Highly Non-contractive Iterated Function Systems on Euclidean Space Can Have an Attractor

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Abstract

Iterated function systems (IFSs) and their attractors have been central to the theory of fractal geometry almost from its inception. Moreover, contractivity of the functions in the IFS has been central to the theory of iterated functions systems. If the functions in the IFS are contractions, then the IFS is guaranteed to have a unique attractor. The converse question, does the existence of an attractor imply that the IFS is contractive, originates in a 1959 work by Bessaga which proves a converse to the contraction mapping theorem. Although a converse is true in that case, it is known that it does not always hold for an IFS. In general, there do exist IFSs with attractors and which are not contractive. However, in the context of IFSs in Euclidean space, this question has been open. In this paper we show that a highly non-contractive iterated function system in Euclidean space can have an attractor. In order to do that, we introduce the concept of an L-expansive map, i.e., a map that has Lipschitz constant strictly greater than one under any remetrization. This is necessitated by the absence of positively expansive maps on the interval.

Keywords: iterated function system, attractor, contractive, L-expansive

MSC Classification: 28A80

1 Introduction

A seminal result of Hutchinson [16] states that, if the functions in an iterated function system (IFS) F on a complete metric space are contractions, then F has a unique attractor. This paper concerns the converse: does the existence of a unique attractor of an IFS imply that the IFS is contractive? In particular, whether or not a highly non-contractive IFS on Euclidean space can possess a unique attractor was an open question. Classes of such IFSs, as well as particular examples, appear in Sections 4, 5, and 6 of this paper. The remainder of this section contains definitions of all terms used above, a short history of results on the subject, and an overview of subsequent sections of the paper.

Let $(\mathbb{X}, d)$ be a metric space. An *iterated function system* (IFS) is a set

$$F = \{f_1, f_2, \dots, f_N\}$$

of continuous functions from $\mathbb{X}$ into itself. Throughout this paper we denote the n-fold composition of a function f with itself by $f^{(n)}$. The *Lipschitz constant of a function* f is

$$\mathrm{Lip}_d(f) = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}.$$

A function $f : \mathbb{X} \to \mathbb{X}$ is

(a) a *contraction with respect to d* if $\mathrm{Lip}_d(f) < 1$;
(b) *non-expansive with respect to d* if $\mathrm{Lip}_d(f) \leq 1$;
(c) *L-expansive with respect to d* if $\mathrm{Lip}_d(f) > 1$.

Definition 1 Two metrics on $\mathbb{X}$ are *equivalent* if they induce the same topology on $\mathbb{X}$. Given a metric space $(\mathbb{X}, d)$, an IFS F on $\mathbb{X}$ is *contractive* if there is an equivalent metric with respect to which all the functions in F are contractions. A function f on $(\mathbb{X}, d)$ is *L-expansive* if f is L-expansive with respect to any metric d' on $\mathbb{X}$ that is equivalent to d. An IFS F is *L-expansive* if all the functions in F are L-expansive. "L-expansive IFS" is what is meant by the informal phrase "highly non-contractive IFS" in the title of this paper.

If $\mathbb{Y} \subset \mathbb{X}$ and $f(\mathbb{Y}) \subset \mathbb{Y}$, then f is said to be *L-expansive on* $\mathbb{Y}$ if the restriction $f_{|\mathbb{Y}}$ is L-expansive. If $f(\mathbb{Y}) \subset \mathbb{Y}$ for every $f \in F$, then F is said to be *L-expansive on* $\mathbb{Y}$, if all maps in F are L-expansive on $\mathbb{Y}$. Clearly, if an IFS is L-expansive on $\mathbb{Y} \subset \mathbb{X}$, then it is L-expansive.

Remark 1 The term "expansive" has a different meaning in dynamical systems theory. For a compact infinite metric space $\mathbb{X}$, a continuous surjection $f : \mathbb{X} \to \mathbb{X}$ is called *positively expansive* if there is a constant $e > 0$ such that if $x \neq y$ then $d(f^{(n)}(x), f^{(n)}(y)) > e$ for some nonnegative integer n. Note that this condition is independent of the choice of equivalent metric on $\mathbb{X}$. Furthermore, there are no positively expansive maps on a closed interval, see [2, Chap. 2.2], [25]. Note also that every positively expansive map is L-expansive but not vice versa. In this paper we are primarily interested in maps on Euclidean space. Since the notion of positively expansive maps is too restrictive in this setting, we introduced the broader notion of L-expansion.

For the collection $\mathcal{K}(\mathbb{X})$ of non-empty compact subsets of $\mathbb{X}$ and an IFS F on $\mathbb{X}$, the classical Hutchinson operator $F : \mathcal{K}(\mathbb{X}) \to \mathcal{K}(\mathbb{X})$ is given by

$$F(K) = \bigcup_{f \in F} f(K).$$

By abuse of language, the same notation F is used for the IFS and for the Hutchinson operator; the meaning should be clear from the context. Furthermore, for simplicity we write $F(x)$ instead of $F(\{x\})$.

Denote the Hausdorff metric on $\mathcal{K}(\mathbb{X})$ by h_d. Convergence in $\mathcal{K}(\mathbb{X})$ is always with respect to the Hausdorff metric. Note that equivalent metrics d, d' on $\mathbb{X}$ lead to equivalent Hausdorff metrics $h_d, h_{d'}$ on $\mathcal{K}(\mathbb{X})$.

Definition 2 A compact set A is the *attractor* of IFS F if

- (*invariance*) $F(A) = A$, and
- (*attraction*) $F^{(n)}(K) \to A$ for every $K \in \mathcal{K}(\mathbb{X})$,

where $F^{(n)}$ denotes the n-fold composition of F.

An iterated functions system, as a method for constructing fractals, was introduced by John Hutchinson [16] in 1981. In that paper is the fundamental result that, if the functions in an IFS F on a complete metric space are contractions, then F has a unique attractor. Inquiry into a converse of Hutchinson's theorem, asking whether contractivity is necessary for the existence of an IFS attactor, can be traced back to 1959, well before the notion of an IFS was introduced. The contraction mapping theorem, first stated by Stefan Banach [4] in 1922, is a special case of Hutchinson's theorem applied to an IFS consisting of a single function. A 1959 paper by C. Bessaga [10] was titled "On the converse of the Banach fixed-point principle." Subsequently, a slew of versions of a converse to Banach's fixed point theorem appeared; see [11] for an extensive list of references.

For an IFS consisting of at least two functions, the first converse of Hutchinson's theorem was proved for an affine IFS [3] (see also [23]). Specifically, if all the functions of an IFS F on $\mathbb{R}^s$ are affine and F has an attractor, then F is contractive, although the equivalent metric on $\mathbb{R}^s$, for which the functions in F are contractions, may not be the standard Euclidean metric. This result was followed by analogous converses for (1)

IFSs consisting of projective transformations on n-dimensional real projective space [8] and (2) for IFSs consisting of Möbius transformations on the extended complex plane, equivalently the Riemann sphere [26]. Also, as proved in [5] and [24], if an attractor A exists and admits a suitable fibering structure (so that the single-valued coding map exists), then there is a remetrization of A making all the maps of the IFS *weakly* contractive, in particular non-expansive. In [21] the authors survey results on the existence of an attractor for IFSs that are, in various senses, weakly contractive.

The first counterexample to the converse of Hutchinson's theorem probably appeared in the 2000 paper of A. Kameyama [17]. He introduces the notion of a *topological self-similar set* and asks whether there is a metric inducing the underlying topology such that the functions associated with the topological self similar set (IFS in the terminology of this paper) are contractions with respect to this metric. He proves that the answer to his question is "no" by providing an IFS consisting of two functions on an abstractly defined space.

Less abstract, more geometric counterexamples to the converse of Hutchinson's theorem, have subsequently been found. In particular, an L-expansive IFS consisting of two functions on the circle with an attractor appears in [20]. See also [7, 18]. The following question, however, has gone unanswered up to now.

Question 1 Does there exist an L-expansive IFS on $\mathbb{R}^s$ that has an attractor?

