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Distortion Reversal in Aperiodic Tilings

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Abstract
It is proved that homeomorphic images of certain two-dimensional aperiodic tilings, such as Ammann A2 tilings, are recognizable, in both mathematical and practical senses. One implication of the results is that it is possible to search for distorted aperiodic structures in nature, where they may be hiding in plain sight.

Keywords Ammann tiling · Aperiodic tiling · Distortion

Mathematics Subject Classification 52C20 · 05B45

1 Introduction

Herein we prove that homeomorphic images of certain two-dimensional self-similar aperiodic tilings are recognizable, in both mathematical and practical senses. In the practical sense our results say it is possible to search for and recognize distorted aperiodic structures, for example in natural and physical settings.

In this paper we focus on Ammann A2 tilings because of their simplicity. But the ideas are more generally applicable. Our results illustrate that aperiodic tilings can be retrieved from their images distorted by *unknown* homeomorphisms. Figure 1 shows

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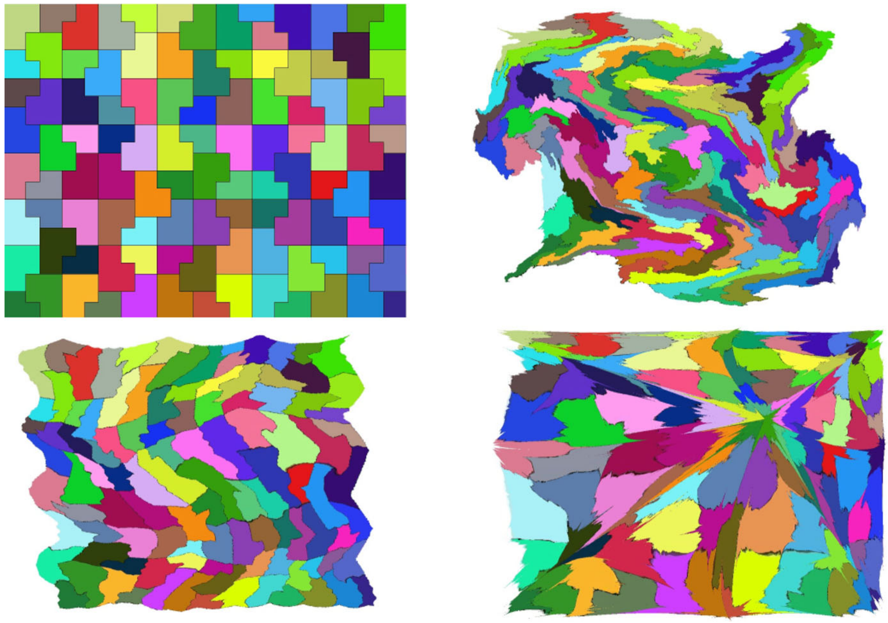

Fig. 1 This illustrates a patch of an Ammann A2 tiling, top left, and three different homeomorphisms of it. Properties of the Ammann A2 tiling may be determined from a distorted image

an undistorted Ammann A2 tiling and three distortions of it. In Fig. 2 an algorithm (without knowledge of the distorting homeomorphism) is applied to the distorted tiling to successively retrieve a patch of the original Ammann A2 tiling. The situation is a bit strange: the unknown distortion may be such as to change the Hausdorff dimension of the boundaries of the tiles in a very rough way, or to transform an Ammann A2 tiling to a tiling by triangles. Nevertheless, the key properties of the tiling may still be discernable.

There is a substantial literature on the occurrence of two-dimensional tilings in physics [10]. Many of these are distortions of tilings by hexagons [9]. Many naturally occurring tilings have an average of six faces per tile, and typical tiles are hexagons. Ammann A2 tiles are hexagonal and comprise among the simplest of self-similar tilings. Despite this, they do not seem to show up in nature. Are they hiding in plain sight?

The main result of this paper addresses the following mathematical question. Suppose that we are given a distorted version T' of an Ammann A2 tiling T. Specifically, there is an unknown (not revealed to us) homeomorphism h of the plane such that $T' = h(T)$. Is it possible to retrieve the undistorted Ammann A2 tiling T from its distorted image T' - without knowledge of h?

