A COMBINATORIAL APPROACH TO SELF-REP TILINGS OF EUCLIDEAN SPACE

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Abstract. A graph directed iterated function system, originally introduced to construct deterministic fractals, serves here as a framework for a tiling theory. The tiling space in this paper is the closure, in an appropriate metric space, of the set of self-similar tilings introduced by W. Thurston. The method applies to most of the self-referential tiles that appear in the literature.

1. Introduction

The tilings in this paper are tilings of Euclidean space by isometric copies of a finite set of prototiles, in particular those that have global self-replicating properties. The mathematical investigation of such tilings has a long history; consider, for example, the prescient 1619 "monster" tiling of Johannes Kepler in Figure 1. Research has been particularly robust since the discovery of the Penrose tilings [13] in 1974 and quasicrystals in 1984. Although fascinating in their own right, these tilings have connections to a myriad of areas such as symbolic dynamics, β-numeration, and toral automorphism.

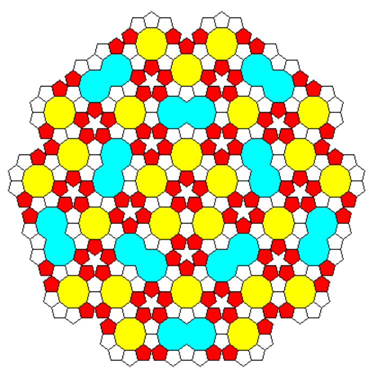

Figure 1. Kelpler's monster

A dynamical systems approach to tiling has been actively investigated over the past three decades, employing the action on the set of tilings by translations; see [14, 17] and references therein. This paper instead takes a combinatorial perspective. It uses a graph directed iterated function system (GIFS) as the basic tool rather than notions like inflation-deflation and substitution. GIFSs have have been used previously in the study of tiles and tiling, for example in the study of Rauzy fractals (see Section 9.2) and in the paper [1]. The goal of the current paper is to simplify and extend tiling theory from the GIFS point of view. Because the paper is intended to be self-contained, the presentation is a bit expository.

The concept of a GIFS originated in the construction of deterministic fractals [12] but is also related to the notion of a rep-tile, the term coined by S. Golomb [3] and popularized by Martin

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Gardner in 1963 in Scientific American [4]. A rep-tile was originally defined as a single polygon that can be tiled by smaller similar and congruent copies of itself. If we generalize from a single polygon to a finite prototile set Q such that each tile in Q is, in turn, tiled by smaller similar copies of prototiles in Q, then we are close to the concept of a GIFS (see Proposition 3.1).

Formally, a GIFS $\mathcal{F} = (G, F)$ in this paper consists of a finite directed graph (digraph) G, each edge of which is labeled by a similarity transformation taking Euclidean space $\mathbb{R}^d$ onto itself. The set F is the set of all such edge functions. If each similarity in F is a contraction, then associated with the GIFS is an n-tuple $\mathbf{A} := (A_1, A_2, \ldots, A_n)$ of compact sets, called the attractor of $\mathcal{F}$, where n is the order (number of vertices) of the digraph G. See Definition 3.1 for a formal definition of a GIFS and its attractor. The functions in F explicitly indicate how each attractor component is the union of smaller similar copies of components of $\mathbf{A}$. Figure 2 shows, in the top row, the three attractor components of the last GIFS in Example 9.1. The second row of the figure illustrates that each attractor component is the union of smaller similar copies of the attractor components. In this example, the boundary of each tile is a fractal.

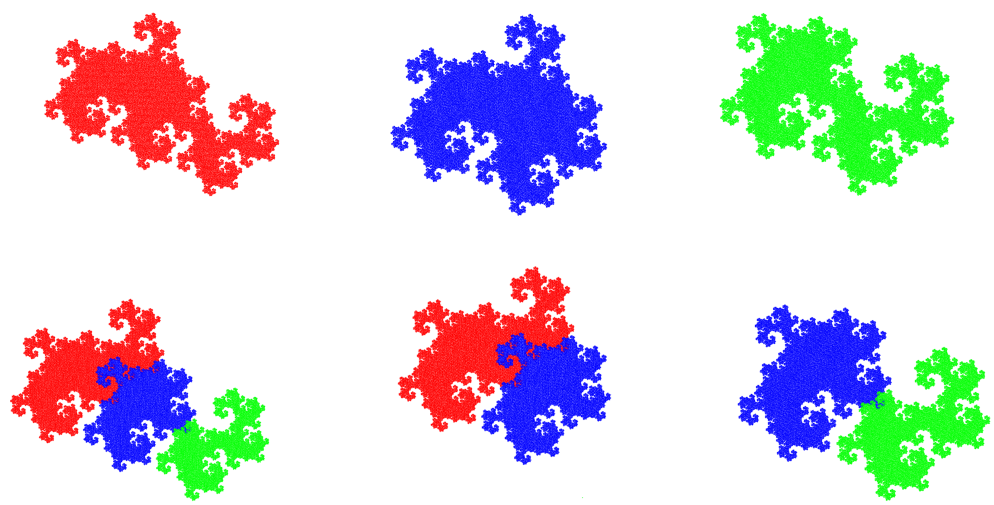

FIGURE 2. Prototile set for the TGIFS of Example 9.1.

If the components of the attractor of a GIFS have nonempty interior, then tilings of $\mathbb{R}^d$ by copies of the attractor components may be possible. The primary concepts in this paper are that of a *tiling-GIFS* (TGIFS) (see Definition 4.1) and the associated TGIFS-tilings of $\mathbb{R}^d$ (see Definition 4.2). More precisely, associated with any TGIFS $\mathcal{F}$ with attractor $(A_1, A_2, \ldots, A_n)$ is

- an uncoutable *parameter space* $\mathcal{P} := \mathcal{P}(\mathcal{F})$,
- a *tiling space* $\mathbb{T} := \mathbb{T}(\mathcal{F})$ consisting of a set of tiling of $\mathbb{R}^d$, each with prototile set $Q(\mathcal{F}) := \{A_1, A_2, \ldots, A_n\}$, and
- a *tiling map* $\mathcal{T} : \mathcal{P} \to \mathbb{T}$ from the parameter space to the tiling space.

Both the parameter space and the tiling space are metric spaces, and the tiling map is continuous with respect to these metrics. Given a TGIFS $\mathcal{F}$, each tiling in $\mathbb{T}(\mathcal{F})$ will be referred to as an *$\mathcal{F}$-tiling*. For each parameter $\theta \in \mathcal{P}$, Definition 4.2 provides a systematic construction of the corresponding tiling $\mathcal{T}(\theta) \in \mathbb{T}$.

A tiling T can be either periodic or non-periodic, periodic if there is a translational symmetry of T, otherwise non-periodic. Figure 3 shows two well known tilings, the twin dragon tiling and the original Rauzy tiling [16]. The twin dragon tiling is periodic and the Rauzy tiling is non-periodic. Both tilings can be obtained from our TGIFS method (see Section 9.2). The two additional fractal TGIFS-tilings that appear in Figure 4 are also non-periodic and will be discussed in Section 9.2.

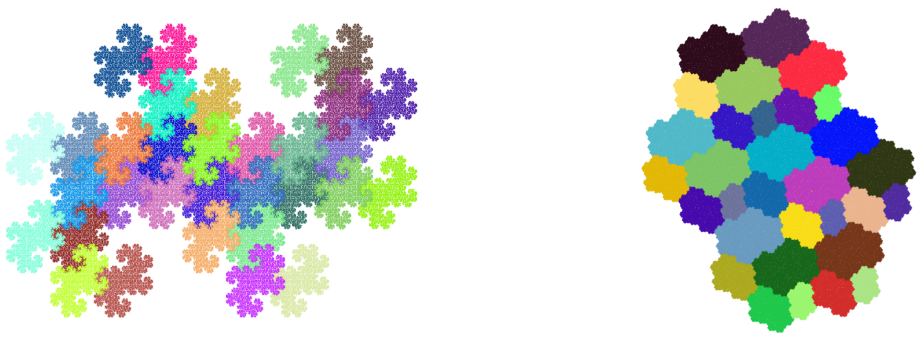

FIGURE 3. Twin Dragon (periodic) and Rauzy (non-periodic) TGIFS-tilings.

