because AA^{T} and BB^{T} are equivalent matrices, they have the same set of elementary divisors. Theorem 1 states that the matroid definition of the set of elementary divisors coincides with the definition for graphs given in the introduction.

Theorem 1. *The elementary divisors of the cycle matroid of a graph G are the elementary divisors of the Laplacian matrix $L(G)$.*

Proof. Orient the edges of G arbitrarily. The vertex-edge incidence matrix A of G ($+1$ for an outward edge and -1 for an inward edge) gives the usual unimodular co-ordinatization of the cycle matroid. Moreover, $AA^{\mathrm{T}} = L(G)$. □

Theorem 2. *The dual M^* of a unimodular matroid M is also unimodular, and M and M^* have the same elementary divisors.*

Proof. A unimodular matroid M of rank r has a totally unimodular co-ordinatization of the form $A = (I_r \mid L)$, where I_r is the $r \times r$ identity matrix [7], and it is easy to check that $B = (-L^{\mathrm{T}} \mid I_{n-r})$ is a totally unimodular co-ordinatization of M^*. Elementary row and column operations performed on the matrix

$$\begin{pmatrix} I_r & L \\ -L^{\mathrm{T}} & I_{n-r} \end{pmatrix}$$

result in

$$\begin{pmatrix} I & 0 \\ 0 & I + L^{\mathrm{T}}L \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} I + LL^{\mathrm{T}} & 0 \\ 0 & I \end{pmatrix}.$$

Therefore $AA^{\mathrm{T}} = I + LL^{\mathrm{T}}$ and $BB^{\mathrm{T}} = I + L^{\mathrm{T}}L$ have the same elementary divisors. But the elementary divisors of AA^{T} and BB^{T} are, by definition, the elementary divisors of M and M^*, respectively. □

An *integral lattice* Λ is defined as any subgroup of the additive group Z^n. Since an integral lattice is a free group, it has a basis, and the number of elements r in a basis is called the *dimension* of the lattice. The elements of Λ will be considered as row vectors and a basis as a $r \times n$ full rank matrix. An integral lattice Λ will be called *unimodular* if there exists a unimodular basis matrix for Λ. Note that if Λ is unimodular, then actually every basis matrix for Λ is a unimodular matrix. To see this let A and B be two bases for the same integral lattice. Then $B = PA$ for some invertible integral matrix P. Hence if A is unimodular, then so is B.

The set of minimal supports of the elements of an integral lattice Λ in Z^n are the circuits of a matroid on $S = \{1, 2, \ldots, n\}$, denoted $M[\Lambda]$. The next theorem characterizes unimodular matroids as the matroids of unimodular lattices. The lattice representation of a matroid is related to Tutte's chain group [5]. Let $\Pi(n)$ be the group of $n \times n$ permutation matrices—the matrices having exactly one ± 1 in each row and each column. The elements of $\Pi(n)$ act on the points of Z^n by multiplication on the right. This action is equivalent to simply permuting the $\pm$ co-ordinate axes of E^n. Let $\Lambda^\perp$ denote the orthogonal complement of Λ with respect to the standard linear map $Z^n \times Z^n \to Z$ defined by $(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i y_i$. It is easy to verify that $\Lambda^{\perp\perp} = \Lambda$ for any unimodular lattice Λ.

THEOREM 3. *The mapping $\Lambda \to M[\Lambda]$ induces a bijection between the equivalence classes under $\Pi(n)$ of unimodular sublattices of Z^n and the isomorphism classes of unimodular matroids of cardinality n.*

PROOF. We use the fact, essentially due to Whitney [6], that any basis for a lattice Γ is a co-ordinatization of $M[\Gamma^\perp]$. We first show that if Λ is a unimodular lattice, then $M[\Lambda]$ is a unimodular matroid. If $A = (I \mid L)$ is a unimodular basis for Λ, then $B = (-L^{\mathrm{T}} \mid I)$ is a unimodular basis for $\Lambda^\perp$. Hence B is a unimodular co-ordinatization of $M[\Lambda^{\perp\perp}] = M[\Lambda]$.