An affirmative answer, Theorem 4.6, and its proof appear in Sect. 4. This result depends on a *distortion reversing operator* $\mathcal{Q}$ that is defined in Sect. 3. Section 2 provides background on Ammann A2 tilings. Restricted to a finite region of the plane, which is by necessity the case for all tiling illustrations, the distortion reversal mapping $\mathcal{Q}$ can be easily and efficiently carried out. This is how Fig. 2 was obtained.

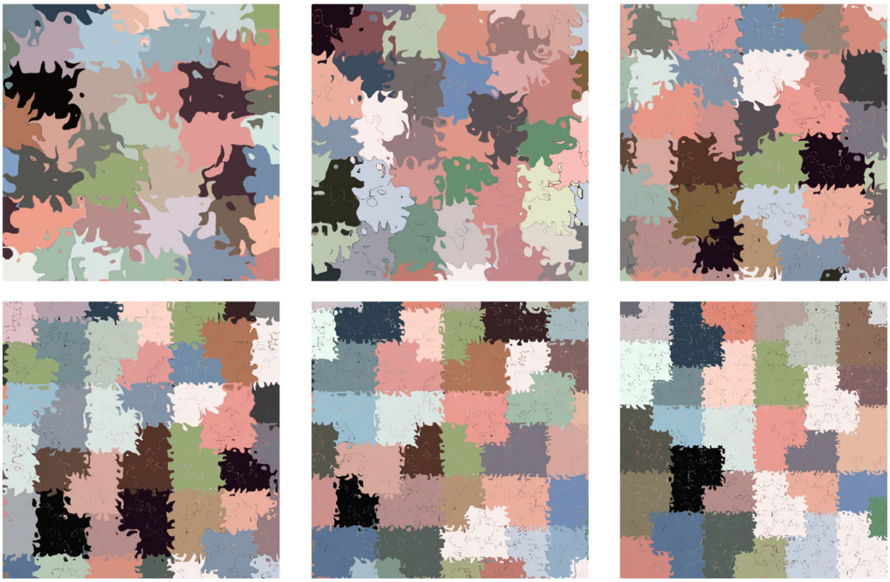

Fig. 2 Successive images, obtained using a distortion reversing operator $\mathcal{Q}$, converge towards patches of an Ammann A2 tiling

2 Ammann A2 Tilings

In this paper a *tile* is defined as a set in the plane homeomorphic to a closed disk. A *tiling* of the plane is a countable set of non-overlapping tiles whose union is $\mathbb{R}^2$. A tiling of a subset of the plane is defined similarly. By *non-overlapping* is meant that the intersection of any two distinct tiles has empty interior. For all tilings in this paper, the intersection of any two distinct tiles is connected (possibly empty). A *patch* of a tiling is a finite number of tiles whose union is a topological disk. Two tilings T and T' are said to be *congruent* if there is an isometry U such that $T' = U(T)$. A finite set of tiles P, called a *prototile set* is *aperiodic* if there exist tilings of the plane by copies of the tiles in P, but every such tiling has no translational symmetry. A tiling T is *repetitive*, also called *quasiperiodic*, if, for every finite patch P of T, there is a real number R such that every ball of radius R contains a patch congruent to P. Two tilings are *locally isomorphic* if any patch in either tiling also appears in the other tiling.

A vertex of a tiling is a point where three or more tiles intersect; an edge is a connected subset of the boundary of a tile that contains no vertices in its interior and that is bounded by two vertices. For any tiling T of the plane, denote by $\Gamma(T)$ the graph whose vertices and edges are the vertices and edges of T. Note that if T and T' are homeomorphic, then $\Gamma(T)$ and $\Gamma(T')$ are isomorphic graphs.