The concept of a self-similar tiling was introduced by Thurston [19] in 1989. The main defining property of a *self-similar tiling* T is the existence of an expanding similarity ϕ such that, for each tile $t \in T$, the larger tile $\phi(t)$ is, in turn, tiled by tiles in T. See Definition 6.2 for a precise definition of self-similar in the context of this paper. Not all TGIFS-tilings are self-similar. We prove, however, that

- every self-similar tiling is a TGIFS-tiling (Theorem 6.1), and
- for a given TGIFS $\mathcal{F}$, the set of self-similar tilings in $\mathbb{T}(\mathcal{F})$ is dense in $\mathbb{T}(\mathcal{F})$ (Proposition 8.1).

In a sense, the set of self-similar tilings in $\mathbb{T}$ is analogous to the set of rationals in $\mathbb{R}$; see Proposition 6.1.

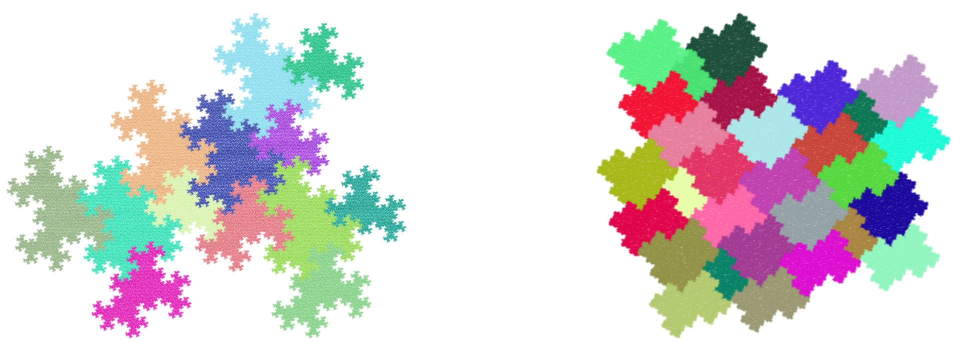

FIGURE 4. TGIFS-tilings from Examples 9.1.

Locally, any two $\mathcal{F}$-tilings look the same; they are locally isomorphic (Definition 4.4). For a given TGIFS $\mathcal{F}$, there may be an uncountable number of $\mathcal{F}$-tilings, but a finite patch, as in Figure 4, gives no clue as to which tiling it is. Global information about $\mathcal{F}$-tilngs is reflected in a certain dynamical system $(\mathbb{T}, \widehat{S})$ (Definition 8.2). For a large class of TGIFS, including most of those that give rise to non-periodic tilings, this dynamical system is topologically conjugate to a subshift of finite type that depends only on the unlabeled digraph of $\mathcal{F}$. This allows for the application of certain dynamical system invariants to the study of the tilings (see Section 8.1).

The subject of this paper is the existence and properties of TGIF-tilings. The next section provides an overview of results.

2. Organization and Results

Section 3.

Basic notions about tiling of Euclidean space, about directed graphs and their adjacency matrices, and about graph directed iterated function systems (GIFS) are discussed in Section 3. Every GIFS (G, F) for which the functions in F are constractions has a unique attractor consisting of a set of compact subsets of $\mathbb{R}^d$, one for each vertex of the digraph G of the GIFS. And every self-replicating set Q as discussed in Section 1 is the attractor of a GIFS.

Section 4.

The subjects of this section are *tiling-GIFSs* (TGIFSs) (Definition 4.1) and their associated TGIFS-tilings. The parameter space $\mathcal{P}(\mathcal{F})$ of a TGIFS $\mathcal{F} = (G, F)$ is defined so that a tiling is assigned to each member of the parameter space, the construction encapsulated in Definition 4.2. The set of prototiles of a TGIFS-tiling is the set of attractor components of the TGIFS. Also associated with $\mathcal{F}$ is the scaling ratio $\lambda(\mathcal{F})$ which is the contraction constant $\lambda(f)$ of all of the similarity transformations in F. For a TGIFS $\mathcal{F}$, the following tiling properties hold.

- The scaling ratio must be $\lambda(\mathcal{F}) = 1/\sqrt[d]{\rho}$, where d is the dimension of the Euclidean space and ρ is the Perron-Frobenius eigenvalue of the adjacency matrix of the digraph G of $\mathcal{F}$ (Proposition 4.1).
- Theorem 4.1 provides a sufficient condition for two $\mathcal{F}$-tilings to be isometric (congruent).
- Every $\mathcal{F}$-tiling is repetitive (quasiperiodic), and every pair of $\mathcal{F}$-tilings are locally isomorphic (Theorems 4.1 and 7.2).
- The proportion of tiles of each prototile type in any $\mathcal{F}$-tiling is given by the coordinates of the normalized left Perron-Frobenius eigenvector of the adjacency matrix of the digraph G of $\mathcal{F}$ (Theorem 4.2).

Section 5.

This section provides motivation for our definition of a TGIFS-tiling. Naturally associated to a GIFS are finite patches of tiles that we call admissible patches. To avoid randomness in the patches, we can restrict the admissible patches to be uniform (Definition 5.4). We prove the following:

- Given a GIFS, if a tiling contains admissible patches of arbitrarily large cardinality, then the GIFS must be commensurable (Definition 5.2 and Theorem 5.1).
- Any admissible patch of a commensurable GIFS is the admissible patch of a corresponding "companion" TGIFS (Proposition 5.2).
- The nested union of uniform patches of a TGIFS $\mathcal{F}$ must be an $\mathcal{F}$-tiling (Theorem 5.2).

Taken together, the above three results informally mean that any tiling that one may reasonable consider self-replicating must be a TGIFS-tiling as defined in Section 4.

Section 6.

The notion of a self-similar tiling is slightly extended from tilings by translates of a set of prototiles as formulated by Thurston and Kenyon [7, 19], to tilings by isometric copies. We show that

- every self-similar tiling is a TGIFS tiling (Theorem 6.1), and
- if a parameter of a TGIFS is eventually periodic (see Section 4.3), then the associated tiling is self-similar (Proposition 6.1).

Section 7.

A hierarchy for a TGIFS tiling T is a sequence of tilings $T_0 = T, T_1, T_2, \dots$ with the property that, for every k, every tile in T_{k+1} is tiled, in turn, by a patch of tiles in T_k. See Definition 7.1 for the complete definition. It is proved in this section that every TGIFS-tiling possess a hierarchy

(Theorem 7.1). The hierarchy is used in the proof that all TGIFS-tilings are quasiperiodic (Theorem 7.2). If, for a given TGIFS $\mathcal{F}$, each $\mathcal{F}$-tiling has a unique hierarchy, then

- every $\mathcal{F}$-tiling is non-periodic (Corollary 7.1) and
- there are an uncountable number of $\mathcal{F}$-tilings (see Corollary 7.2 for the precise statement).

Section 8.