Question 1 is answered in this paper. As previously mentioned, any such IFS cannot be affine. In Sections 4, 5, and 6 we provide classes of IFSs and concrete examples showing that there do exist L-expansive IFS on $\mathbb{R}^s$, $s \geq 1$, possessing an attractor. Moreover, our examples are L-expansive *on the attractor* (see Definition 1), which means that the lack of contractivity is not artificially induced on the complement of the attractor. Section 4 contains results and examples in 1-dimensional Euclidean space, i.e., L-expansive IFSs on $\mathbb{R}$ that have an attractor. Theorem 15 provides an example of an L-expansive IFS on $\mathbb{R}^2$ whose attractor is the closed unit disk. Moreover, according to Theorem 16, every set in $\mathbb{R}^2$ homeomorphic to a closed disk is the attractor of an L-expansive IFS. Theorem 10, Theorem 18 and Corollaries 19 and 20 provide methods to construct new L-expansive IFSs having an attractor from more basic ones. The attractor in Figure 1 was obtained in this way (see Example 6).

Our Definition 1 of an L-expansive IFS is intended to capture the intuitive notion of a "highly non-contractive" IFS. It may be problematic to provide a stronger, yet useful, notion of "high non-contractivity". Specifically, in Section 7 it is shown that, for any IFS F with an attractor and such that $Lip_d(f) < \infty$ for all $f \in F$, the following is true: For any $\epsilon > 0$, there exists an equivalent metric d' such that $Lip_{d'}(f) < 1 + \epsilon$, and d' is complete provided d is complete.

2 *L*-expansive Functions

We begin with results that will be used to show that the IFSs in Sections 4, 5, and 6 are L-expansive. The proof of the following lemma is routine.

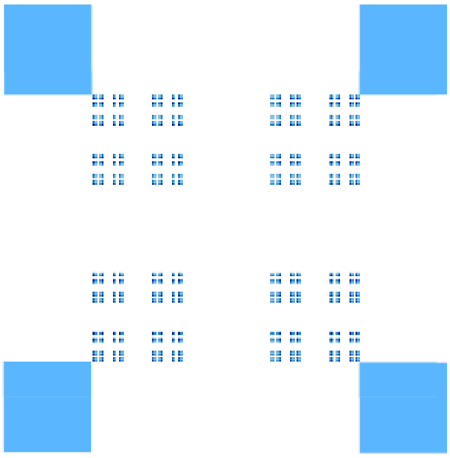

Fig. 1 The attractor of the L-expansive IFS of Example 6.

Lemma 1 *Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous function on a metric space $\mathbb{X}$. If, for some positive integer n, the composition $f^{(n)}$ is L-expansive, then f itself is L-expansive.*

A function f on a metric space $\mathbb{X}$ and a function g on a metric space $\mathbb{Y}$ are called *conjugate* if there is a homeomorphism $h : \mathbb{X} \to \mathbb{Y}$ such that $g = h \circ f \circ h^{-1}$.

Lemma 2 *If functions $f : \mathbb{X} \to \mathbb{X}$ and $g : \mathbb{Y} \to \mathbb{Y}$ are conjugate, then f is L-expansive if and only if g is L-expansive.*

Proof Let d be a fixed metric on $\mathbb{X}$ and ϱ be a fixed metric on $\mathbb{Y}$. Let $h : \mathbb{X} \to \mathbb{Y}$ be the homeomorphism conjugating f with g. It is sufficient to observe that for any metric d' on $\mathbb{X}$ equivalent to d, there is a metric ϱ' on $\mathbb{Y}$ equivalent to ϱ such that $Lip_{\varrho'}(hfh^{-1}) = Lip_{d'}(f)$. Namely, $\varrho'(x, y) := d'(h^{-1}(x), h^{-1}(y))$, $x, y \in \mathbb{Y}$, satisfies this requirement. □

Let $x_0 \in \mathbb{X}$ be a non-isolated fixed point of a mapping $f : \mathbb{X} \to \mathbb{X}$ of a metric space $\mathbb{X}$. We say that x_0 is:

(a) *repelling* if there exists an open set $U \ni x_0$ such that for all $x \in U \setminus \{x_0\}$ there is an $n \in \mathbb{N}$ for which $f^{(n)}(x) \notin U$;

(b) *partially repelling* if there exists an open set $U \ni x_0$ such that for all open sets V with $x_0 \in V \subset U$, there is an $x \in V$ and an $n \in \mathbb{N}$ for which $f^{(n)}(x) \notin U$.

Note that a repelling fixed point is partially repelling, but the converse implication does not hold. For instance, a fixed point of a map of the real line that is repelling only from one side, is partially repelling, though it is not repelling.

Proposition 3 (Repelling Criterion for L-expansiveness) *If $f : \mathbb{X} \to \mathbb{X}$ has a partially repelling fixed point, then f is L-expansive.*

Proof Suppose, by way of contradiction, that f is not L-expansive with respect to some metric d equivalent to the original metric on $\mathbb{X}$. Let x_0 be the partially repelling fixed point. Then there exists $\varepsilon > 0$ and sequences $x_n \to x_0$ and $k_n \in \mathbb{N}$ such that $d(f^{(k_n)}(x_n), x_0) \geq \varepsilon$ for all $n \in \mathbb{N}$. This leads to a contradiction:

$$\varepsilon \leq d(f^{(k_n)}(x_n), x_0) = d(f^{(k_n)}(x_n), f^{(k_n)}(x_0)) \leq d(x_n, x_0) \to 0.$$

□

Remark 2 A fixed point x_0 of f is partially repelling if and only if the family of iterates $\{f^{(n)}\}_{n\in\mathbb{N}}$ is not equicontinuous at the point x_0. Therefore, if $\mathbb{X}$ is a compact metric space, then Proposition 3 is immediate from the Markov criterion (see [22, Chap. 1.5]): f is non-expansive under some equivalent metric in $\mathbb{X}$ if and only if the family of iterates $\{f^{(n)}\}_{n\in\mathbb{N}}$ is uniformly equicontinuous. For various notions of equicontinuity we refer to [27, A.7.2].

For a real function, one can quickly determine if a fixed point is repelling whenever the derivative at that point exists. Below we denote by $f'(x_0)$ the derivative of the function $f : [a, b] \to [a, b]$ at point $x_0 \in [a, b]$. In case $x_0 = a$, $f'(x_0)$ stands for the right derivative, and in case $x_0 = b$, $f'(x_0)$ stands for the left derivative. We allow the derivatives to be infinite.

Proposition 4 (Derivative Test for L-expansiveness, [1]) *Let $f : [a, b] \to [a, b]$ be a continuous map and $x_0 = f(x_0)$. If $|f'(x_0)| > 1$, then x_0 is a repelling fixed point. In particular, f is L-expansive.*

Example 1 The following functions are easily seen to be L-expansive using the previous results in this section. Several will be used later in the paper.

1. $s(x) = \sqrt{x}$

   The function $s(x)$ is L-expansive on $[0, 1]$ by Proposition 4 ($x_0 = 0$, $f = s$, $f'(x_0) = \infty$).

2. $\widehat{s}(x) = 1 - \sqrt{x}$

   Note that $\widehat{s} \circ \widehat{s}(x) = 1 - \sqrt{1 - \sqrt{x}}$ is L-expansive by Proposition 4 ($f := \widehat{s} \circ \widehat{s}$ with $x_0 = 0$, $f'(x_0) = \infty$). By Lemma 1, the map $\widehat{s}$ is L-expansive on $[0, 1]$.

   One cannot apply Proposition 4 directly to $f := \widehat{s}$, because $x_0 = \frac{1}{2}(3 - \sqrt{5})$ is the only fixed point of f in $[0, 1]$, but $|f'(x_0)| < 1$.

3. $\widetilde{s}(x) = \sqrt{1 - x}$

   Let $h(x) = 1 - x$ and note that $\widetilde{s}(x) = h \circ \widehat{s} \circ h^{-1}$. That $\widetilde{s}$ is L-expansive on $[0, 1]$ follows from Lemma 2.

   One cannot apply Proposition 4 directly to $f := \widetilde{s}$, because $x_0 = \frac{1}{2}(\sqrt{5} - 1)$ is the only fixed point of f in $[0, 1]$, but $|f'(x_0)| < 1$.

4. $\overline{s}(x) = 1 - \sqrt{1 - x}$

The function $\overline{s}(x)$ is L-expansive on $[0,1]$ by Proposition 4 ($f := \overline{s}$ with $x_0 = 1$, $f'(x_0) = \infty$).

5. $p(x) = x^\alpha$, $\alpha > 1$

   The function $p(x)$ is L-expansive on $[0,1]$ by Proposition 4 ($x_0 = 1$, $f = p$, $f'(x_0) = \alpha > 1$).