The classical Ammann A2 hexagon G, sometimes referred as the *golden bee*, is depicted in Fig. 3. It is the only polygon, other than any right triangle and any parallelogram with side lengths in the ratio $\sqrt{2} : 1$, that is the non-overlapping union of two smaller similar copies of itself [8]. These two smaller copies are isometric to sG and

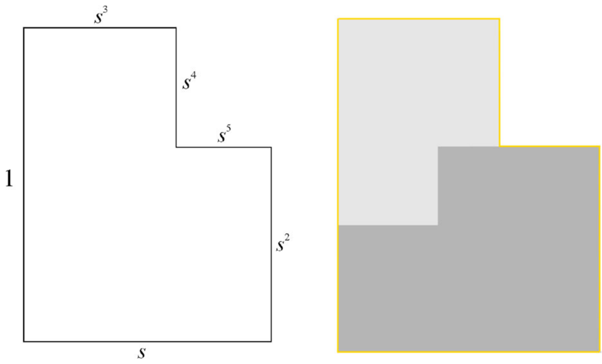

Fig. 3 Golden bee G

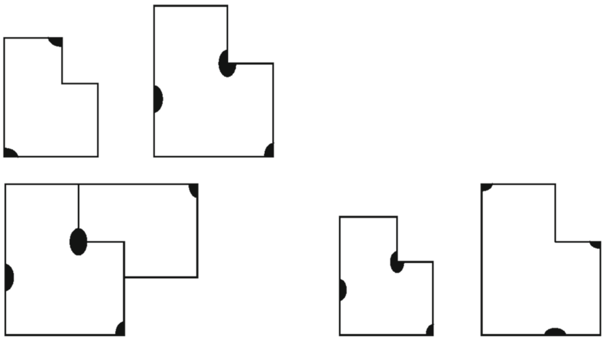

Fig. 4 Two equivalent matching rules for the Amman tilings

s^2G, where $s = 1/\tau$ and τ is the square root of the golden ratio, i.e., τ is the positive real root of $x^4 - x^2 - 1 = 0$.

An *Ammann A2 tiling* is a tiling of the plane by non-overlapping isometric copies of sG and s^2G, which we will refer to as the *large* and *small* tiles, respectively. The tiling must obey matching rules dictated by the elliptical markings on the upper tiles at the left in Fig. 4. A patch of tiles with marked matchings are shown in Fig. 5. Other alternative but equivalent definitions of the Ammann A2 tiling are also possible. For example, the matching rules indicated by markings shown on the right in Fig. 4 [5, p. 551]. Hereafter we will call an Ammann A2 tiling simply an *Ammann tiling*. A portion of an Ammann tiling is shown at the upper left in Fig. 1. There are three (up to congruence) tilings of the first quadrant of $\mathbb{R}^2$ by Ammann tiles [4]. These give rise, via reflections in the x and y-axes, to special Ammann tilings of $\mathbb{R}^2$.

Much has been written on Ammann tilings both in mathematics journals, for example [1, 2, 4, 5] and in recreational sources, notably [6, 7]. The pair of Ammann tiles, like the Penrose tiles, is an aperiodic prototile set. There are uncountable many Ammann tilings. Every Ammann tiling is repetitive and every pair of Ammann tilings are locally isomorphic. A property of every Ammann tiling T that is significant for our results is the following.

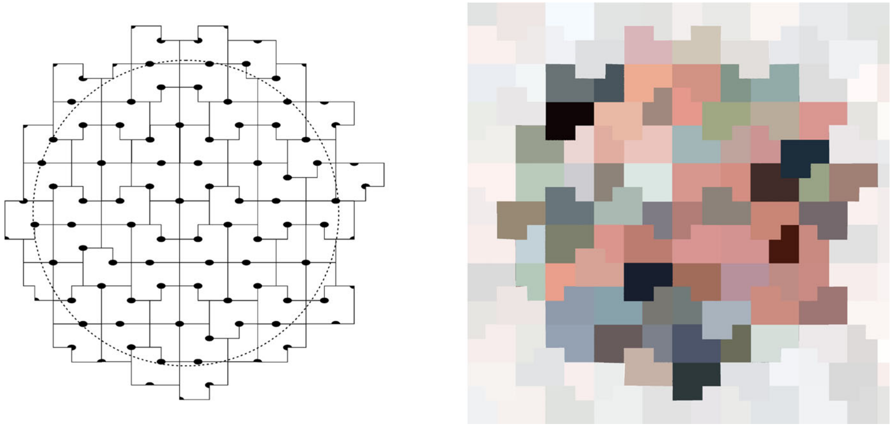

Fig. 5 Left: a circular patch of an Ammann A2 tiling, with markings as in Fig. 4. Right: the same patch, with tiles in various colors, and part of a possible continuation using faded tiles