The subject of this section is the tiling dynamical system $(\mathbb{T}, \widehat{S})$ of a TGIFS as mentioned in Section 1. The function $\widehat{S}$ acts on the tiling space $\mathbb{T}$ by taking a tiling $T \in \mathbb{T}$ one level up (after scaling) in its hierarchy (Definition 8.2). Theorem 8.1 states that, for a TGIFS $\mathcal{F}$ satisfying natural assumptions, $(\mathbb{T}, \widehat{S})$ is topologically conjugate to the discrete dynamical system $(\mathcal{P}, S)$, where S is the shift map acting on the parameter space $\mathcal{P}$ (Definition 4.3). The section ends with results on the global structure of TGIFS-tilings (Theorems 8.2 and 8.3), whose proofs rely on dynamical system invariants, in particular on the topology entropy and on the Artin-Mazur zeta function.

Section 9.

Although every contractive GIFS has an attractor, the attractor components may have empty interior, rendering them inadmissible as tiles. This is not an issue in dimenion 1:

- For every strongly connected digraph G, not a directed cycle, there exists a TGIFS on $\mathbb{R}$ whose digraph is G (Theorem 9.1).

In dimensions $d \geq 2$, however, TGIFSs and their tilings are not as plentiful. Examples come from tilings discovered over the past few decades including certain polygonal tilings [2], digit tilings [21], crystallographic tilings [5], Rauzy tilings [16], and sporadic examples like the Penrose tilings. All of these fit into the TGIFS framework. For a given strongly connected digraph G and dimension d, we conjecture (Conjecture 9.1) that the set of TGIFSs is nowhere dense in a space of all contractive GIFSs, which may help explain their scarsity. In Section 9 we show the following.

- If $\mathcal{F}$ is a TGIFS on $\mathbb{R}^d$, $d \geq 2$, with scaling ratio $\lambda := \lambda(\mathcal{F})$, then $1/\lambda$ is a weak-Perron number. (Proposition 9.1).
- For every $\lambda < 1$ such that $1/\lambda$ is a Perron number, there exists a TGIFS on $\mathbb{R}^d$ with scaling ratio λ (Theorem 9.2).

The section concludes with tiling examples based on two algebraic methods that produce TGIFSs, digit sets and an elegant notion of GIFS duality due to Rao, Wen and Yang [15] that produces Rauzy fractals.

3. Graph Directed Iterated Function System and the Attractor

3.1. **Tilings of Euclidean Space.** In this paper, a *tile* is a compact subset of $\mathbb{R}^d$, and a *tiling of a set* $X \subseteq \mathbb{R}^d$ is a set of pairwise non-overlapping tiles whose union is X. *Non-overlapping* means that the intersection of any two distinct tiles has measure zero. Two tilings T and T' are *isometric* or *congruent* if there is an isometry of $\mathbb{R}^d$ taking one onto the other, and this is denoted $T \cong T'$. Two tilings are *equal*, denoted $T = T'$, if they are identical. A *patch* of a tiling T is a finite subset of T. For a finite set Q of tiles, a Q-*tiling* T is a tiling of $\mathbb{R}^d$ in which each tile is congruent to a tile in Q. The set Q is called a *prototile set* for T. A tile t that is congruent to $q \in Q$ will be referred to as a *type* q tile.

3.2. **Directed Graphs and Adjacency Matrices.** Let $G = (V, E)$ be a finite, strongly connected, directed graph (digraph) with vertex set $V = \{1, 2, \dots, n\}$ and edge set E. A digraph G is *strongly connected* if, for any two vertices i and j, there is a directed path from i to j. In this paper, path always means a directed path, and a path can have repeated vertices and/or edges. The digraph G may have loops and/or multiple edges. A strongly connected digraph G

will be called *primitive* if the greatest common divisor of the lengths of all closed paths of G is 1. Equivalently, a strongly connected digraph is primitive if and only if, for k sufficiently large, there is a path of length k joining any two vertices.

For an edge $e = (i, j)$, directed from vertex i to vertex j, the vertex i is denoted e^- and the the vertex j is denoted e^+. Let E_i denote the set of all edges e such that $e^- = i$, i.e., the set of vertices directed out of vertex i.

Associated to a digraph G is its *adjacency matrix* $M := M(G) = (m_{i,j})$, where $m_{i,j}$ is the number of edges from vertex i to vertex j. It is well known that G is strongly connected if and only if the matrix M is irreducible, and G is primitive if and only if M is primitive. A square non-negative matrix M is *primitive* if there is an integer $k \geq 0$ such that all entries of M^k are positive, and M is *irreducible* if for all i, j there is a $k = k(i, j)$ such that $M^k_{i,j} > 0$. Clearly a primitive matrix is irreducible.

3.3. **Graph Directed Iterated Function Systems.** To generalize the notion of a rep-tile as mentioned in Section 1, call a finite multiset Q of tiles a *rep-set* if each tile in Q can be tiled by smaller similar copies of tiles in Q. We allow distinct tiles in Q to be congruent to allow for the same shape to be tiled by similar copies of tiles in Q in different ways. For example, a 2×2 square can be tiled by four 1×1 squares or the 2×2 square can be tiled by two 1×1 squares and a 1×2 rectangle. Proposition 3.1 below shows that a graph directed iterated function captures this notion of a rep-set.

Definition 3.1 (GIFS)**.** A *graph directed iterated function system (GIFS)* on $\mathbb{R}^d$ is a pair $\mathcal{F} = (G, F)$, where $G = (V, E)$ is a strongly connected digraph and

$$F = \{f_e \,:\, e \in E\},$$

where each function $f_e : \mathbb{R}^d \to \mathbb{R}^d$ is continuous. The function f_e can be considered as a label on the edge e.

Let $\mathbb{H}$ denote the set of nonempty compact subsets of $\mathbb{R}^d$, and define $\mathbf{F} : \mathbb{H}^n \to \mathbb{H}^n$ as follows. If $\mathbf{X} = (X_1, X_2, \dots, X_n) \in \mathbb{H}^n$, then

$$\mathbf{F}(\mathbf{X}) = \big(F_1(\mathbf{X}), F_2(\mathbf{X}), \dots, F_n(\mathbf{X})\big),$$

where, for $i = 1, 2, \dots, n$,

$$F_i(\mathbf{X}) = \bigcup_{e \in E_i} f_e(X_{e^+}).$$

The following is a well-known result in the theory of graph iterated function systems, which is a generalization of a fundamental result of Hutchinson [6]. In the theorem $\mathbf{F}^k$ denotes the k-fold iteration of $\mathbf{F}$.

Theorem 3.1 ([12])**.** *If (G, F) is a GIFS such that each function in F is a contraction, then there exists a unique $\mathbf{A} = (A_1, A_2, ..., A_n) \in \mathbb{H}^n$ such that*

$$\mathbf{A} = \mathbf{F}(\mathbf{A}) \qquad \textit{and} \qquad \mathbf{A} = \lim_{k\to\infty} \mathbf{F}^k(\mathbf{B}) \tag{3.1}$$

independent of $\mathbf{B} \in \mathbb{H}^n$, where convergence is with respect to the Hausdorff metric on $\mathbb{H}^n$.

Definition 3.2 (Attractor)**.** The set $\mathbf{A}$ is called the *attractor* of the GIFS, and $\{A_1, A_2, ..., A_n\}$ is its set of *attractor components*, each of which is compact. The first condition in Equation (3.1) can be restated as

$$A_i = \bigcup_{e \in E_i} f_e(A_{e^+}) \tag{3.2}$$

for $i \in \{1, 2, \dots, n\}$. If, in Equation (3.2), each distinct pair $f_e(A_{e^+})$, $f_{e'}(A_{e'^+})$ is non-overlapping, then A_i is called *non-overlapping*. In this case, $\{f_e(A_{e^+}) : e \in E_i\}$ is a tiling of A_i. If every attractor component is non-overlapping, then the GIFS is called *non-overlapping*.

Remark 3.1. An ordinary iterated function system (IFS) is the special case of a GIFS whose graph consists of a single vertex with loops and whose attractor consists of a single component.