To show that the mapping is injective, assume that $M[\Lambda]$ and $M[\Gamma]$ are isomorphic. If A and B are bases of Λ and Γ, respectively, then $A = PBDQ$ for some unimodular matrix P, diagonal matrix D with ± 1 on the diagonal, and permutation matrix Q with exactly one 1 in each row and each column [7]. But PB is also a basis for Γ and DQ is just a ± 1 permutation matrix.

To show that the mapping is surjective, assume that M is a unimodular matroid. Let $A = (I \mid L)$ be a totally unimodular co-ordinatization of M and let Λ be the lattice generated by the rows of A. Then A is a co-ordinatization of $M[\Lambda^\perp]$ and hence $M = M[\Lambda^\perp]$. To see that $M[\Lambda^\perp]$ is unimodular, note that the totally unimodular matrix $B = (-L^{\mathrm{T}} \mid I)$ is a basis for $\Lambda^\perp$. □

THEOREM 4. *Let Λ be a unimodular lattice and A and B any two bases for Λ. Then the elementary divisors of $M[\Lambda]$ are the elementary divisors of AB^{T}. In particular, the elementary divisors of $M[\Lambda]$ are the elementary divisors of AA^{T}.*

PROOF. Since the rows of both A and B form bases for the unimodular lattice Λ they are unimodular matrices. As in the proof of Theorem 3, B is a co-ordinatization of $M[\Lambda^\perp] = M^*[\Lambda]$, the last equality also due to Whitney [6]. By definition, the elementary divisors of $M^*[\Lambda]$ are the elementary divisors of BB^{T}. But, by Theorem 1, $M[\Lambda]$ and $M^*[\Lambda]$ have the same elementary divisors. So the elementary divisors of $M[\Lambda]$ are the elementary divisors of BB^{T}. Any two bases A and B of Λ are related by $A = PB$ where P is a unimodular matrix. Then $AB^{\mathrm{T}} = (PB)B^{\mathrm{T}} = P(BB^{\mathrm{T}})$. Hence AB^{T} and BB^{T} are equivalent and, consequently, have the same elementary divisors. □

3. Graphs and the Characterization Problem

In this section $M[G]$ denotes the cycle matroid of a graph G. Theorem 5 below and its corollary (due to Oberst [3]) determine the elementary divisors of a graph in terms of intersection matrices. Let $\gamma = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{q-n+1}\}$ and $\gamma^* = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n-1}\}$ be bases for the cycle and cocycle spaces of G, respectively, where n and q are the number of vertices and edges in G. For each cycle in γ orient the edges arbitrarily in one of the two directions around the cycle. Similarly, each cutset in γ^* partitions the vertex set $V = V_1 \cup V_2$, and orients the edges arbitrarily in one of the two directions, from V_1 to V_2 or the reverse. Define the *cycle intersection matrix* $W(\gamma) = (c_{ij})$ and *cocycle intersection matrix* $W(\gamma^*) = (b_{ij})$, where entries c_{ij} and b_{ij} are the number of edges that cycles $\mathbf{c}_i$ and $\mathbf{c}_j$ have in common or cocycles $\mathbf{b}_i$ and $\mathbf{b}_j$ have in common, respectively. If a pair of common edges have the same orientation, the intersection is counted $+1$; if the orientation is reverse then -1.

Theorem 5. *The elementary divisors of G are the elementary divisors of the intersection matrix $W(\gamma)$ (or $W(\gamma^*)$) for any basis γ of the cycle space (or basis γ^* of the cocycle space).*