6. $g(x) = \alpha x + \beta$ for $x \in [a,b]$, where $\alpha > 1$ and $a \le \frac{\beta}{1-\alpha} \le b$.

   The function g is L-expansive on $[a,b]$ by Proposition 4 ($x_0 = \frac{\beta}{1-\alpha}$, $f = g$, $f'(x_0) = \alpha > 1$).

7. For $0 \le c < d \le 1$ and any continuous L-expansive $f : [0,1] \to [0,1]$, let

$$g(x) = f\Big(\frac{x-c}{d-c}\Big) \cdot (d-c) + c.$$

   Then $g(x)$ is L-expansive on $[c,d]$. This follows from Lemma 2 because the restriction $g\big|_{[c,d]} = h \circ f \circ h^{-1}$, for the linear increasing homeomorphism $h : [0,1] \to [c,d]$.

3 The Existence of an Attractor

In subsequent sections, a proof will be required that a certain set $A \in \mathcal{K}(\mathbb{X})$ is the attractor of a certain IFS F on $\mathbb{X}$. The results gathered in this section will be helpful in this regard. In particular, Theorem 7 provides a way to reduce the attraction property in Definition 2 to showing that it holds for a single point in A. Lemma 6, in turn, reduces that requirement to showing that a certain set is dense in A. Theorem 10 is used in Example 5, where the attractor is partitioned into three parts.

The following proposition has appeared in various guises; see [14] for an early reference.

Proposition 5 *Assume that F is an IFS on a metric space $\mathbb{X}$ with attractor A, and let $h : \mathbb{X} \to \mathbb{Y}$ be a homeomorphism. Then $h(A)$ is the attractor of the IFS $hFh^{-1} = \{h \circ f \circ h^{-1} : f \in F\}$.*

Lemma 6 ([15], Chap. 7, pp. 663–664, Propositions 1.19 and 1.20) *If $S_1 \subset S_2 \subset S_3 \subset \dots$ is a nested sequence of compact sets in a metric space $\mathbb{X}$ and if $S := \overline{\bigcup_{n \ge 1} S_n}$ is compact, then $S_n \to S$.*

Theorem 7 *Let F be an IFS on compact metric space $\mathbb{X}$. If there is an $x_0 \in \mathbb{X}$ such that*

1. *for every $x \in \mathbb{X}$ there is a sequence (x_n) with $x_n \in F^{(n)}(x)$ such that $x_n \to x_0$, and*
2. *$F^{(n)}(x_0) \to \mathbb{X}$,*

then $F^{(n)}(x) \to \mathbb{X}$ for all $x \in \mathbb{X}$. In particular, $\mathbb{X}$ is the attractor of F.

Proof Fix any $x \in \mathbb{X}$ and any $\varepsilon > 0$. By assumption (2) we can find an $n_0 \in \mathbb{N}$ so that

$$h_d\big(F^{(n_0)}(x_0), \mathbb{X}\big) < \frac{\varepsilon}{2}. \tag{1}$$

Clearly, the map $x \mapsto F^{(n_0)}(x)$ is continuous with respect to the Hausdorff metric h_d. In particular, it is continuous at x_0, so we can find $\eta > 0$ such that

$$h_d\big(F^{(n_0)}(y), F^{(n_0)}(x_0)\big) < \frac{\varepsilon}{2} \tag{2}$$

for every y in the ball $B(x_0, \eta)$ centered at x_0 of radius η.

Take any $x \in \mathbb{X}$. By assumption (1) there is a sequence (x_n) such that $x_n \in F^{(n)}(x)$ for all $n \in \mathbb{N}$ and $x_n \to x_0$. Let $k_0 \in \mathbb{N}$ be such that $x_k \in B(x_0, \eta)$ for all $k \geq k_0$. By equation (2) we have

$$h_d\big(F^{(n_0)}(x_k), F^{(n_0)}(x_0)\big) < \frac{\varepsilon}{2}. \tag{3}$$

Let $n \geq k_0 + n_0$ and express $n = n_0 + k$ for some $k \geq k_0$. Because $x_k \in F^{(k)}(x)$ we have

$$F^{(n_0)}(x_k) \subset F^{(n_0+k)}(x) = F^{(n)}(x). \tag{4}$$

Hence

$$\begin{aligned} h_d(F^{(n)}(x), \mathbb{X}) &\leq h_d(F^{(n_0)}(x_k), \mathbb{X}) \\ &\leq h_d(F^{(n_0)}(x_k), F^{(n_0)}(x_0)) + h_d(F^{(n_0)}(x_0), \mathbb{X}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The first inequality is from Equation (4); the last inequality from Equations (1) and (3). The result follows. □

Below we provide natural conditions for IFS on the interval $[0, 1]$ which guarantee conditions (1) and (2) of the above lemma.

Proposition 8 *Let F be an IFS on the interval $[0,1]$ and $x_0 \in [0,1]$ be a fixed point of one of the maps from F. Assume that there exist closed intervals $I_1, \ldots, I_k \subset [0,1]$ and one-to-one maps $f_{p_1}, \ldots, f_{p_k} \in F$ such that*

(1) $\bigcup_{i=1}^{k} f_{p_i}(I_i) = [0,1]$;
(2) $|f'_{p_i}(x)| < 1$ *for* $i = 1, \ldots, k$, *and for all* $x \in \mathrm{Int}(I_i)$;
(3) *for distinct* $i, j \in \{1, \ldots, k\}$, *if* $f_{p_i}(I_i) \cap f_{p_j}(I_j) \neq \emptyset$, *then* $F^{(n)}(x_0) \cap f_{p_i}(I_i) \cap f_{p_j}(I_j) \neq \emptyset$ *for some* $n \in \mathbb{N}$.

Then $F^{(n)}(x_0) \to [0,1]$.

Proof Since $x_0 \in F(x_0)$, we see that $(F^{(n)}(x_0))$ is increasing, and hence it is enough to show that

$$S := \bigcup_{n \in \mathbb{N}} F^{(n)}(x_0)$$

is dense in $[0,1]$. Assume, on the contrary, that it is not the case. Let

$$\mathcal{J} := \{J : J \text{ is an open interval and } S \cap J = \emptyset\}$$

and set $L := \sup_{J \in \mathcal{J}} |J|$. Then there exists a sequence of intervals $J_n = (a_n, b_n)$, $n \in \mathbb{N}$, such that $b_n - a_n \to L$ and $S \cap J_n = \emptyset$. By passing to a subsequence, we can assume that $a_n \to a_0$, $b_n \to b_0$ for some a_0, b_0. Necessarily $a_0 < b_0$, $S \cap (a_0, b_0) = \emptyset$ and $b_0 - a_0 = L$.

Define $J_0 := (a_0, b_0)$ and consider two cases:
Case 1. $J_0 \subset f_{p_i}(I_i)$ for some $i = 1, \ldots, k$ and corresponding p_i. Then $f_{p_i}^{-1}(J_0) \subset I_i$, so by condition (2) we have that $|f_{p_i}^{-1}(J_0)| > |J_0|$. This contradicts the maximality of L as $S \cap J = \emptyset$ implies that $f_j^{-1}(J) \cap S = \emptyset$ for every $j = 1, \ldots, k$.
Case 2. Case 1 does not hold. We claim that

$$\emptyset \neq f_{p_i}(I_i) \cap f_{p_j}(I_j) \subset J_0$$

for some distinct $i, j \in \{1, \ldots, k\}$. Then, by condition (3), there is an element of $F^{(n)}(x_0)$ in J_0, which contradicts $S \cap J_0 = \emptyset$, completing the proof. To prove the claim, denote by c_i, d_i the respective endpoints of $f_{p_i}(I_i)$, $i = 1, \ldots, k$. From the family of intervals $[c_i, d_i]$ which contain a_0, choose $[c_{i_0}, d_{i_0}]$ with the largest right endpoint. Then $d_{i_0} < b_0$; otherwise it would contradict the assumption that we are not in Case 1. By condition (1), there is a $j_0 \neq i_0$ such that $d_{i_0} \in [c_{j_0}, d_{j_0})$. Then $a_0 < c_{j_0}$; otherwise that $d_{j_0} > d_{i_0}$ contradicts the assumption that d_{i_0} is the largest right endpoint containing a_0. Now

$$\emptyset \neq [c_{i_0}, d_{i_0}] \cap [c_{j_0}, d_{j_0}] \subset (a_0, b_0) = J_0,$$

proving the claim. □

The next proposition guarantees assumption (1) of Theorem 7. The proof is straightforward.