Proposition 3.1. *Every rep-set is the attractor of a GIFS.*

Proof. Given a rep-set Q, let (G, F) be the GIFS where the vertices of G are the elements of Q. By the definition of a rep-set, each $p \in Q$ is tiled by smaller similar copies of a muiti-subset $Q_p \subseteq Q$. Add an edge in G directed from p to each tile in Q_p. For each such edge e directed from p to $q \in Q_p$, let the function f_e be a similarity transformation such that $p = \bigcup_{e \in E_p} f_e(q)$. In view of Equation (3.2), the rep-set Q is the set of attractor components of the GIFS (G, F). Therefore, every rep-set is the attractor of a GIFS whose functions are contractive similarity transformations. □

4. GIFS-Tilings

A method for obtaining tilings of $\mathbb{R}^d$ from a GIFS is provided in this section. Results in Section 5 show that these are the tilings that satisfy conditions that one would expect of a tiling based on a GIFS.

4.1. **Tiling-GIFS.**

According to Theorem 3.1, any GIFS whose functions are contractions has an attractor. An issue, however, is that an attractor component that has empty interior cannot serve as a tile. Even if the attractor components have non-empty interior, it may occur that the GIFS is overlapping (Definition 3.2), resulting in overlap in a tiling obtained from the GIFS. The scaling ratio of a similarity transformation $f : \mathbb{R}^d \to \mathbb{R}^d$ is denote $\lambda(f)$. The restiction on the scaling ratios in condition (1) of the following definition is justified in Section 5.

Definition 4.1 (TGIFS)**.** Call a GIFS $\mathcal{F} = (G, F)$ a *tiling-GIFS* or simply a *TGIFS* if

1. every function $f \in F$ is a similarity transformation with $\lambda(f)$ independent of $f \in F$, the common value denoted $\lambda(\mathcal{F}) < 1$;
2. each component of the attractor of $\mathcal{F}$ has nonempty interior; and
3. $\mathcal{F}$ is non-overlapping.

The set $\{A_1, \dots, A_n\}$ of attractor components of $\mathcal{F}$ will be called the *prototile set* of $\mathcal{F}$, denoted $Q(\mathcal{F})$. *Without loss of generality, it will be assumed throughout that the origin is contained in the interior of each attractor component and, if two distinct attractor components are congruent, then they are placed so as not to coincide.*

For an irreducible non-negative matrix M with spectral radius ρ, in particular for the adjacency matrix of a strongly connected digraph, implications of the Perron-Frobenius theorem include:

- ρ is an eigenvalue of M, called the *Perron-Frobenius eigenvalue*;
- the left and right eignespaces corresponding to ρ are one-dimensional;
- ρ has a corresponding left and right eigenvector all of whose components are positive; moreover the only eigenvectors of M that have all positive components are those corresponding to ρ;
- if M is primitive, then $|\zeta| < \rho$ for all other eigenvalues ζ of M.

Proposition 4.1. *Let $\mathcal{F}$ be a GIFS on $\mathbb{R}^d$ such that all of its functions have common scaling ratio and all of its attractor components have nonempty interior. Then $\mathcal{F}$ is non-overlapping if and only if $\lambda(\mathcal{F}) = 1/\sqrt[d]{\rho}$, where ρ is the Perron-Frobenius eigenvalue of $M(G)$. In particular, if $\mathcal{F}$ is a TGIFS, then $\lambda(\mathcal{F}) = 1/\sqrt[d]{\rho}$.*

Proof. Assume that $\lambda(f) = 1/\sqrt[d]{\rho}$ for all $f \in F$. Denote by x_i the Lebesgue measure of the attractor component A_i, $i = 1, 2, \dots, n$, of $\mathcal{F}$, and let $x = (x_1, \dots, x_n)^t$, where t denotes the transpose. The Lebesgue measure of $f_e(A_{e^+})$ is then $(1/\rho)x_i$. If A_i is overlapping for some i, then $x < (1/\rho)Mx = x$, a contradiction. Here the vector inequality $x < y$ means that $x_i \leq y_i$ for all i and $x_i < y_i$ for at least one i.

Conversely, assume that $\mathcal{F} = (G, F)$ is non-overlapping. Then $M(\lambda^d\, x) = x$, where $\lambda := \lambda(\mathcal{F})$ and $M = M(G)$ is the adjacency matrix of G. Equivalently the eigen-equation $Mx = (1/\lambda^d)x$ holds. Since x is positive, it must be an eigenvector corresponding to ρ. Therefore $\lambda(\mathcal{F}) = 1/\rho$. $\square$

4.2. **Notation.** Denote by $\Sigma^* := \Sigma^*(G)$ the set of paths of finite length in a digraph $G = (V, E)$ and $\Sigma := \Sigma(G)$ the set of all infinite paths. An infinite path has a starting vertex but no terminal vertex. A path $\sigma = e_1\, e_2\, \cdots$ will be written as its ordered string of edges $e_i \in E,\ i = 1, 2 \dots.$ The starting vertex of a path σ will be denoted σ^-, and the terminal vertex of a finite path by σ^+. The length of a finite path σ, i.e., the number of edges, will be denoted $|\sigma|$. A path consisting of a single vertex has length zero.

For $\sigma = e_1 e_2 \cdots \in \Sigma$ let

$$\sigma|k = e_1 e_2 \cdots e_k \in \Sigma^*,$$

and $\sigma|0$ the path that is just the vertex σ^-. For any edge e in G, let $\overleftarrow{e}$ be the oppositely directed edge.

Let $\mathcal{F} = (G, F)$ be a GIFS. For any function $f_e \in F$ define

$$f_{\overleftarrow{e}} := (f_e)^{-1}.$$

Denote by $\overleftarrow{G}$ the digraph obtained from G by reversing the direction on all edges. Define $\overleftarrow{\Sigma}^* := \overleftarrow{\Sigma}^*(G)$ and $\overleftarrow{\Sigma} := \overleftarrow{\Sigma}(G)$ as the set of all finite and infinite paths, respectively, in $\overleftarrow{G}$. For $\sigma = e_1 e_2 e_3 \cdots e_k \in \Sigma^*$, define

$$f_\sigma := f_{e_1} \circ f_{e_2} \circ f_{e_3} \circ \cdots \circ f_{e_k}.$$

For $\overleftarrow{\sigma} = \overleftarrow{e_1}\,\overleftarrow{e_2}\,\overleftarrow{e_3} \cdots \overleftarrow{e_k} \in \overleftarrow{\Sigma}^*$, let

$$f_{\overleftarrow{\sigma}} := f_{\overleftarrow{e_1}} \circ f_{\overleftarrow{e_2}} \circ f_{\overleftarrow{e_3}} \circ \cdots \circ f_{\overleftarrow{e_k}} = f^{-1}_{e_1} \circ f^{-1}_{e_2} \circ f^{-1}_{e_3} \circ \cdots \circ f^{-1}_{e_k}.$$

4.3. **The Parameter Space.** Let $\mathcal{F} = (G, F)$ be a TGIFS. Any path $\overleftarrow{\theta} \in \overleftarrow{\Sigma}$ will be referred to as a *parameter* of $\mathcal{F}$. To simplify notation, denote the set of perameters by

$$\mathcal{P} = \mathcal{P}(\mathcal{F}) := \overleftarrow{\Sigma}.$$

Define a metric d on $\mathcal{P}$ by

$$d(\overleftarrow{\sigma}, \overleftarrow{\omega}) = \begin{cases} 0 & \text{if } \overleftarrow{\sigma} = \overleftarrow{\omega} \\ 2^{-k} & \text{otherwise, where } k \text{ is the first integer such that } \overleftarrow{\sigma}_k \neq \overleftarrow{\omega}_k \end{cases}.$$