Proof. Orient the edges of G arbitrarily and consider, from simplicial homology and cohomology theory, the usual boundary and cobounary operators on 1-chains with coefficients in Z. The set Λ of incidence vectors of 1-chains with 0 boundary and the set Λ^* of incidence vectors of 1-chains that are coboundaries are both integer lattices, the matroids of which are $M[G]$ and $M^*[G]$, respectively. Furthermore, $\Lambda^\perp = \Lambda^*$. The rows of the vertex-edge incidence matrix of G spans Λ^*, and this matrix is totally unimodular. Therefore $M[\Lambda^*]$ is a unimodular matroid. Since Λ^* is unimodular, $\Lambda^{*\perp} = \Lambda^{\perp\perp} = \Lambda$ is also unimodular, as in the proof of Theorem 3. Any basis γ (γ^*) of the cycle (cocycle) space is a basis, with the appropriate $\pm$ signs for orientation, of Λ (Λ^*). Let A be a matrix the rows of which form a basis for Λ (or, equivalently, for the cycle space when considered modulo 2). By Theorem 4 the elementary divisors of $M[G]$ are the elementary divisors of AA^{T}. However, $AA^{\mathrm{T}} = W(\gamma)$. Similarly, if A is a matrix the rows of which form a basis for Λ^*, then the elementary divisors of $M^*[G]$ are the elementary divisors of $W(\gamma^*) = AA^{\mathrm{T}}$. But, by Theorem 2, $M[G]$ and $M^*[G]$ have the same elementary divisors. □

Example. For the cycle C_n on n vertices a basis γ of the cycle space consists of a single cycle and $W(\gamma) = (n)$. Therefore the set of elementary divisors is $D(C_n) = \{n\}$.

Special choices for the cycle and cocycle spaces of a graph in Theorem 5 facilitate the computation of the elementary divisors. Let T be the set of edges in any spanning forest of a graph G with edge set S. Then each $e \in S - T$ determines a unique *basic cycle* of G, i.e. the unique cycle in $T \cup \{e\}$. Similarly, each $e \in T$ determines a unique *basic bond*, i.e. the unqiue bond in $(S - T) \cup \{e\}$. Let $W(T)$ and $W^*(T)$ be the intersection matrices of the basic cycles and basic bonds, respectively. Corollary 1 follows from the fact that the basic cycles and basic bonds form bases for the cycle and cocycle spaces, respectively.

Corollary 1. *The elementary divisors of a graph G are the elementary divisors of the intersection matrix $W(T)$ (or $W^*(T)$), for any spanning forest T.*

Theorem 6. *The product of the elementary divisors of a connected graph G is equal to the number of spanning trees of G.*

Proof. Let n denote the order of G. The Laplacian $L(G)$ has the form $\left(\begin{smallmatrix} L & c \\ r & * \end{smallmatrix}\right)$, where the last row $(r \mid *)$ and the last column $\left(\begin{smallmatrix} c \\ * \end{smallmatrix}\right)$ are integral linear combinations of the other rows and columns, respectively. Hence $L(G)$ is equivalent to the matrix $\left(\begin{smallmatrix} L & 0 \\ 0 & 0 \end{smallmatrix}\right)$, which has the same elementary divisors as L. By the Matrix-Tree Theorem [2] the number of spanning trees of G is equal to the determinant of any cofactor of $L(G)$, in particular to $\det(L)$. But elementary row operations preserve the absolute value of the determinant, so $\det(L)$ is the product of its elementary divisors. □

For a connected graph G of order n let $A(G)$ denote the quotient of Z^n by the group generated by the rows of the Laplacian $L(G)$. Note that the elementary divisors of G are simply the elementary divisors d_i in the canonical direct sum decomposition $Z \oplus_1^r Z_{d_i}$ of $A(G)$ guaranteed by the classification theorem for Abelian groups. In this sense it is the Abelian group $A(G)$ that is the invariant of G. If $L(G)$ is put into equivalent diagonal form (not necessarily Smith normal form), then the diagonal entries are also orders of cyclic summands in some direct sum decomposition of $A(G)$ and, therefore, completely determine the elementary divisors. So, for convenience in expressing the next result, we allow $D(G)$ to be a multiset of diagonal elements in any diagonal matrix equivalent to $L(G)$, and understand it to mean the corresponding uniquely determined multiset of elementary divisors. For example, $D(G) = \{60, 18, 75\}$ means the same as $D(G) = \{3, 30, 900\}$. In Theorem 7, $D(G)$ is denoted in the form $t_1\{m_1\} + t_2\{m_2\} + \cdots + t_k\{m_k\}$, where the distinct elementary divisors m_i appear with multiplicity t_i.