- For each small tile s in T, there is a *partner*, defined as the unique large tile b in T such that the union of s and b, called the *amalgamation* of s and b, is a hexagon congruent to G. Any two distinct small tiles have distinct partners.

The tiling obtained from T by amalgamating each small tile with its partner is denoted $A(T)$. The scaled tiling $a(T) := sA(T)$ is an Ammann tiling called the *amalgamation* of T.

Proposition 2.1 *If T is an Ammann tiling then its amalgamation $a(T)$ is an Ammann tiling. Conversely, each Ammann tiling is the amalgamation of a unique Ammann tiling $a^{-1}(T)$. That is, a provides a bijection from the set of all Ammann tilings to itself.*

Proof The upper left and right panels of Fig. 4 show two different decorations on the small and large tiles, call them the $D1$ and the $D2$ decorations, respectively. Each small tile in $a(T)$ (left tile in the right panel of Fig. 4) is a (scaled by s) large tile in T (right tile in the left panel); and each large tile in $a(T)$ (right tile in the right panel) is the scaled amalgamation of a small and large tile in T (bottom of the left panel). As previously noted, the matching rules dictated by the $D1$ and $D2$ decorations are equivalent in that they both define the Ammann tilings. Therefore if T is an Ammann tiling, then so is $a(T)$. Conversely, let T be an Ammann tiling. Then form the tiling $s^{-1}T$ and split each of the resulting largest tiles (uniquely) into a small tile and a large tile, yielding $a^{-1}(T)$. □

Remark 2.2 Let $T' = h(T)$ be the image of an Ammann tiling T under a homeomorphism h. The small and large tiles in T' are defined to be the images of the small and large tiles in T. If s′ is a small tile in T' corresponding to a small tile s in T, then the partner of s′ is the image under h of the partner of s. Therefore the amalgamation mapping a is meaningful for the distorted Ammann tiling T' using exactly the same definitions as are used for an undistorted Ammann tiling. We employ the notation $a(T')$ for the amalgamation of T'. Note, however, that from the picture alone, say in

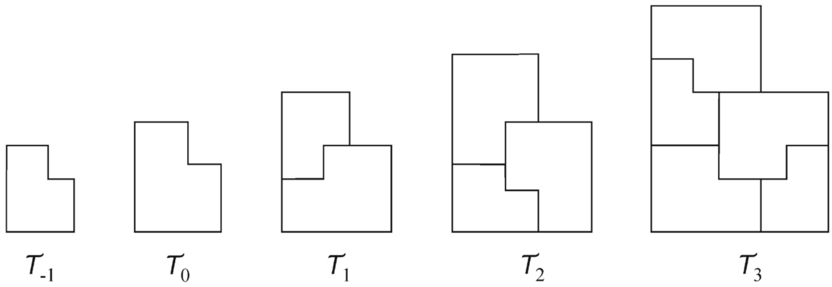

Fig. 6 The patches $\mathcal{T}_n$ for $n = -1, 0, 1, 2, 3$

the distorted Ammann tilings in Fig. 1 ignoring the colors, it is not apparent which tile is the partner of a given tile.

3 The Hierarchy and the Distortion Reversal Mapping

Let A be the amalgamation operator on the Ammann tilings. Given an Ammann tiling T, the sequence of tilings $H(T) := \{T, A(T), A^2(T), \ldots\}$, consisting of Ammann tilings at larger and larger scales, will be referred to as the *hierarchy of* T. If t is any tile at any level of the hierarchy, then t is the non-overlapping union of tiles in T. Denote this patch of T by $\mathcal{T}(T, t)$. Because the amalgamation of a small tile with its large partner is unique, Proposition 3.1 below follows by a routine induction argument.