This makes $(\mathcal{P}, d)$ a compact metric space, which we call the *parameter space* of the TGIFS. A parameter $\overleftarrow{\theta} \in \mathcal{P}$ is *eventually periodic* if there exist $\overleftarrow{\theta}_0, \overleftarrow{\theta}_1 \in \overleftarrow{\Sigma}^*$ such that $\overleftarrow{\theta} = \overleftarrow{\theta}_0\, \overleftarrow{\theta}_1\, \overleftarrow{\theta}_1\, \overleftarrow{\theta}_1 \cdots.$

4.4. **$\mathcal{F}$-Tilings.**

Definition 4.2 (TGIFS-tiling)**.** Given a TGIFS $\mathcal{F}$, for each parameter $\overleftarrow{\theta} \in \mathcal{P}$, let

$$X_k = \{\sigma \in \Sigma^* : \sigma^- = \theta_k^-,\, |\sigma| = k\}.$$

Thus X_k is the set of all paths of length k in the digraph of $\mathcal{F}$ that start at the last vertex of the path $\overleftarrow{\theta}|k$. Define a tiling $T(\overleftarrow{\theta}) := T(\mathcal{F}, \overleftarrow{\theta})$ as follows. For $\sigma \in X_k$, $k \geq 1$, let

$$\begin{aligned} &\text{(tile)} && T(\overleftarrow{\theta}, k, \sigma) = (f_{\overleftarrow{\theta}|k} \circ f_\sigma)(A_{\sigma^+}) \\ &\text{(patch of tiles)} && T(\overleftarrow{\theta}, k) = \{(f_{\overleftarrow{\theta}|k} \circ f_\sigma)(A_{\sigma^+})\ :\ \sigma \in X_k\} \\ &\text{(tiling)} && T(\overleftarrow{\theta}) = \bigcup_{k=0}^{\infty} T(\overleftarrow{\theta}, k). \end{aligned} \tag{4.1}$$

In Equation (4.1) the composition $f_{\overleftarrow{\theta}|k} \circ f_\sigma$ is a homeomorphism that takes a prototile (attractor component) to a tile in $T(\overleftarrow{\theta})$. That $T(\overleftarrow{\theta}, k)$ is a patch, i.e. non-overlapping, will become clear in Section 5 (see Definition 5.1, Equation (5.2)), and it is routine to verify that $T(\overleftarrow{\theta})$ is the nested union of these patches. The tiling $T(\overleftarrow{\theta})$ will be referred to as a *TGIFS-tiling*. For a particular TGIFS $\mathcal{F}$, the tiling $T(\overleftarrow{\theta})$ will be referred to as an *$\mathcal{F}$-tiling*. For each TGIFS $\mathcal{F}$ there are potentially uncountably many tilings $T(\mathcal{F}, \overleftarrow{\theta})$, $\overleftarrow{\theta} \in \mathcal{P}$, although some, and possible all, may coincide.

Almost all TGIFS-tilings fill the whole space $\mathbb{R}^d$. More specifically, for all θ in a dense subset of the parameter space $\mathcal{P}$, the tiling $T(\overleftarrow{\theta})$ covers $\mathbb{R}^d$ [1]. There are cases, however, where $T(\overleftarrow{\theta})$ tiles a subset of $\mathbb{R}^d$. For example, consider the TGIFS (G, F) on $\mathbb{R}$, where G consists of a single vertex and two loops e_1, e_2 and $F = \{f_{e_1}, f_{e_2}\}$, where $f_{e_1}(x) = (1/2)\,x$, $f_{e_2}(x) = (1/2)\,x + 1/2$, and $\overleftarrow{\theta} = \overleftarrow{e}_1 \overleftarrow{e}_1 \cdots$. Then $T(\overleftarrow{\theta})$ tiles the half line $\{x \in \mathbb{R} : x \geq 0\}$. *We assume hereafter, unless stated otherwise, that TGIFS-tiling means a tiling of the whole space.*

Example 4.1 (Ammann Chair Tiling)**.** We illustrate Definition 4.2 with the Ammann chair tiling. The shape, called the A2 tile or sometimes the "golden bee", was discovered by R. Ammann in 1977 and is shown on the left in Figure 6. Figure 5 shows two digraphs. The digraph on the left will be relevant in Section 5. Consider now only the digraph G on the right. With $s = 1/\sqrt{\tau}$, where $\tau = (1+\sqrt{5})/2$ is the golden ratio, the functions are:

$$f_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} s \\ 0 \end{pmatrix}, \quad f_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The respective scaling ratios are $\lambda(f_1) = \lambda(f_2) = \lambda(f_3) = s$. The two attractor components, shown in orange at the left in Figure 6 have nonempty interior. The GIFS (G, F) is non-overlapping. Therefore $\mathcal{F} = (G, \{f_1, f_2, f_3\})$ is a TGIFS. The patch $T(\overleftarrow{\theta}, 2)$ and an $\mathcal{F}$-tiling are shown at the right in Figure 6.

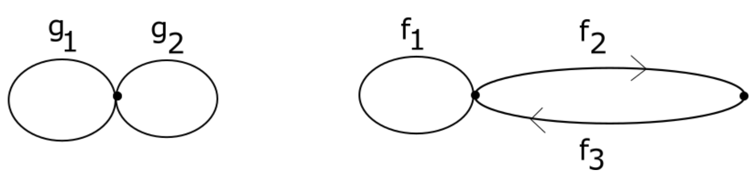

FIGURE 5. A digraph of a GIFS and a digraph of a TGIFS.

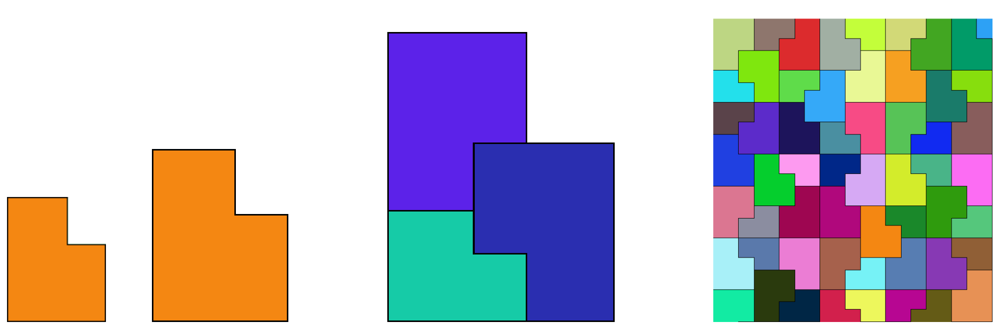

FIGURE 6. Ammann chair tiling: prototiles, second level patch, TGIFS-tiling.

4.5. **Properties of $\mathcal{F}$-Tilings.**

Definition 4.3. The *shift map* $S : \mathcal{P} \to \mathcal{P}$ is defined by

$$(S \overleftarrow{\theta})_i = \theta_{i+1},$$

i.e., $S(e_1 e_2 e_3 \cdots) = e_2 e_3 \cdots$, and S^k denotes its k^{th} iterate.

Definition 4.4 (Repetitive, Local Isomorphism)**.** A tiling T is *repetitive*, also called *quasiperiodic*, if, for every patch T_0 of T, there is a real number R such that every ball of radius R contains a patch congruent to T_0. Two repetitive tilings are *locally isomorphic* if every patch in one also occurs in the other.