Theorem 7. *For trees, cycles, the complete and complete bipartite graphs, and wheels with m* spokes, we have:

(1) $D(T) = \{\ \}$;

(2) $D(C_n) = \{n\}$;

(3) $D(K_n) = (n-2)\{n\}$;

(4) $D(K_{m,n}) = \{mn\} + (m-2)\{n\} + (n-2)\{m\}$;

(5) $D(W_m) = \begin{cases} \{F_m, 5F_m\}, & \textit{if } m \textit{ even}, \\ 2\{F_{m-1} + F_{m+1}\}, & \textit{if } m \textit{ odd}; \end{cases}$

where F_n is the nth Fibonacci number with $F_1 = F_2 = 1$.

Proof. If T is a tree, then the cycle space is trivial, and therefore the set of elementary divisors is empty by Theorem 5. The cycle intersection matrix of C_n is just (n).

To show (3) consider the matrix consisting of all rows and columns of $L(G)$ except the first and last and perform the following row and column operations: (i) add all other rows to the first; (ii) add the first row to all others; (iii) subtract the first column from all the others. The matrix is now diagonal with entries $\{1\} + (n-2)\{n\}$.

For $K_{m,n}$ it is sufficient to find the elementary divisors of the matrix

$$\begin{pmatrix} mI_n & -1 \\ -1 & nI_{m-1} \end{pmatrix},$$

where mI_n and nI_{m-1} are diagonal submatrices of sizes n and $m-1$ respectively. All the entries in submatrix (-1) are -1. Now (i) add all other rows to the first; (ii) add the first to the last $m-1$ rows; and (iii) subtract the first column from all the others. This results in a matrix of the form

$$\begin{pmatrix} mI_{n-1} & -1 \\ 0 & nI_{m-1} \end{pmatrix}$$

Now (iv) add all the other rows to the last; (v) add the last to the first n row; and (vi) subtract the last column from all others. This results in elementary divisors $(m-2)\{n\}$ and the submatrix $m(I+J)$, of size $n-1$, where J is the all-1 matrix. Finally (vii) subtract the first from every other row; (viii) add every other column to the first; and (ix) subtract every other row from the first. This results in the elementary divisors given in the theorem.

In computing the elementary divisors of the wheels the following fact is used [1]. If δ_i is the greatest common divisor of all $i \times i$ subdeterminants of an $m \times m$ matrix A, then the elementary divisors of A are given by $\delta_1, \delta_2/\delta_1, \ldots, \delta_m/\delta_{m-1}$. Start with the matrix $L(W_m)$ and delete the row and column corresponding to the hub of the wheel, leaving an $m \times m$ matrix. For this matrix $\delta_i = 1$ for all $i \leq m-2$. The remaining δ_m and δ_{m-1} are computed by induction. The details are left as an exercise. □

Let $\lambda(G)$ denote the number of elementary divisors (counting multiplicity). The next result gives bounds on the number of elementary divisors, and characterizes trees as the only connected loopless graphs with an empty set of elementary divisors and the complete graphs as the only simple graphs that attain an upper bound.

Corollary 2. (1) *For any graph G with n vertices and q edges, we have $0 \leq \lambda(G) \leq \min\{n-1, q-n+1\}$. Moreover, $\lambda(G)=0$ iff G is a forest, with possible loops.*

(2) *For any simple graph G with n vertices, we have $\lambda(G) \leq n-2$, with equality iff $G = K_n$.*

Proof. The intersection matrices W and W^* of Theorem 5 have maximum sizes $q-n+1$ and $n-1$, proving the inequality. Note that, by Corollary 1, adding loops at vertices does not effect the elementary divisors of a graph. In one direction the statement about forests is implied by formula (1) of Theorem 7. Conversely, if $\lambda(G)=0$, then by Theorems 6, each connected component of G has exactly one spanning tree. Therefore G must be a forest, with possible loops.