Proposition 3.1 *If t_1 and t_2 are congruent tiles at any level of the respective hierarchies of Ammann tilings T_1 and T_2, then $\mathcal{T}(T_1, t_1)$ and $\mathcal{T}(T_2, t_2)$ are congruent.*

For an Ammann tiling T and $n \geq 0$, let t_n be any large tile in $A^n(T)$. By Proposition 3.1 the patch $\mathcal{T}(T, t_n)$ is independent of T and t_n. Denote this patch, up to isomorphism, by $\mathcal{T}_n$ and call $\mathcal{T}(T, t_n)$ *the $\mathcal{T}_n$-patch corresponding to the tile* $t_n \in A^n(T)$.

Definition 3.2 (*The red tiling*) Let T be a tiling of the plane and $\Gamma = \Gamma(T)$, the associated graph as defined in Sect. 2. A *red graph* $\widehat{\Gamma}$ is constructed as follows. Color red each edge of Γ (and its two incident vertices) that joins two vertices of degree 4. For any red vertex lying on only one red edge, remove the color red from that vertex and remove the incident edge. Let $\widehat{\Gamma}$ be the graph induced by the red edges and vertices, i.e., by removing all edges and vertices not colored red. If each face of $\widehat{\Gamma}$ is homeomorphic to a disk, then let $\widehat{T}$ denote the tiling induced by $\widehat{\Gamma}$. Note that vertices of degree 2 in $\widehat{\Gamma}$ are not vertices of $\widehat{T}$. The red tiling $\widehat{T}$ of an Ammann tiling T is illustrated on the right in Fig. 7. For any region f of $\widehat{T}$ (face of the red graph) its red boundary together with all enclosed original tiles of T is a finite tiling which is denoted $T(f)$.

The following property of Ammann tilings is known as *border forcing*, see for example [1, 3].

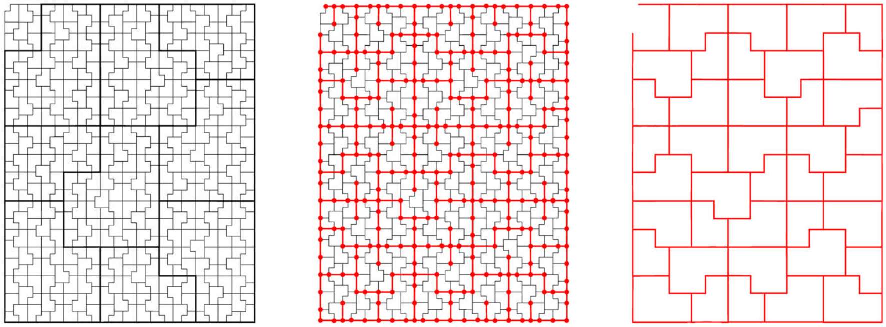

Fig. 7 The left panel shows a patch taken from an Ammann tiling of the plane. It had been subdivided into $\mathcal{T}_6$'s and $\mathcal{T}_7$'s to illustrate (part of) a tiled member of a hierarchy. The middle panel shows the same patch with some of the vertices and edges coloured red, according to the first step in the construction in Definition 3.2. The right panel shows the visible part of the red tiling obtained after the construction in Definition 3.2 is completed

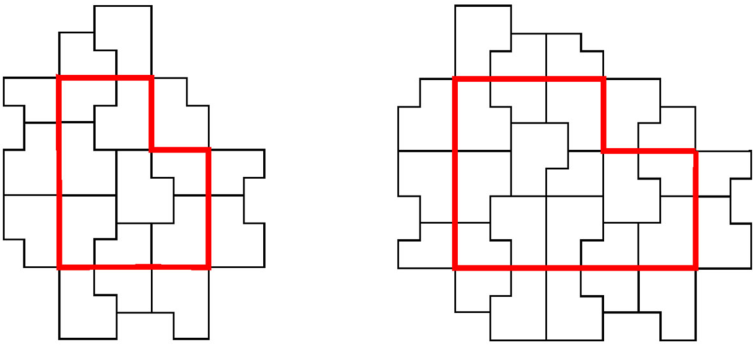

Fig. 8 The forced tiles on the boundaries of any $\mathcal{T}_4$ and $\mathcal{T}_5$. See text