Proposition 9 *Let F be an IFS on a metric space $\mathbb{X}$ and $x_0 \in \mathbb{X}$. Fix a map $f \in F$, and let $I_f = \{x \in \mathbb{X} : f^{(n)}(x) \to x_0\}$. Assume that for every $x \in \mathbb{X} \setminus I_f$ there exists a $g \in F$ so that $g(x) \in I_f$. Then for every $x \in \mathbb{X}$, there is a sequence (x_n) with $x_n \in F^{(n)}(x)$ such that $x_n \to x_0$.*

Theorem 10 (Gluing IFSs) *Let $\mathbb{X}$ be a metric space and $\mathbb{X}_1, \ldots, \mathbb{X}_k$ nonempty compact subsets of $\mathbb{X}$ so that $\mathbb{X} = \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_k$. Let*

$$F_j = \{f_1^j, \ldots, f_{N_j}^j\},\ j = 1, \ldots, k,\ N_j \in \mathbb{N}$$

be IFSs on $\mathbb{X}$ such that for every $j = 1, \ldots, k$ we have

(1) the restriction $\overline{F}_j := \{(f_1^j)_{|\mathbb{X}_j}, \ldots, (f_{N_j}^j)_{|\mathbb{X}_j}\}$ is an IFS on $\mathbb{X}_j$ with attractor A_j;
(2) $F_j(x) \cap \mathbb{X}_j \neq \emptyset$ for every $x \in \mathbb{X}$;
(3) $F_j(\mathbb{X}_i \cap K) \subset A_1 \cup \cdots \cup A_k \cup F_i(\mathbb{X}_i \cap K)$ for all $i \neq j$ and all $K \in \mathcal{K}(\mathbb{X})$.

Then $A = \bigcup_{j=1}^k A_j$ is the attractor of the IFS $F = \bigcup_{j=1}^k F_j$.

Proof For convenience we denote $F_i^{(0)}(S) := S$ for $S \subset \mathbb{X}$. Clearly, for every $n \in \mathbb{N}$ and $K \in \mathcal{K}(\mathbb{X})$,

$$\bigcup_{i=1}^k F_i^{(n-1)}(F(K) \cap \mathbb{X}_i) \subset F^{(n)}(K). \tag{5}$$

We will also show that for $n \in \mathbb{N}$ and all $K \in \mathcal{K}(\mathbb{X})$, we have

$$F^{(n)}(K) \subset A \cup \bigcup_{i=1}^k F_i^{(n-1)}(F(K) \cap \mathbb{X}_i). \tag{6}$$

By (3), we have

$$F(A) = \bigcup_{j=1}^{k} F_j(A) = \bigcup_{j=1}^{k} \bigcup_{i=1}^{k} F_j(A_i) \subset A \cup \bigcup_{i=1}^{k} F_i(A_i) = A \cup \bigcup_{i=1}^{k} A_i = A. \tag{7}$$

For $n = 1$, the inclusion (6) is evident. Assume that (6) holds for some $n \in \mathbb{N}$. We will verify it for $n + 1$. By (3) and (7) we get, for all $K \in \mathcal{K}(\mathbb{X})$, that

$$F^{(n+1)}(K) \subset \bigcup_{j=1}^{k} F_j \left(A \cup \bigcup_{i=1}^{k} F_i^{(n-1)}(F(K) \cap \mathbb{X}_i) \right) \subset A \cup \bigcup_{j=1}^{k} \bigcup_{i=1}^{k} F_j(F_i^{(n-1)}(F(K) \cap \mathbb{X}_i))$$
$$\subset A \cup \bigcup_{i=1}^{k} F_i^{(n)}(F(K) \cap \mathbb{X}_i).$$

The last inclusion is valid because, according to (1) and (3) for $j \neq i$, n and K replaced with $F_i^{(n-1)}(F(K) \cap \mathbb{X}_i)$, we have

$$F_j(F_i^{(n-1)}(F(K) \cap \mathbb{X}_i)) = F_j(F_i^{(n-1)}(F(K) \cap \mathbb{X}_i) \cap \mathbb{X}_i) \subset A \cup F_i(F_i^{(n-1)}(F(K) \cap \mathbb{X}_i) \cap \mathbb{X}_i)$$
$$= A \cup F_i^{(n)}(F(K) \cap \mathbb{X}_i).$$

Therefore (6) holds for $n + 1$ and all $K \in \mathcal{K}(\mathbb{X})$.

Finally, since $F(K) \cap \mathbb{X}_j$ is nonempty, according to (2) we have that $(F_i^{(n)}(F(K) \cap \mathbb{X}_i))$ converges to A_i for $i = 1, \dots, k$, and also that $A \cup (F_i^{(n)}(F(K) \cap \mathbb{X}_i))$ converges to A. Thus conditions (5) and (6) imply that $F^{(n)}(K) \to A$. □

We finish this section with the following straightforward observation which will be used throughout the rest of the paper.

Proposition 11 *Let F be an IFS on $\mathbb{X}$ and $\mathbb{Y} \subset \mathbb{X}$ be such that $F(\mathbb{X}) \subset \mathbb{Y}$. Suppose that the restricted IFS $F_{|\mathbb{Y}}$ has an attractor A. Then A is the attractor of F.*

4 *L*-expansive IFSs on $\mathbb{R}$ having an Attractor

Examples of L-expansive IFSs on $\mathbb{R}$ that possess a unique attractor appear in this section.

Example 2 Consider the IFS $F = \{f_0, f_1, f_2\}$ on $\mathbb{R}$ consisting of maps:

$$f_0(x) = \begin{cases} \sqrt{x} & \text{if } x \in [0,1] \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 1 \end{cases} \quad f_1(x) = \begin{cases} 1 - \sqrt{x} & \text{if } x \in [0,1] \\ 1 & \text{if } x < 0 \\ 0 & \text{if } x > 1 \end{cases} \quad f_2(x) = \begin{cases} \frac{1}{16} & \text{if } x < \frac{1}{8} \\ \frac{7}{2}x - \frac{3}{8} & \text{if } \frac{1}{8} \le x \le \frac{1}{4} \\ \frac{1}{2} & \text{if } x > \frac{1}{4} \end{cases}$$

Referring to items (1), (2) and (6) of the Example 1, F is L-expansive on $[0, 1]$. Now observe that

- $|f_0'(x)| < 1$ on $\left(\frac{1}{4}, 1\right]$ and $f_0\left(\left[\frac{1}{4}, 1\right]\right) = \left[\frac{1}{2}, 1\right]$;
- $|f_1'(x)| < 1$ on $\left(\frac{1}{4}, 1\right]$ and $f_1\left(\left[\frac{1}{4}, 1\right]\right) = \left[0, \frac{1}{2}\right]$;

- $F(1) \ni f_2(1) = \frac{1}{2}$;
- $f_0^{(n)}(x) \to 1$ for all $x \in (0, 1]$, and $f_2(0) > 0$.

Hence the assumptions of Propositions 8 and 9 are satisfied, and thus Theorem 7 implies that $[0, 1]$ is the attractor of $F_{|[0,1]}$. Hence, by Proposition 11, the interval $[0, 1]$ is the attractor of F.

Example 3 Consider the IFS $F = \{f_0, f_1, f_2\}$ on $\mathbb{R}$ consisting of maps:

$$f_0(x) = \begin{cases} \sqrt{x} & \text{if } x \in [0,1] \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 1 \end{cases} \qquad f_1(x) = \begin{cases} 1 - \sqrt{1-x} & \text{if } x \in [0,1] \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 1 \end{cases} \qquad f_2(x) = \begin{cases} \frac{1}{16} & \text{if } x < \frac{1}{8} \\ \frac{7}{2}x - \frac{3}{8} & \text{if } \frac{1}{8} \le x \le \frac{1}{4} \\ \frac{1}{2} & \text{if } x > \frac{1}{4} \end{cases}$$

As in the previous example, we can see that F is L-expansive and $[0, 1]$ is its attractor (observe that $|f_1'(x)| < 1$ on $\left[0, \frac{3}{4}\right)$ and $f_1\left(\left[0, \frac{3}{4}\right]\right) = \left[0, \frac{1}{2}\right]$).