Theorem 4.1. *For a given TGIFS $\mathcal{F}$ with prototile set $Q(\mathcal{F})$ and for every pair $\overleftarrow{\theta}, \overleftarrow{\theta}' \in \mathcal{P}(\mathcal{F})$ the following properties hold.*

1. *Each tile in $T(\overleftarrow{\theta})$ is congruent to a tile in $Q(\mathcal{F})$,*
2. *If $\mathcal{F}$ is primitive, then a congruent copy of each tile in $Q(\mathcal{F})$ occurs in $T(\overleftarrow{\theta})$.*
3. *If $S^k(\overleftarrow{\theta}) = S^k(\overleftarrow{\theta}')$ for some k, then $T(\overleftarrow{\theta}) \cong T(\overleftarrow{\theta}')$*
4. *$T(\overleftarrow{\theta})$ is repetitive for all $\overleftarrow{\theta} \in \mathcal{P}$.*
5. *Every pair of $\mathcal{F}$-tilings are locally isomorphic.*

Proof. Concerning property (1), that each tile in $T(\overleftarrow{\theta})$ is congruent to a tile in $Q(\mathcal{F})$ follows from Definition 4.2 because for all k

$$\lambda(f_{\overleftarrow{\theta}|k} \circ f_\sigma) = \lambda(f_{\overleftarrow{\theta}|k})\, \lambda(f_\sigma) = \Big(\frac{1}{\lambda(\mathcal{F})}\Big)^k \big(\lambda(\mathcal{F})\big)^k = 1.$$

Concerning property (2), primitivity implies that there is an positive integer k such that, for every two vertices i and j in G, there is a path of length k from i to j. This implies that for any parameter $\overleftarrow{\theta}$, patch $T(\overleftarrow{\theta}, k)$, and therefore $T(\overleftarrow{\theta})$, contains a congruent copy of each tile in $Q(\mathcal{F})$.

Concerning property (3), an isometry taking $T(\overleftarrow{\theta})$ onto $T(\overleftarrow{\theta}')$ is $f_{\overleftarrow{\theta}'|k} \circ (f_{\overleftarrow{\theta}|k})^{-1}$.

The proof of property (4) is deferred to Section 7, becasue the notion of a hierarchy is used in the proof. The proof of property (5) is similar to the proof of property (3). $\square$

Remark 4.1 (Necessity of primitivity in statement (2) of Theorem 4.1)**.** The following is an example where a prototile does not appear in a TGIFS-tiling. Let $\mathcal{F} = (G, F)$ be the 1-dimensional TGIFS with digraph given by its adjacency matrix M and $F = \{f_{1,2}, f_{1,3}, f_{2,1}, f_{3,1}\}$ where

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} f_{1,2}(x) &= \frac{x}{\sqrt{2}} \qquad f_{1,3}(x) = \frac{x+1}{\sqrt{2}} \\ f_{2,1}(x) &= f_{3,1}(x) = \frac{x}{\sqrt{2}}, \end{aligned}$$

and $f_{i,j}$ denotes the function on the edge (i,j). The attractor components are intervals $\{[0, \sqrt{2}], [0,1], [0,1]\}$, but $T(\overleftarrow{\theta})$ is a tiling of the real line by intervals of just length $\sqrt{2}$ if $\overleftarrow{\theta} = 1\,2\,1\,2 \cdots$.

4.6. **Tiling Frequencies.** Let $\mathcal{F} = (G, F)$ be a TGIF and $T := T(\overleftarrow{\theta})$ an $\mathcal{F}$-tiling. For a prototile $p \in Q(\mathcal{F})$ define $N_{k,\theta}$ and $N_{k,\theta}(p)$ as the number of tiles in $T(\overleftarrow{\theta}, k)$ and the number of tiles of type p in $T(\overleftarrow{\theta}, k)$, respectively. Letting $Q = Q(\mathcal{F}) = \{p_1, p_2, \dots, p_n\}$, define the *frequency* $\beta_\theta(p)$ of prototile $p \in Q$ in tiling $T(\overleftarrow{\theta})$ and the *frequence vector* $\boldsymbol{\beta}_\theta(\mathcal{F})$ as

$$\beta_\theta(p) := \lim_{k\to\infty} \frac{N_{k,\theta}(p)}{N_{k,\theta}} \qquad \text{and} \qquad \boldsymbol{\beta}_\theta(\mathcal{F}) := \big(\beta_\theta(p_1), \beta_\theta(p_2), \dots, \beta_\theta(p_n)\big),$$

respectively.

A vector $x \in \mathbb{R}^n$ is *normalized* if it is a unit vector with respect to the 1-norm, i.e., $\|x\|_1 = 1$. Recall that the power method gives a normalized left eigenvector y of $M(G)$ corresponding to the Perron-Frobenius eigenvalue $\rho(\mathcal{F})$ as the limit $y = \lim_{k\to\infty} xM^k/\|xM^k\|$, independent of x as long as the component of x in the direction of y is non-zero. Let e_i be the standard basis vector whose i^{th} coordinate is 1, all others 0.

Theorem 4.2. *Let $M(G)$ be the $n \times n$ adjacency matrix of a TGIFS $\mathcal{F}$ and let $\mathbf{y}(\mathcal{F})$ denote the normalized positive left eigenvector corresponding to the Perron-Frobenius eigenvalue of $M(G)$. Assume that the component of standard basis vector e_i in the direction of $\mathbf{y}(\mathcal{F})$ is non-zero for $i = 1, 2, \dots, n$. If $T(\overleftarrow{\theta})$ is an $\mathcal{F}$-tiling, then $\boldsymbol{\beta}_\theta(\mathcal{F}) = \mathbf{y}(\mathcal{F})$, independent of the parameter $\overleftarrow{\theta} \in \mathcal{P}$.*

Proof. Denote the (i,j) entry of M^k by $m_{i,j}^{(k)}$, which is the number of paths in the digraph of $\mathcal{F}$ of length k from vertex i t vertex j. For a fixed parameter $\overleftarrow{\theta}$, let $i(k) = \theta_k^-$, i.e., the last vertex in the path $\overleftarrow{\theta}|k$. For ease of notation, let $N_k = N_{k,\theta}$ and $N_k(p) = N_{k,\theta}(p)$. Referring to the Definition 4.2 of an $\mathcal{F}$-tiling we have

$$N_k(p_j) = m_{i(k),j}^{(k)} \qquad \text{and} \qquad N_k = \sum_{j=1}^{n} m_{i(k),j}^{(k)}.$$

Noting that $N_k = \|\big(N_k(p_1), N_k(p_2), \dots, N_k(p_n)\big)\|_1$ we have

$$\Big(\frac{N_k(p_1)}{N_k}, \frac{N_k(p_2)}{N_k}, \dots, \frac{N_k(p_n)}{N_k}\Big) = \frac{\big(N_k(p_1), N_k(p_2), \dots, N_k(p_n)\big)}{\|\big(N_k(p_1), N_k(p_2), \dots, N_k(p_n)\big)\|_1} = \frac{e_{i(k)}M^k}{\|e_{i(k)}M^k\|_1}.$$

Referring to the power method for finding an eigenvector correspoinding to the dominant eigenvalue of a square matrix, we have

$$\lim_{k\to\infty} \frac{eM^k}{\|eM^k\|_1} = \mathbf{y}(\mathcal{F})$$

independent of the particular standard basis vector e. Therefore

$$\boldsymbol{\beta}_\theta(\mathcal{F}) = \lim_{k\to\infty} \frac{e_{i(k)}M^k}{\|e_{i(k)}M^k\|_1} = \mathbf{y}(\mathcal{F}).$$

Note that $\boldsymbol{\beta}_\theta(\mathcal{F})$ does not depend on the parameter θ. □

Example 4.2 (Tile Frequencies for the Ammann chair tiling of Example 4.1)**.** The adjacency matrix of the TGIFS digraph is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The Perron-Frobenius eigenvalue is $\tau = (1+\sqrt{5})/2$, the golden ratio. The normalized corresponding left eigenvector is $(1/\tau, 1/\tau^2)$. Therefore, asymptotically about 61.80% of the tiles in an Ammann chair tiling are the large prototile, and about 38.20% are the small prototile.