Proposition 3.3 *Let T be an Ammann tiling. For $n \geq 4$, let $\mathcal{T}(T, t_n)$ be the $\mathcal{T}_n$-patch corresponding to a tile $t_n \in A^n(T)$. If $t \in T \backslash \mathcal{T}(T, t_n)$ is a tile that shares an edge with t_n, then t is the refection across part of the boundary of t_n of a tile in $\mathcal{T}(T, t_n)$.*

Proof Using tile markings it can be established that there is a small finite set of neighborhoods of any small tile in any Ammann tiling. See Fig. 9. By expanding and splitting these neighborhoods five times we reveal all ways in which t can meet the boundary of any $\mathcal{T}_4$, up to isometry. We find that t is the reflection across part of the boundary of $\mathcal{T}_4$ of a tile that is contained in $\mathcal{T}_4$, up to isometry. The general case follows by induction, using the facts that a is a bijection and the reflection property is invariant under application of a^{-1}. $\square$

Figure 8 shows the forced positions of the tiles that are adjacent to any $\mathcal{T}_4$ and to any $\mathcal{T}_5$ that appear in any Ammann tiling of the plane. Not only is the tiling of the regions with red boundaries unique, but the surrounding layer of tiles is also unique, or "forced".

Theorem 3.4 *If T is an Ammann tiling, then $\widehat{T} = A^5(T)$.*

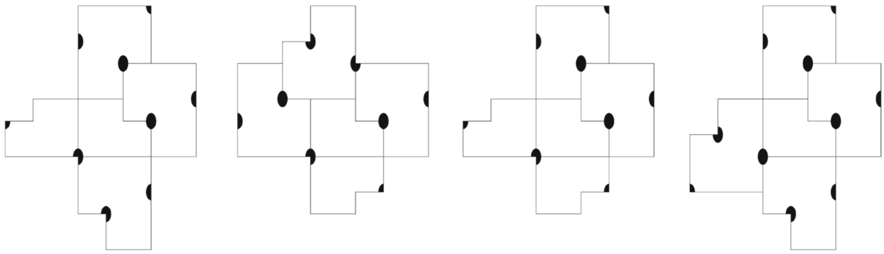

Fig. 9 The only ways in which a small tile can share segments of its boundary with other tiles in any Ammann tiling of the plane, up to isometry

Proof By Proposition 2.1 any Ammann tiling T of the plane by s and b can equivalently be considered as a tiling by $\mathcal{T}_4$'s and $\mathcal{T}_5$'s, according to $T = a^{-5}\widehat{T}$. By Proposition 3.3 each tile that shares a segment of the boundary of an isometric copy of $\mathcal{T}_4$ or $\mathcal{T}_5$ is reflected across a segment of the boundary. This implies that every vertex of the outer boundary of each (isometric copy of) $\mathcal{T}_4$, and also of each $\mathcal{T}_5$, is a meeting point of four edges, including two successive edges of the outer boundary of the $\mathcal{T}_4$ or $\mathcal{T}_5$, respectively. This implies that the red tiling includes the boundaries of all isometric copies of$\mathcal{T}_4$'s and $\mathcal{T}_5$'s that it contains. It is straightfoward to check that the red tiling construction algorithm removes all edges and vertices in the interior of any copies of $\mathcal{T}_4$ and of any copies of $\mathcal{T}_5$. Since $a^{-5}T = \widehat{T}$ can be treated as a unique union of nonoverlapping copies of $\mathcal{T}_4$'s and $\mathcal{T}_5$'s, again by Proposition 2.1, the red tiling has to be $s^{-5}a^5\widehat{T} = A^5(T)$. □

In Fig. 10, the red tiling construction is applied to a distorted Ammann tiling.