Example 4 Consider the IFS $F = \{f_0, f_1, f_2\}$ on $\mathbb{R}$ consisting of maps:

$$f_0(x) = \begin{cases} \sqrt{x} & \text{if } x \in [0,1] \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 1 \end{cases} \qquad f_1(x) = \begin{cases} \sqrt{1-x} & \text{if } x \in [0,1] \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 1 \end{cases} \qquad f_2(x) = \begin{cases} \frac{1}{16} & \text{if } x < \frac{1}{8} \\ \frac{7}{2}x - \frac{3}{8} & \text{if } \frac{1}{8} \le x \le \frac{1}{4} \\ \frac{1}{2} & \text{if } x > \frac{1}{4} \end{cases}$$

Proposition 12 *The IFS of Example 4 is L-expansive, and $[0, 1]$ is its attractor.*

Proof As previously, F is L-expansive and condition (1) of Theorem 7 is satisfied. However, here we cannot use Proposition 8 as $|f_1'(x)| < 1$ on $\left[0, \frac{3}{4}\right)$ and $f_1\left(\left[0, \frac{3}{4}\right]\right) = \left[0, \frac{1}{2}\right]$. More delicate reasoning is required to show that assertion (1) of Theorem 7 is satisfied. As previously, it is sufficient to show that $S := \bigcup_{n \ge 0} F^{(n)}(1)$ is dense in $[0, 1]$. By way of contradiction, assume that there is an open interval $I_1 \subset [0, 1]$, such that $S \cap I_1 = \emptyset$. By the definition of the function f_3, we have $1/2 \in S$, hence $1/2 \notin I_1$. We claim that also $1/\sqrt{2} \notin I_1$. By way of contradiction, assume that $1/\sqrt{2} \in I_1$. Since $1/2 \in S$, also $1/\sqrt{2} = f_0(1/2) \in S$. Therefore $1/\sqrt{2} \in S \cap I_1$, contradicting $S \cap I_1 = \emptyset$.

Since $1/2 \notin I_1$, there are two possibilities. Either $I_1 \subset [0, 1/2]$ or $I_1 \subset [1/2, 1]$. Assume that the first case holds, namely $I_1 \subset [0, 1/2]$. Note that, for any open interval $I \subset [0, 1]$, if $S \cap I = \emptyset$, then $S \cap f_i^{-1}(I) = \emptyset$ for $i = 0, 1$. In particular, if $I_2 := f_1^{-1}(I_1)$, then $S \cap I_2 = \emptyset$ and $I_2 \subset [1/2, 1]$. Therefore, there is no loss of generality in assuming that the second case holds, namely $I_1 \subset [1/2, 1]$.

Now let I_3 be an open interval in $[1/2, 1]$ whose length $|I_3|$ is maximum among all open intervals $I \subset [1/2, 1]$ such that $S \cap I = \emptyset$. Because $1/\sqrt{2} \notin I_3$, there are two possibilities. Either $I_3 \subset [1/2, 1/\sqrt{2}]$ or $I_3 \subset [1/\sqrt{2}, 1]$. In the first case let $I_0 := f_1^{-1}(I_3)$, and in the second case let $I_0 := f_0^{-1}(I_3)$. In either case $I_0 \subset [1/2, 1]$ and $S \cap I_0 = \emptyset$. Because the derivatives satisfy $|(f_0^{-1})'(x)| > 1$ for $x \in (1/\sqrt{2}, 1]$ and $|(f_1^{-1})'(x)| > 1$ for $x \in (1/2, 1/\sqrt{2})$, it must be the case that $|I_0| > |I_3|$, contradicting the maximality of $|I_3|$. □

In the example below, the attractor involves a Cantor set.

Example 5 Let $F = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ be the IFS on the interval $[0,1]$ consisting of the six functions given by

$$f_1(x) = \begin{cases} \frac{1}{2}\sqrt{x} & \text{if } x < \frac{1}{4} \\ \frac{1}{3}x + \frac{1}{6} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ \frac{5}{12} & \text{if } x > \frac{3}{4} \end{cases} \qquad f_2(x) = \begin{cases} \frac{1}{4} & \text{if } x < 0 \\ \frac{1}{4} - \frac{1}{2}\sqrt{x} & \text{if } x \in [0, \frac{1}{4}] \\ 0 & \text{if } x > \frac{1}{4} \end{cases}$$

$$f_3(x) = \begin{cases} \frac{1}{64} & \text{if } x < \frac{1}{32} \\ \frac{7}{2}x - \frac{3}{32} & \text{if } \frac{1}{32} \le x \le \frac{1}{16} \\ \frac{1}{8} & \text{if } x > \frac{1}{16} \end{cases} \qquad f_4(x) = \begin{cases} \frac{7}{12} & \text{if } x < \frac{1}{4} \\ \frac{1}{3}x + \frac{1}{2} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ \frac{1}{4}\sqrt{4x-3} + \frac{3}{4} & \text{if } x > \frac{3}{4} \end{cases}$$

$$f_5(x) = \begin{cases} 1 & \text{if } x < \frac{3}{4} \\ 1 - \frac{1}{4}\sqrt{4x-3} & \text{if } x \in [\frac{3}{4}, 1] \\ \frac{3}{4} & \text{if } x > 1 \end{cases} \qquad f_6(x) = \begin{cases} \frac{49}{64} & \text{if } x < \frac{25}{32} \\ \frac{7}{2}x - \frac{63}{32} & \text{if } x \in [\frac{25}{32}, \frac{13}{16}] \\ \frac{7}{8} & \text{if } x > \frac{13}{16} \end{cases}$$

The graphs of the functions in F are shown in Figure 4.

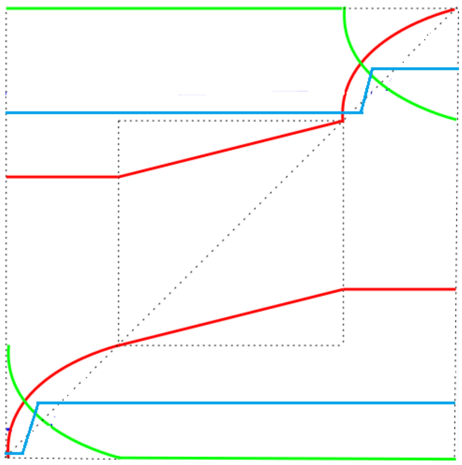

Fig. 2 Graphs of the functions in the IFS of Example 5.

Theorem 13 *The IFS F of Example 5 has attractor $A := [0, \frac{1}{4}] \cup C \cup [\frac{3}{4}, 1]$, where C is the Cantor ternary set built on the interval $[\frac{1}{4}, \frac{3}{4}]$. Moreover, F is L-expansive on A.*

Proof Since all the maps in F are of the form of those in Example 1, the IFS F is L-expansive on A (use items (1), (2), (6) and (7), for $h_1(x) = \frac{1}{4}x$ and $h_2(x) = \frac{1}{4}x + \frac{3}{4}$).

To show that A is the attractor, it is enough to prove that A is the attractor of the IFS G on $[0,1]$ consisting of the restrictions of maps in F to the interval $[0,1]$. Thus, in the rest of the proof we deal with IFS G and use Theorem 10.

Set $I_l = [0, \frac{1}{4}]$, $I_c := [\frac{1}{4}, \frac{3}{4}]$ and $I_r := [\frac{3}{4}, 1]$ and consider the IFSs $F_l = \{f_1, f_2, f_3\}$, $F_r = \{f_4, f_5, f_6\}$ and $F_c = \{f_1, f_4\}$. We will show that the IFSs F_l, F_r, F_c satisfy the assumptions of Theorem 10. We first verify (1). Referring to Example 2 and Proposition 5, we have the following:

(L) The interval $[0, \frac{1}{4}]$ is the attractor of $\overline{F}_l$ (use $h(x) = \frac{1}{4}x$).
(R) The interval $[\frac{3}{4}, 1]$ is the attractor of $\overline{F}_r$ (use $h(x) = \frac{1}{4}x + \frac{3}{4}$).
(C) The Cantor set C is the attractor of $\overline{F}_c$ (use $h(x) = \frac{1}{2}x + \frac{1}{4}$).

Hence (1) is satisfied.