Any simple graph G, except K_n, has a row in the adjacency matrix with at least one 1 and one 0, neither on the diagonal. It may also be assumed, without loss of generality, that these entries occur in the 2nd and 3rd columns, respectively. Since the 3rd column does not consist entirely of 0's it may also be assumed that the $(2,3)$ entry is 1. The elementary divisors of G are the elementary divisors of the matrix of size $n-1$ obtained by deleting the last row and column from $L(G)$. Row 1 and column 2 (except entry $(1,2)$) can be zeroed out in the matrix by row and column operations, resulting in a matrix of size $(n-2)$. In the resulting matrix entry $(1,2)$ is again 1, and again row 1 and column 2 can be zeroed out by row and column operations, resulting in a matrix of size $n-3$. This shows that $\lambda(G) < n-2$. The statement concerning K_n follows from Theorem 7. □

The upper bound $n-1$ is achieved, from example, by the multigraph on n vertices with every two vertices joined by 2 edges. The upper bound $q-n+1$ is achieved by cycles and, for example, the graphs in Figure 3.

The remainder of this section addresses the question of how well the elementary divisors distinguish between graphs. From the theory in Section 2, the elementary divisors do not distinguish between graphs with the same underlying cycle matroid. The two graphs in Figure 1 with the same set of elementary divisors, for example, have the same matroid. A theorem of Whitney [8] states that two graphs (without isolated vertices) have the same cycle matroid iff one can be transformed into the other by a

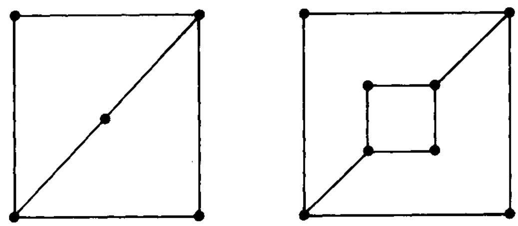

FIGURE 3. Graphs for which $\lambda(G) = q - p + 1$.

sequence of the following operations: (i) identification of two vertices in distinct components or the inverse operation, splitting at a cut-vertex and (ii) twisting. (If a 2-connected graph G can be obtained from two disjoint graphs G_1 and G_2 by identifying vertices u_1 and v_1 in G_1 with vertices u_2 and v_2, respectively, in G_2, then a *twisting* of G is obtained when u_1 is identified with v_2 and v_1 with u_2.) Corollary 3 below is a consequence of Whitney's theorem. From Theorem 2 and the fact that $M^*[G] = M[G^*]$ when G is planar with dual graph G^*, it follows that the elementary divisors also do not distinguish between a planar graph and its dual.

COROLLARY 3. *If* $G = G_1 \cup G_2$, *where* $\cup$ *denotes the disjoint or one vertex union, then* $D(G) = D(G_1) \cup D(G_2)$.

A pair of unimodular matroids of the same cardinality and rank will be called *co-invariant* if they have the same elementary divisors. The matroids $M[G]$ and $M[H]$ of a pair of connected graphs G and H have the same cardinality and rank iff they have the same number n of vertices and same number q of edges. Therefore, a pair of (n, q)-graphs will be called *co-invariant* if they have the same elementary divisors. The two $(8, 9)$-graphs in Figure 4 are co-invariant with $D(G) = \{24\}$, although they are not Whitney equivalent. Call a matroid M *simple* (*dual simple*) if M (M^*) has no loops or parallel elements (no one or two element dependent sets). Note that the graphs in Figure 4 are not dual simple. It is natural to ask how well the elementary divisors distinguish between matroids that are both simple and dual simple. In particular, we have no example of a pair of distinct, simple, non-dual, 3-connected, co-invariant graphs.

Conditions for a pair of unimodular matroids to be co-invariant is given in Theorem 8 in terms of the lattices that represent them. Let $A(Z^n)$ denote the automorphism group of Z^n; that is, the group of all group isomorphisms of lattice Z^n to itself. Then $A(Z^n)$ can be identified with the special linear group over Z acting on vectors by matrix multiplication on the right. More precisely, there is a group isomorphism $\phi: SL(n, Z) \to A(Z^n)$ defined by $\phi(B)(\mathbf{x}) = \mathbf{x}B$ for $B \in SL(n, Z)$ and $x \in Z^n$. An n-dimensional sublattice of Z^n is called *full*. Two full lattices Λ and Γ of Z^n are called *equivalent* if there is an automorphism of Z^n, i.e. an element of $SL(n, Z)$, taking Λ to

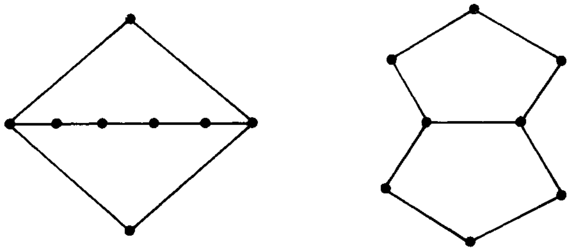

FIGURE 4. Co-invariant graphs with $D(G) = \{24\}$.