5. Every GIFS-Based Tiling is an $\mathcal{F}$-Tiling

The scaling ratios of similarity functions in a GIFS can take different values. In our definition of a TGIFS, however, the scaling ratios are equal. It may seem that this is excessively restrictive, resulting in a less than universal set of tilings. In our definition of a TGIFS-tiling, it is assumed that the lengths of all paths in X_k (see Definition 4.2) are equal. This may also seem restrictive. In this section it will be shown that if a tiling is even loosely based on a GIFS, then there is no loss of generality in assuming that the scaling ratios be equal and that the paths in X_k have the same length, thus that the GIFS is a TGIFS and also that the associated tilings are TGIFS-tilings.

We begin with a notion of what it means for a tiling to be based on a GIFS. *We assume in this section that every GIFS $\mathcal{F}$ is non-overlapping, that the attractor components have nonempty interior.*

Definition 5.1 (Admissible Patch)**.** For a GIFS $\mathcal{F} = (G, F)$, let r be any vertex of G, referred to as the *root*. Referring to Definition (3.2),

$$W_r(\mathcal{F}) := \{f_e(A_{e^+}) : e \in E_r\} \tag{5.1}$$

is a tiling of the attractor component A_r. If X_r is a set of finite paths in G starting at r, then by applying Equation (3.2) recursively we arrive at the fact that

$$W_r(\mathcal{F}, X_r) := \big\{f_\sigma(A_{\sigma^+}) : \sigma \in X_r \subset \Sigma^*\big\} \tag{5.2}$$

is a tiling of A_r if

(1) no proper subpath of a path in X_r lies in X_r, and
(2) if $\sigma \in X_r$ and σ' is any proper subpath of σ starting at vertex r, then $\sigma' e$ is a subpath (not necessarily proper) of a path in X_r for all edges e such that $\sigma'^+ = e^-$.

Call a set X_r of finite paths rooted at r and satisfying conditions (1) and (2) an *admissible set of paths*. For any vertex r and any admissible set X_r of paths, call $W_r(\mathcal{F}, X_r)$ an *admissible patch* or *admissible $\mathcal{F}$-patch*. Note that A_r is itself an admissible patch (X_r being the single point path r), and $W_r(\mathcal{F})$ is an admissible patch (X_r being the set of all paths of length 1 starting at r). If T is a tiling, call a patch T_0 of T *admissible* if it is isometric to $\mu\, W_r(\mathcal{F}, X_r)$ for some positive real "scaling up" constant μ.

A minimal requirement for a tiling T to be based on a GIFS $\mathcal{F}$ is that

- T contain admissible $\mathcal{F}$-patches of arbitrary large cardinality.

Consequences of this assumption appear in Section 5.1. To avoid randomness in the tilings, we subsequently impose the stronger requirement that

- T be the nested union of admissible $\mathcal{F}$-patches.

5.1. **Commensurable GIFS.**

Definition 5.2 (Commensurable GIFS)**.** Let F be a set of similarity transformations from $\mathbb{R}^d$ onto $\mathbb{R}^d$. For $f, g \in F$, call f and g *commensurable* if

$$\frac{\log\big(\lambda(f)\big)}{\log\big(\lambda(g)\big)} \in \mathbb{Q}.$$

Call F *commensurable* if every pair of functions in F is commensurable, and call a GIFS (G, F) *commensurable* if F is commensurable.

Proposition 5.1. *A set F of similarities is commensurable if and only if there is a real number $s > 0$ and a set $\{b_f \in \mathbb{N} : f \in F\}$ of positive integers such that $\lambda(f) = s^{b_f}$ for $f \in F$.*

Proof. The existence of a real $s > 0$ and a set $\{b_f \in \mathbb{N} : f \in F\}$ of positive integers such that $\lambda(f) = s^{b_f}$ for all $f \in F$ clearly implies that F is commensurable.

In the other direction, let

$$\alpha_f = \log_s\big(\lambda(f)\big) \qquad \text{so that} \qquad \lambda(f) = s^{\alpha_f}.$$

Let $f_0 \in F$. By the assumption that F is commensurable, there is a $d \in \mathbb{N}$ and $b_f \in \mathbb{N}$ for all $f \in F$ such that $\alpha_f/\alpha_{f_0} = b_f/d$. Let $s' = s^{\frac{\alpha_{f_0}}{d}}$. Then, for all $f \in F$,

$$\lambda(f) = s^{\alpha_f} = \Big(s^{\frac{\alpha_{f_0}}{d}}\Big)^{b_f} = (s')^{b_f}.$$

□

Theorem 5.1. *Let $\mathcal{F}$ be a primitive GIFS. If there exists a tiling T of $\mathbb{R}^d$ having a finite prototile set and containing admissible $\mathcal{F}$-patches of arbitrary large cardinality, then $\mathcal{F}$ must be commensurable.*

Example 5.1 (Primitivity in Theorem 5.1 is necessary)**.** The followig is a counterexample to Theorem 5.1 if the assumption of primitivity is removed. Consider the GIFS $\mathcal{F} = (G, F)$ on $\mathbb{R}$, where G is the graph consisting of two vertices r, r', with edges e_1, e_2 from r to r' and edges e_3, e_4 from r' to r. Note that G is not primitive; the lengths of all closed paths are divisible by 2. Let $F = \{f_{e_1}, f_{e_2}, f_{e_3}, f_{e_4}\}$ where $f_{e_1}(x) = 3/4\,x$, $f_{e_2}(x) = 3/4\,x + 1/2$, $f_{e_3}(x) = 1/3\,x$, $f_{e_4}(x) = 1/3\,x + 1/3$. The attractor components of $\mathcal{F}$ are the intervals $A_r = [0, 1]$, $A_{r'} = [0, 2/3]$. Note that the scaling ratios $3/4$ and $1/3$ are not commensurable; thus $\mathcal{F}$ is not commensurable. Let $X_r(k)$ be the set of all paths in G rooted at r of length $2k$; this set of paths is admissible. It is routine to check that the admissible patch $W(\mathcal{F}, X_r(k))$ consists of the interval $[0, 4^k]$ subdivided into 4^k unit intervals. Let T be the tiling of the line by unit intervals. Thus T contains admissible patches of arbitrary large cardinality.

The following graph theoretic result will be used in the proof of Theorem 5.1. Let G be a strongly connected digraph whose edges are colored using q colors, $q \geq 2$. For an admissible set X_r of paths rooted at a vertex r of G, call two paths *equivalent* if they contain the same number of edges of each color, and let $|X_r|_\equiv$ denote the number of equivalence classes.

Lemma 5.1. *Let G be a strongly connected, primitive digraph whose edges are colored using q colors, $q \geq 2$. For every integer N there exists an M such that, if X_r is an admissible set of paths rooted at vertex r with $|X_r| \geq M$, then $|X_r|_\equiv \geq N$.*

Proof. If the lemma holds for every 2-coloring, then it holds for every q-coloring. To see this, let the colors be $\{1, 2, \dots, q\}$. For all edges colored $3, 4, \dots q$, change the colors to color 2. The number $|X_r|$ does not change, and $|X_r|_\equiv$ cannot increase.

We now prove the result for every 2-coloring (say red and blue) of its edge set. By way of contradiction assume that there is a 2-coloring, a vertex r, a natural number N, and a sequence $(X_k)_{k\geq 1}$ of admissible paths rooted at r such that

(1) $|X_k| \to \infty$ as $k \to \infty$, and
(2) $|X_k|_\equiv \leq N$.