To verify (2) note that:

(L) For every $x \in [0, 1]$ we have $F_l(x) \cap I_l \neq \emptyset$ as $f_3([0, 1]) \subset I_l$.
(R) For every $x \in [0, 1]$ we have $F_r(x) \cap I_r \neq \emptyset$ as $f_6([0, 1]) \subset I_r$.
(C) For every $x \in [0, 1]$ we have $F_c(x) \cap I_c \neq \emptyset$ as $f_1([\frac{1}{4}, 1]) \subset I_c$ and $f_4([0, \frac{3}{4}]) \subset I_c$.

To show (3), take any nonempty and compact $K \subset [0, 1]$. We have

$$\begin{aligned}
F_l(I_r \cap K) &\subset \left[0, \frac{1}{4}\right] \cup \left\{\frac{5}{12}\right\} \subset A, \\
F_l(I_c \cap K) &\subset \left[0, \frac{3}{4}\right] \cup f_1(I_c \cap K) \subset A \cup F_c(I_c \cap K), \\
F_r(I_l \cap K) &\subset \left[\frac{3}{4}, 1\right] \cup \left\{\frac{7}{12}\right\} \subset A, \\
F_r(I_c \cap K) &\subset \left[\frac{3}{4}, 1\right] \cup f_4(I_c \cap K) \subset A \cup F_c(I_c \cap K), \\
F_c(I_l \cap K) &\subset \left[0, \frac{1}{4}\right] \cup \left\{\frac{7}{12}\right\} \subset A, \\
F_c(I_r \cap K) &\subset \left[\frac{3}{4}, 1\right] \cup \left\{\frac{5}{12}\right\} \subset A.
\end{aligned}$$

□

5 L-expansive IFSs on $\mathbb{R}^2$ having an attractor

We begin with a lemma used in the proof of Theorem 15 below.

Lemma 14 ([20], Example 3.3) *Let the IFS $G = \{g_1, g_2\}$ on the unit circle $\mathbb{S}^1$ centered at the origin in $\mathbb{R}^2$ be defined as follows in terms of the polar angle $\theta \in [0, 2\pi)$:*

$$g_1(\theta) = 2\theta, \qquad g_2(\theta) = 2\theta + \alpha,$$

where α/π is irrational. Then G is L-expansive and the attractor of G is $\mathbb{S}^1$.

The Möbius transformation $M(z) = \frac{2z+1}{z+2}$ on the complex plane takes the unit disk D centered at the origin onto itself and has attracting fixed point 1 and repelling fixed point -1 (see [9, Chap. 4.3]). Define

$$h_0(z) = \begin{cases} M(z) & \text{if } z \in D \\ M\left(\frac{z}{|z|}\right) & \text{if } z \notin D. \end{cases}$$

Then -1 is a partially repelling fixed point of h_0. Additionally, define the following four functions on $\mathbb{R}^2$ in terms of polar coordinates (r, θ) as follows:

$$h_1(r,\theta) = \begin{cases} (2r,\theta) & \text{if } r < \frac{1}{4} \\ (\frac{1}{2},\theta) & \text{if } r \geq \frac{1}{4} \end{cases} \qquad h_2(r,\theta) = \begin{cases} (1-\sqrt{1-r},2\theta) & \text{if } 0 \leq r \leq 1 \\ (1,2\theta) & \text{if } r > 1 \end{cases}$$

$$h_3(r,\theta) = \begin{cases} (\sqrt{r},2\theta+\alpha) & \text{if } 0 \leq r \leq 1 \\ (1,2\theta+\alpha) & \text{if } r > 1 \end{cases} \qquad h_4(r,\theta) = \begin{cases} (\sqrt{r},0) & \text{if } 0 \leq r \leq 1 \\ (1,0) & \text{if } r > 1 \end{cases}$$

Theorem 15 *The IFS $H = \{h_0, h_1, h_2, h_3, h_4\}$ with functions as defined above is L-expansive on its attractor D.*

Proof The function h_0 is L-expansive by Proposition 3, since -1 is a partially repelling fixed point. That h_1 and h_4 are L-expansive on the interval from $(0,0)$ to $(1,0)$ follows from Example 1. Therefore h_1 and h_4 are L-expansive on D. The proof that h_2 and h_3 are L-expansive on the circle $\mathbb{S}^1$ follows from Lemma 14. Therefore h_2 and h_3 are L-expansive on D.

Theorem 7 with $x_0 = (1,0)$ (polar coordinates) will be used to prove that D is the attractor of H restricted to D. Using Proposition 9 we see that condition (1) of Theorem 7 holds.

It remains to show that $H^{(n)}(x_0) \to D$, where $x_0 = (1,0)$. Because $x_0 \in H(x_0)$, we have $H^{(n)}(x_0) \subset H^{n+1}(x_0)$ for all $n \in \mathbb{N}$. By Lemma 6 it is sufficient to show that $\bigcup_{n\geq 1} H^{(n)}(x_0)$ is dense in D.

Consider the IFS $\widehat{H} := \{\widehat{h}_0, \widehat{h}_1, \widehat{h}_2, \widehat{h}_4\}$, where the four functions $\widehat{h}_0, \widehat{h}_1, \widehat{h}_2, \widehat{h}_4$ are h_0, h_1, h_2, h_4 with domain restricted to the interval $[0,1]$ on the x-axis. Using the IFS from Example 3 (with functions h_2, h_4 playing the role of f_0, f_1) we get that $\bigcup_{n\geq 1} \widehat{H}^{(n)}(x_0)$ is dense in $[0,1]$. Therefore if $H_4 := \{h_0, h_1, h_2, h_4\}$, then $\bigcup_{n\geq 1} H_4^{(n)}(x_0)$ is dense in $[0,1]$. This implies that $h_3\big(\bigcup_{n\geq 1} H_4^{(n)}(x_0)\big)$ is dense in the ray from the origin to $(1,\alpha)$.

By Lemma 14 we know that $\bigcup_{n\geq 1} G^{(m)}(x_0)$ is dense in $\mathbb{S}^1$, where G is the IFS in Lemma 14. With h_2, h_3 playing the role of g_1, g_2 in Lemma 14, we conclude that if $H_2 = \{h_2, h_3\}$, then $\bigcup_{m\geq 1} H_2^{(m)}([0,1])$ is dense in D. Therefore, using continuity of maps from H, we have that $\bigcup_{m\geq 1} H_2^{(m)}\big(\bigcup_{n\geq 1} H_4^{(n)}(x_0)\big)$ is dense in D. Since

$$\bigcup_{m\geq 1} H_2^{(m)}\Big(\bigcup_{n\geq 1} H_4^{(n)}(x_0)\Big) \subset \bigcup_{n\geq 1} H^{(n)}(x_0),$$

it must be the case that $H^{(n)}(x_0)$ is dense in D.

We have thus shown that H restricted to D is an L-expansive IFS with attractor D. Hence, by Proposition 11, the disk D is the attractor of H. □

Theorem 16 *If $A \subset \mathbb{R}^2$ is homeomorphic to the closed unit disk, then A is the attractor of an L-expansive IFS on $\mathbb{R}^2$.*

Proof Let D denote the closed unit disk in $\mathbb{R}^2$. By the Jordan-Schoenflies Theorem, there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(D) = A$. By Theorem 15 there is an IFS F' on $\mathbb{R}^2$ which is L-expansive on its attractor D. Let $F = hF'h^{-1} := \{hfh^{-1} : f \in F'\}$. By Lemma 2 and Proposition 5, A is the attractor of F, and F is L-expansive on A. □

By Theorem 16, many fractal sets, the twin dragon for example, are the attractors of L-expansive IFSs. The usual IFS used to generate the twin dragon, however, is not L-expansive. In [6, Theorem 2.1, Theorem 2.2], sufficient conditions are given for an affine digit tile, like the twin dragon, to be a topological disk. Theorem 16 may fail in dimension greater than 2 because the Jordan-Schoenflies Theorem is not true in higher dimensions.

6 *L*-expansive IFSs with attractors on Cartesian products

The metric space results in this section show, in particular, that the examples of IFSs on the real line $\mathbb{R}$ in Section 4 can be lifted to examples on the Euclidean space $\mathbb{R}^s$ of any dimension $s \geq 2$. These results concern the product $\prod_{j=1}^{s} \mathbb{X}_j$ of metric spaces $\mathbb{X}_1, \ldots, \mathbb{X}_j$. The metric on the product space can be any metric that induces the product topology.