Definition 3.5 If T is any tiling such that $\widehat{T}$ is also a tiling, then the *distortion reversal operator* $\mathcal{Q}$ is defined by

$$\mathcal{Q}(T) = s^5\,\widehat{T}.$$

Corollary 3.6 *If T' is the homeomorphic image of an Ammann tiling, then $\mathcal{Q}(T') = a^5(T')$.*

Proof By Theorem 3.4 we have $\mathcal{Q}(T) = s^5\,\widehat{T} = s^5A^5(T) = a^5(T)$. As noted in Sect. 2, the graph $\Gamma(T')$ is isomorphic to $\Gamma(T)$. Therefore, $\widehat{T'}$ is homeomorphic to $\widehat{T}$ and $\mathcal{Q}(T') = s^5\widehat{T'} = s^5A^5(T') = a^5(T')$. □

4 Distortion Reversal

Definition 4.1 A homeomorphism h of the plane has *bounded distortion* if there is a constant C such that

$$|x - h(x)| \leq C$$

for all $x \in \mathbb{R}^2$.

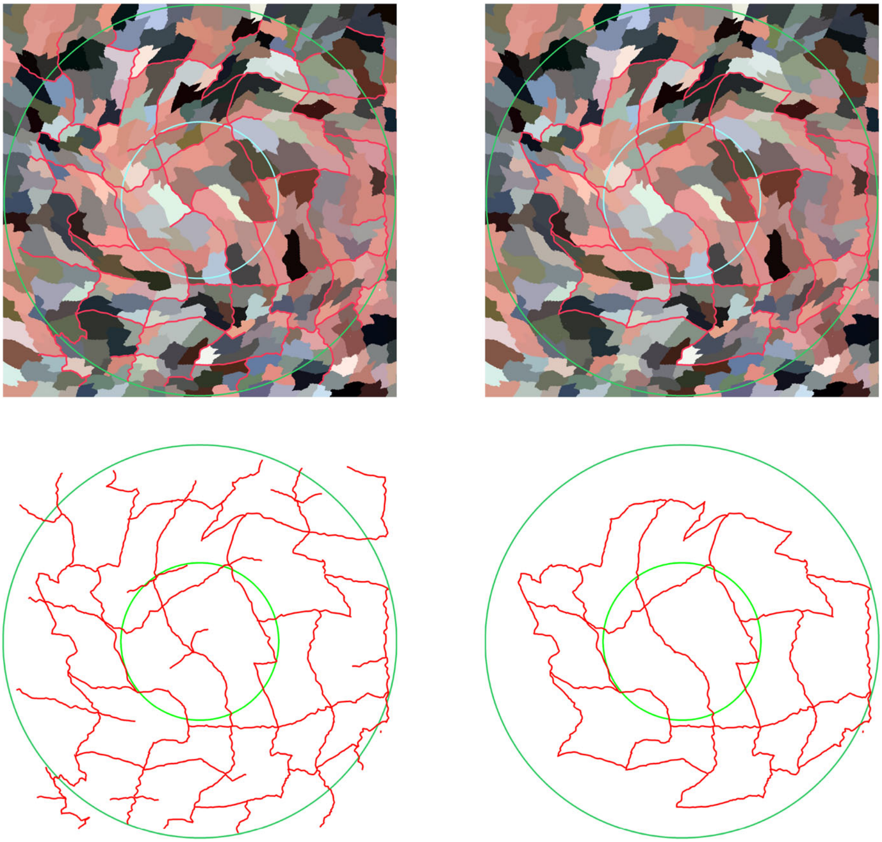

Fig. 10 Left: a distorted Ammann tiling with edges that connect vertices of degree 4 marked in red. Right: the same tiling after removal of red edges that do not lie on red cycles

If C is the constant in Definition 4.1 of a homeomorphism h of bounded distortion, then h is said to have *bounded distortion* C.