Γ. Let Λ be any unimodular lattice of dimension r and $\Lambda^\perp$ the orthogonal complement of dimension $n-r$ as defined in Section 2. Then the lattice $\Lambda \oplus \Lambda^\perp$, spanned by Λ and $\Lambda^\perp$, is a full lattice.

THEOREM 8. *Let Λ and Γ be unimodular lattices of the same dimension. Then the following statements are equivalent*:
(1) *$M[\Lambda]$ and $M[\Gamma]$ are co-invariant*;
(2) *$\Lambda \oplus \Lambda^\perp$ is equivalent to $\Gamma \oplus \Gamma^\perp$*;
(3) *there exists basis matrices A_Λ, B_Λ of Λ and A_Γ, B_Γ of Γ such that $A_\Lambda B_\Lambda^{\mathrm{T}} = A_\Gamma B_\Gamma^{\mathrm{T}}$.*

PROOF. We first show that the elementary divisors of $M[\Lambda]$ are the elementary divisors of any basis for $\Lambda \oplus \Lambda^\perp$. Let $A = (I \mid L)$ be a basis for Λ in canonical form. By Theorem 4, the elementary divisors of $M[\Lambda]$ are the elementary divisors of $AA^{\mathrm{T}} = I + LL^{\mathrm{T}}$. Now $B = (-L^{\mathrm{T}} \mid I)$ is a basis for $\Lambda^\perp$ and hence

$$\begin{pmatrix} I_r & L \\ -L^{\mathrm{T}} & I \end{pmatrix}$$

is a basis of $\Lambda \oplus \Lambda^\perp$. But, as in the proof of Theorem 2, the latter matrix has the same elementary divisors as $I + LL^{\mathrm{T}}$, i.e. the same elementary divisors as $M[\Lambda]$. All bases for $\Lambda \oplus \Lambda^\perp$ are equivalent; therefore, the elementary divisors of any basis of $\Lambda \oplus \Lambda^\perp$ are the elementary divisors of $M[\Lambda]$. Concerning the equivalence of statements (1) and (2), we have shown that if A and B are bases for lattices $\Lambda \oplus \Lambda^\perp$ and $\Gamma \oplus \Gamma^\perp$, respectively, then $M[\Lambda]$ and $M[\Gamma]$ are co-invariant iff A and B are equivalent. However, A and B are equivalent iff $\Lambda \oplus \Lambda^\perp$ and $\Gamma \oplus \Gamma^\perp$ are equivalent lattices. To see this, note that A and B are equivalent iff $A = PBQ$, where $P \in SL(n, Z)$ and $Q \in SL(n, Z)$. But $B' = PB$ is also a basis for $\Gamma \oplus \Gamma^\perp$ and $A = B'Q$ means that $\Lambda \oplus \Lambda^\perp$ and $\Gamma \oplus \Gamma^\perp$ are equivalent.

Concerning the equivalence of statements (1) and (3), $M[\Lambda]$ and $M[\Gamma]$ have the same elementary divisors, according to Theorem 4, iff $A_\Lambda B_\Lambda^{\mathrm{T}}$ and $A'B'^{\mathrm{T}}$ are equivalent matrices, where A', B' are bases for Γ. This means that $A_\Lambda B_\Lambda^{\mathrm{T}} = PA'B'^{\mathrm{T}}Q^{\mathrm{T}} = (PA')(QB')^{\mathrm{T}}$ for some unimodular matrices P and Q. However, $A_\Gamma = PA'$ and $B_\Gamma = QB'$ also form bases for Λ and Γ, respectively. □

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