Proposition 17 *Let $f_j : \mathbb{X}_j \to \mathbb{X}_j$, $j \in \{1, 2, \ldots, s\}$, be continuous functions. Define $\widehat{f} : \mathbb{X} \to \mathbb{X}$ on the product $\mathbb{X} = \prod_{j=1}^{s} \mathbb{X}_j$ by*

$$\widehat{f}(x_1, \ldots, x_s) := (f_1(x_1), \ldots, f_s(x_s)).$$

If

1. *one of the maps, say f_{j_0}, is L-expansive, and*
2. *each map f_j, $j \neq j_0$, has a fixed point in $\mathbb{X}_j$,*

then the product map $\widehat{f}$ is L-expansive.

Proof Let $z_j = f_j(z_j)$ be a fixed point of f_j for $j \neq j_0$, as warranted by (2). By way of contradiction, suppose that $\widehat{f}$ is non-expansive under some metric d on $\mathbb{X}$ equivalent to the initial metric on $\mathbb{X}$. In particular,

$$\begin{aligned} d\big((z_1, \ldots z_{j_0-1}, f_{j_0}(x), z_{j_0+1} \ldots, z_s), (z_1, \ldots, z_{j_0-1}, f_{j_0}(y), z_{j_0+1}, \ldots, z_s)\big) &= \\ d\big(\widehat{f}(z_1, \ldots, z_{j_0-1}, x, z_{j_0+1}, \ldots, z_s), \widehat{f}(z_1, \ldots, z_{j_0-1}, x, z_{j_0+1}, \ldots, z_s)\big) &\leq \\ d\big((z_1, \ldots, z_{j_0-1}, x, z_{j_0+1}, \ldots, z_s), (z_1, \ldots, z_{j_0-1}, y, z_{j_0+1}, \ldots, z_s)\big) & \end{aligned}$$

for all $x, y \in \mathbb{X}_{j_0}$. Thus f_{j_0} is non-expansive with respect to the following metric ρ, which is equivalent to the original metric in $\mathbb{X}_{j_0}$:

$$\rho(x, y) := d((z_1, \ldots, x, \ldots, z_s), (z_1, \ldots, y, \ldots, z_s))$$

for $x, y \in \mathbb{X}_{j_0}$. This contradicts (1). □

Theorem 18 (Cartesian product of IFSs) *Let $F_j = \{f_{1,j}, \ldots, f_{N_j,j}\}$, $j \in \{1, \ldots, s\}$, $s \geq 2$, be IFSs on metric spaces $\mathbb{X}_j$ with attractors $A_j \subset \mathbb{X}_j$. If*

$$F := \{\widehat{f}_{i_1,\ldots,i_s} : (i_1, \ldots, i_s) \in \prod_{j=1}^{s} \{1, \ldots, N_j\}\}$$

is an IFS on the product $\mathbb{X}$ of the $\mathbb{X}_j$'s, defined as follows:

$$f_{i_1,\ldots,i_s}(x_1, \ldots, x_s) := (f_{i_1,1}(x_1), \ldots, f_{i_s,s}(x_s))$$

for all $(x_1, \ldots, x_s) \in \mathbb{X}$, then $\prod_{j=1}^{s} A_j \subset \mathbb{X}$ is the attractor of F.

Proof Fix $K \in \mathcal{K}(\mathbb{X})$, $n \in \mathbb{N}$, and choose an arbitrary $(y_1, \ldots, y_s) \in K$. Then

$$\prod_{j=1}^{s} F_j^{(n)}(y_j) = F^{(n)}(y_1, \ldots, y_s) \subset F^{(n)}(K) \subset F^{(n)}\left(\prod_{j=1}^{s} \pi_j(K)\right) = \prod_{j=1}^{s} F_j^{(n)}(\pi_j(K)),$$

where π_j is the projection from $\mathbb{X}$ to $\mathbb{X}_j$. Since $F_j^{(n)}(y_j) \to A_j$ and $F_j^{(n)}(\pi_j(K)) \to A_j$, an application of a squeezed sequence argument, supported by standard properties of the Hausdorff limit (see [15]), yields $F^{(n)}(K) \to \prod_{j=1}^{s} A_j$. □

A more general version of the above theorem was established recently in [19]. As immediate corollaries of Proposition 17 and Theorem 18, we obtain the following results.

Corollary 19 *Let $F = \{f_1, \ldots, f_N\}$ be an L-expansive IFS on a metric space $\mathbb{X}$ with attractor $A \subset \mathbb{X}$. If $z_2, \ldots, z_s \in \mathbb{X}$ are given points, and $\widetilde{F} := \{\widetilde{f}_1, \ldots, \widetilde{f}_N\}$ is the IFS on the product space $\mathbb{X}^s$, $s \geq 2$, defined as follows:*

$$\widetilde{f}_i(x_1, \ldots, x_s) := (f_i(x_1), z_2, \ldots, z_s)$$

for all $(x_1, \ldots, x_s) \in \mathbb{X}^s$, then $\widetilde{F}$ is L-expansive and $A \times \prod_{j=2}^{s}\{z_j\} \subset \mathbb{X}^s$ is the attractor of $\widetilde{F}$.

Corollary 20 *Let $F = \{f_i : i \in I\}$ be an L-expansive IFS on a metric space $\mathbb{X}$ with attractor $A \subset \mathbb{X}$ so that each map f_i has a fixed point. If $\widehat{F} := \{\widehat{f}_{i_1,\ldots,i_s} : (i_1, \ldots, i_s) \in I^s\}$ is an IFS on the product space $\mathbb{X}^s$ defined as follows:*

$$\widehat{f}_{i_1,\ldots,i_s}(x_1, \ldots, x_s) := (f_{i_1}(x_1), \ldots, f_{i_s}(x_s))$$

for all $(x_1, \ldots, x_s) \in \mathbb{X}^s$, then $\widehat{F}$ is L-expansive and $A^s \subset \mathbb{X}^s$ is the attractor of $\widehat{F}$.

Example 6 The lift of Example 5 from $\mathbb{R}$ to $\mathbb{R}^2$, using Corollary 20, is shown in Figure 1.

7 An upper bound on the Lipschitz constant

Each IFS in Sections 4, 5, and 6 is L-expansive, which means that, for any metric d that induces the standard topology on Euclidean space, we have $Lip_d(f) > 1$ for every $f \in F$. According to Proposition 22 below, it may be problematic to do much better in terms of expansiveness.

A function f is a *Lipschitz function* with respect to a metric d if $Lip_d(f) < \infty$. Let $Lip_d(F) := \max\{Lip_d(f) : f \in F\}$, and call an IFS F on a metric space $(\mathbb{X}, d)$ a *Lipschitz IFS* if $Lip_d(F) < \infty$. For a metric d and real number p, we use the notation $d^p(x, y) := \big(d(x, y)\big)^p$.

Lemma 21 (Snowflake transform, cf. ([13], Chap. 4.1, p.89)) *If d is a metric on $\mathbb{X}$, then for every $p \in (0, 1]$, d^p is a metric on $\mathbb{X}$ equivalent to d. Moreover, $(\mathbb{X}, d^p)$ is complete if and only if $(\mathbb{X}, d)$ is complete.*

Proposition 22 *If F is a Lipschitz IFS on a metric space $(\mathbb{X}, d)$, then for every $\varepsilon > 0$, there is an equivalent metric ρ on $\mathbb{X}$ such that $Lip_\rho(F) < 1 + \varepsilon$. Moreover, if d is complete, then ρ is complete.*

Proof Since $Lip_d(F) < \infty$, for any $\varepsilon > 0$, we can take $p \in (0, 1]$ such that $\big(Lip_d(F)\big)^p \leq (1 + \varepsilon)$. For every $f \in F$ and $x, y \in \mathbb{X}$, we have

$$d^p(f(x), f(y)) \leq (Lip_d(F)\, d(x, y))^p = (Lip_d(F))^p\, d^p(x, y) \leq (1 + \varepsilon)\, d^p(x, y).$$

Hence we can take $\rho = d^p$. □

The hypothesis of Proposition 22 is that F is a Lipschitz IFS. This motivates the following question.

Question 2 Let F be an IFS on a metric space $(\mathbb{X}, d)$. Under what conditions on $(\mathbb{X}, d)$ does there exist an equivalent metric with respect to which F is a Lipschitz IFS?