Lemma 4.2 *Let h be a homeomorphism of the plane of bounded distortion C. If ϕ_r is a similarity transformation with scaling ratio r, then*

$$|x - \phi^{-1} \circ h \circ \phi(x)| \le \frac{1}{r}\, C$$

for all $x \in \mathbb{R}^2$.

Proof Let $x \in \mathbb{R}^2,\ y = \phi_r(x),\ z = h(y),\ w = \phi_r^{-1}(z)$. Then

$$\begin{aligned} |x - \phi_r^{-1} \circ h \circ \phi_r(x)| &= |x - w| = |\phi_r^{-1}(y) - \phi_r^{-1}(z)| = \frac{1}{r}\, |y - z| \\ &= \frac{1}{r}\, |y - h(y)| \le \frac{1}{r}\, C \end{aligned}$$

for all $x \in \mathbb{R}^2$. $\square$

Definition 4.3 For any two tilings T_1 and T_2 of the plane, a distances function is given by

$$d(T_1, T_2) = d_H(\partial T_1, \partial T_2),$$

where ∂T is the union of the boundaries of the tiles in T and d_H is the Hausdorff distance. Intuitively, the d-distance between two tilings is small if the tilings are almost the same over the entire plane.

Corollary 4.4 below follows immediately from Lemma 4.2.

Corollary 4.4 *If T is a tiling of the plane, h a homeomorphism of bounded distortion C, and ϕ_r is a similarity transformation with scaling ratio r, then*

$$d\big(\phi_r^{-1} \circ h \circ \phi_r)(T), T\big) \leq \frac{1}{r} C.$$

Theorem 4.5 *Let T be an Ammann tiling, h a bounded distortion homeomorphism of the plane, and $T' = h(T)$. Then*

$$\lim_{k\to\infty} d\big(a^k(T'), a^k(T)\big) = 0.$$

Proof Let $H(T) = \{T = T_0, T_1, T_2, \ldots\}$ be the hierarchy of T and $H(T') = \{T' = T'_0, T'_1, T'_2, \ldots\}$ the hierarchy of T'. Then

$$\begin{aligned}
a^k(T) &= s^k T_k, \\
a^k(T') &= s^k T'_k = s^k h(T_k) = s^k h(s^{-k} a^k(T)) = (s^k \circ h \circ s^{-k})\big(a^k(T)\big), \\
&= (\tau^{-k} \circ h \circ \tau^k)\big(a^k(T)\big),
\end{aligned}$$

where we recall that $\tau = s^{-1}$. By Corollary 4.4 we have

$$d\big(a^k(T'), a^k(T)\big) = d\big((\tau^{-k} \circ h \circ \tau^k)\big(a^k(T)\big), a^k(T)\big) \leq \frac{C}{\tau^k}.$$

From this it follows that

$$\lim_{k\to\infty} d\big(a^k(T'), a^k(T)\big) = 0.$$

$\square$

Theorem 4.6 *Let T be an Ammann tiling, h a homeomorphism of the plane, and $T' = h(T)$ the distorted image of T. If h has bounded distortion, then there exists a sequence $(T_k)_{k\geq 0}$ of Ammann tilings such that*

$$\lim_{k\to\infty} d\big(\mathcal{Q}^k(T'), T_k\big) = 0.$$

Proof of Theorem 4.6 Let $T_k = a^{5k}(T)$. From Corollary 3.6 and Theorem 4.5, it immediately follows that

$$\lim_{k\to\infty} d\big(\mathcal{Q}^k(T'), T_k\big) = \lim_{k\to\infty} d\big(a^{5k}(T'), a^{5k}(T)\big) = 0.$$

$\square$

Theorem 4.6 does not state that $\mathcal{Q}^k(T') \to T$. It does imply, however, that an arbitrarily large finite patch of the tiling $\mathcal{Q}^k(T')$ becomes arbitrarily close to a patch P_k of T as $k \to \infty$. This follows from the local isomorphism property of Ammann tilings (see Sect. 2 for a definition of local isomorphism).

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