For some results related to Question 2 see [12, Chap. 3.2] and Remark 2. It is not clear to us, however, that standard remetrization techniques for turning continuous maps into Lipschitz maps could be adapted to answer Question 2.

Theorem 23 is a partial answer to Question 2. It provides conditions on a single function f on $[0, 1]$ sufficient for f to be Lipschitz with respect to some metric equivalent to the standard metric. In particular, it shows that the function $x \mapsto \sqrt{x}$ on $[0, 1]$ is Lipschitz.

Theorem 23 *Let $f : [0, 1] \to [0, 1]$ be a map that satisfies the following:*

- *f is a strictly increasing homeomorhism with $f(0) = 0$, and*
- *there is an $a \in (0, 1)$ such that $f(x) > x$ for all $x \in (0, a]$ and f is continuously differentiable on $[f^{-1}(a), 1]$.*

Then there is a metric d equivalent to the standard metric on $[0, 1]$ such that f is Lipschitz.

Proof Let f be a function that satisfies the conditions of the theorem. For $n \geq 0$, define $a_n = f^{(-n)}(a)$ and $I_n = (a_{n+1}, a_n]$, where $f^{(-n)}$ denotes the n-fold composition of f^{-1} with itself. Note that $a_n \to 0$ as $n \to \infty$. For $x \in (0, a]$, let n_x be the unique integer such that $x \in I_{n_x}$. Note that $n_{f(x)} = n_x - 1$ for $x \in (0, a_1]$. For $x \in [0, a]$ define

$$g(x) = \begin{cases} \frac{1}{2^{n_x}} \Big(f^{(n_x)}(x) + (a_0 - 2a_1) \Big) & \text{if } x \in (0, a] \\ 0 & \text{if } x = 0. \end{cases}$$

It is routine to check that $g(x)$ is a continuous strictly increasing function on $[0, a]$. For all $x, y \in [0, a]$, define

$$d_0(x, y) := |g(x) - g(y)|.$$

Then d_0 is a metric on $[0, a]$ equivalent to the standard metric and such that $d_0(x, y) = d_0(x, z) + d_0(z, y)$ for $x \leq z \leq y$. We first show that f is a Lipschitz function, with $Lip_{d_0}(f) = 2$, on the interval $[0, a_1]$. Assume that $a_1 \geq x \geq y > 0$. Then

$$\begin{aligned} d_0(f(x), f(y)) &= g(f(x)) - g(f(y)) \\ &= \frac{1}{2^{n_{f(x)}}} \Big(f^{(n_{f(x)}+1)}(x) + (a_0 - 2a_1) \Big) - \frac{1}{2^{n_{f(y)}}} \Big(f^{(n_{f(y)}+1)}(y) + (a_0 - 2a_1) \Big) \\ &= \frac{1}{2^{n_x - 1}} \Big(f^{(n_x)}(x) + (a_0 - 2a_1) \Big) - \frac{1}{2^{n_y - 1}} \Big(f^{(n_y)}(y) + (a_0 - 2a_1) \Big) \\ &\leq 2 \left(\frac{1}{2^{n_x}} \big(f^{(n_x)}(x) + (a_0 - 2a_1) \big) - \frac{1}{2^{n_y}} \big(f^{(n_y)}(y) + (a_0 - 2a_1) \big) \right) \\ &= 2 \big(g(x) - g(y) \big) \\ &= 2\, d_0(x, y). \end{aligned}$$

By passing to the limit as $y \to 0$ in the inequality obtained above, we also get that $d_0(f(x), f(0)) \leq 2d_0(x, 0)$ for $x \leq a_1$. Define d on $[0, 1]$ by

$$d(x, y) = \begin{cases} d_0(x, y) & \text{if } x, y \leq a \\ |x - y| & \text{if } x, y \geq a \\ (y - a) + d_0(a, x) & \text{if } x < a < y \\ (x - a) + d_0(a, y) & \text{if } y < a < x \end{cases}$$

It is routine to verify that d is a metric on $[0, 1]$ equivalent to the standard metric. That f is continuously differentiable on $[a_1, 1]$ implies that f is Lipschitz on $[a_1, 1]$ with respect to the metric d. Therefore f is Lipschitz with respect to the metric d on $[0, 1]$. □

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References

[1] K. T. Alligood, T. D. Sauer, J. A. Yorke, *Chaos. An Introduction to Dynamical Systems*, Springer, New York, 1996.

[2] N. Aoki, K. Hiraide, *Topological Theory of Dynamical Systems. Recent Advances*, North-Holland, Amsterdam, 1994.

[3] R. Atkins, M. F. Barnsley, D. C. Wilson, A. Vince, A characterization of point-fibred affine iterated function systems, *Topol. Proc.* **38** (2010), 189–211.

[4] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.* **3** (1922), 133–181.

[5] T. Banakh, W. Kubiś, N. Novosad, M. Nowak, F. Strobin, Contractive function systems, their attractors and metrization, *Topol. Methods Nonlinear Anal.* **46** (2015), no. 2, 1029–1066.

[6] C. Bandt, Y. Wang, Disk-like self-affine tiles in $\mathbb{R}^2$, *Disc. Comp. Geometry* **26** (2001), 591–601.

[7] P. G. Barrientos, F. H. Ghane, D. Malicet, A. Sarizadeh, On the chaos game of iterated function systems, *Topol. Methods Nonlinear Anal.* **49** (2017), no. 1, 105132.

[8] M. F. Barnsley, A. Vince, Real projective iterated function systems, *J. Geometric Analysis* **22** (2012), 1137–1172.

[9] A. F. Beardon, *The Geometry of Discrete Groups*, Springer, New York, 1983.

[10] C. Bessaga, On the converse of the Banach fixed point principle, *Colloq. Math.* **7** (1959), 41–43.

[11] S. Cobzaş, Fixed points and completeness in metric and generalized metric spaces. *J. Math. Sci., New York* **250** (2020), no. 3, 475–535; translation from *Fundam. Prikl. Mat.* **22** (2018), no. 1, 127–215.

[12] S. Cobzaş, R. Miculescu, A. Nicolae, *Lipschitz Functions*, Springer, Cham, 2019.

[13] M. M. Deza, E. Deza, *Encyclopedia of Distances. 4th Edition*, Springer, Berlin, 2016.

[14] W. Gilbert, The division algorithm in complex bases, *Can. Math. Bull.* **39**, (1996) 47–54.

[15] S. Hu, N. S. Papageorgiou, *Handbook of Multivalued Analysis. Volume I: Theory*, Kluwer, Dordrecht, 1997.

[16] J. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747.

[17] A. Kameyama, Distances on topological self-similar sets and the kneading determinants, *J. Math. Kyoto Univ.* **40** (2000), no. 4, 601–672.

[18] B. Kieninger, *Iterated Function Systems on Compact Hausdorff Spaces*, PhD thesis, University of Augsburg, Shaker-Verlag, Aachen, 2002.

[19] K. Leśniak, M. Nowak, Split square and split carpet as examples of non-metrizable IFS attractors, *Banach Cent. Publ.* **125** (2023), 71–79.

[20] K. Leśniak, N. Snigireva, F. Strobin, A. Vince, Transition phenomena for the attractor of an iterated function system, *Nonlinearity*, **35** (2022), no. 1, 555–575.

[21] K. Leśniak, N. Snigireva, F. Strobin, Weakly contractive iterated function systems and beyond: A manual, *J. Difference Equ. Appl.* **26** (2020), no. 8, 1114–1173.

[22] B. M. Levitan, V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, Cambridge, 1982.

[23] R. Miculescu, A. Mihail, Alternative characterization of hyperbolic affine infinite iterated function systems, *J. Math. Anal. Appl.* **407** (2013) 56–68.

[24] R. Miculescu, A. Mihail, On a question of A. Kameyama concerning self-similar metrics, *J. Math. Anal. Appl.* **422** (2015), no. 1, 265–271.

[25] W. L. Reddy, Expanding maps on compact metric spaces, *Topology Appl.* **13** (1982), 327–334.

[26] A. Vince, Möbius iterated function systems, *Trans. Amer. Math. Soc.* **365** (2013), 491–509.

[27] J. de Vries, *Topological Dynamical Systems. An Introduction to the Dynamics of Continuous Mappings*, de Gruyter, Berlin, 2014.