In particular, from (2) it follows that there must be a bound m, independent of k, such that the set of lengths satisfies $|\{|\sigma| \,:\, \sigma \in X_k\}| \leq m$. We claim that there is a sequence (Y_k) of admissible sets of paths satisfying properties (1) and (2) above and such that, for each k, all paths in Y_k have the same length. To prove the claim, denote the lengths of paths in X_k by $l_1(k) > l_2(k) > \cdots > l_m(k)$. We now prove the claim by induction on m, the number of distinct lengths of paths in the X_k. The claim is triviially true for $m = 1$. Assume it true for $m - 1$ and let the sequence $(X_k)_{k\geq 1}$ have paths of m different lengths.

Let Y_k be the set of paths obtained from X_k by replacing (pruning) each path σ of length $l_1(k)$ by its subpath σ' rooted at r and having length l_2. Note that Y_k remains a set of admissible paths. We will show that the sequence (Y_k) of sets of admissible paths satisfies conditions (1) and (2). This will complete the induction argument because Y_k has one less path length than X_k.

Concerning condition (1), it follows from the definition of an admissible set of paths that, if v is the second to last vertex of σ and v' is the last vertex of σ', then the outdegree in G of all vertices on the path from v' to v, including v' but not including v, have outdegree 1. This implies that $|Y_k| \leq |X_k|/\Delta$, where Δ is the maximum outdegree of vertices in G. Therefore $|Y_k| \to \infty$ as $k \to \infty$.

Concerning condition (2), first note that there cannot exist a cycle C in G such that the outdegree of every vertex on C is 1. Otherwise there would exist an attractor component with empty interior, which we are assuming in this section is not the case. Therefore, the longest path γ such that every vertex, except the last, has outdegree 1 is less than n, the order of G. Assume that, in going from X_k to Y_k we prune just one prune just one length at a time, obtaining a sequence $X + k = X_k^0, X_k^1, \dots, X_k^q = Y_k$, where $q = l_1(k) - l_1(k) < n$. Since adjoining a single edge to a path can change the number of red edges (or blue edges) by at most 1, we have $|X_k^{i+1}|_\equiv \leq 2|X_k^i|_\equiv$. Therefore $|Y_k|_\equiv \leq 2^n |Y_k|_\equiv \leq 2^n N$.

It remains to prove a contradiction in the case that, for each k, all paths in Y_k have the same length $n(k)$. If all paths in Y_k have the same length, then by the definition of admissible path, Y_k is the set of all paths of length $n(k)$ rooted at vertex r in G. There exists a closed path c_1 in G containing r that is not monochromatic (recall that both red and blue are used in the coloring). Let L_1 denote the length of c_1. By primitivity of G, there exists another closed path c_2 in G containing r whose length L_2 is relatively prime to L_1. Any non-negative integer solution x, y to the equation

$$xL_1 + yL_2 = n(k)$$

provides a path $c_k(x, y)$ in Y_k obtained by winding x times around c_1 followed by winding y times around c_2. Call such a path a (c_1, c_2)-path. For clarity we omit the index k in what follows, i.e., k fixed and, for example, $n = n(k)$. If n is sufficiently large, then from elementary number theory there are positive integers x_0, y_0 such that $x_0L_1 + y_0L_2 = n$. It follows that $x = x_0 - jL_2$, $y = y_0 + jL_1$ is also a solution for any $j \in \mathbb{Z}$. Since we seek non-negative solutions, the condition

$$\frac{x_0}{L_2} \geq j \geq -\frac{y_0}{L_1}$$

must be satisfied, which implies that there are

$$\left\lfloor \frac{x_0}{L_2} + \frac{y_0}{L_1} \right\rfloor = \left\lfloor \frac{n}{L_1 L_2} \right\rfloor \xrightarrow[n\to\infty]{} \infty$$

solutions. In other words, the number of (c_1, c_2)-paths in Y_k goes to infinity with k.

Denote by a_1, a_2 the number of red edges on L_1 and L_2, respectively. Two of the (c_1, c_2)-paths in Y_k are in the same color equivalence class if and only if they contain the same number of red edges. Counting the number of red edges on the path corresponding to solution x, y, i.e., to each valid j, we obtain $xa_1 + ya_2 = (x_0 - ja_2)a_1 + (y_0 + ja_1)a_2$ red edges. Therefore, two (c_1, c_2)-paths, which we denote by $c(i)$ and $d(j)$, are in the same equivalence class if and only if

$$(x_0 - ja_2)a_1 + (y_0 + ja_1)a_2 = (x_0 - ia_2)a_1 + (y_0 + ia_1)a_2$$

which simplifies to $(i - j)(L_1a_2 - L_2a_1) = 0$. If $i = j$, then $c(i) = c(j)$. That $L_1a_2 = L_2a_1$ is impossible since L_1 and L_2 are relatively prime and $0 < a_1 < L_1$. We have shown that (c_1, c_2)-path in Y_k is in its own equivalence class. Since the number of (c_1, c_2)-path in Y_k goes to infinity with k, we have the desired contradiction to condition (2). □

Proof of Theorem 5.1. For a set W of tiles, let $|W|$ denote the cardinality of W, and let $|W|_\equiv$ denote the number of tiles up to congruence. Because it is assumed that T has a finite prototile set and contains admissible (scaled) $\mathcal{F}$-patches of arbitrary large cardinality, there must be a sequence $\{W_r(X_k)\}$ of (unscaled) admissible patches such that $|W_r(X_k)|_\equiv$ is bounded but $\lim_{k\to\infty} |W_r(X_k)| = \infty$.

By way of contradiction, assume that $\mathcal{F}$ is not commensurable. We will show that, for every N there exists a k_0 such that $|W_r(X_{k_0})|_\equiv \geq N$, a contradiction.

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the scaling ratios of the functions in F. Let $s = \lambda_1$, and define α_i, $i = 1, 2, \ldots, m$, by $\lambda_i = s^{\alpha_i}$. Note that $\alpha_1 = 1$. The commensurable relation is an equivalence relation. Partition the set $\{\alpha_1, \ldots, \alpha_m\}$ into equivalence classes, and call two edges of G equivalent if the corresponding $\alpha's$ are equivalent. Let the number of equivalence classes be q, which is at least 2 by Proposition 5.1. Color the edges of G in q colors according to their equivalence class. For an edge e, denote the color by $\alpha(e)$. For a path $\sigma \in \Sigma^*(G)$, define $\alpha(\sigma) := \sum_{e\in\sigma} \alpha(e)$.

Let $\sigma, \omega \in X_k$. Because a set of pairwise incommensurable numbers are linearly independent over $\mathbb{Q}$, we have that $\alpha(\sigma) = \alpha(\omega)$ if and only if, for every color, the number of occurences of that color in σ equals the the number of occurences of that color in ω. Now $\lambda(f_\sigma) = \lambda(f_\omega)$ if and only if $\alpha(\sigma) = \alpha(\omega)$ if and only if σ and ω are in the same color equivalence class. Because $|X_k| = |W_r(X_k)|$ we have $\lim_{k\to\infty} |X_k| = \infty$. Call $\lambda(f_\sigma)$ the scaling ratio of the path σ. By Lemma 5.1, for every N there exists a k_0 such that if $k \geq k_0$, then $|X_k|_\equiv \geq N$. Therefore, for every N there exists a k_0 such that if $k \geq k_0$, then there exists at least N paths in X_k with pairwise different scaling ratios λ. For $\sigma \in X_k$, there are at most n (order of digraph G)

possibilities for σ^+, which implies that, for $k \geq k_0$, there are at least N/n distinct tiles $f_\sigma(A_{\sigma^+})$ in $W_k(X_k)$, i.e., $|W_r(X_{k_0})|_\equiv \geq N/n$. $\square$

5.2. **Companion GIFS.**

Definition 5.3 (Companion GIFS)**.** Let $\mathcal{F} = (G, F)$ be a commensurable GIFS. By Proposition 5.1 there is an $s > 0$ and a set $\{a_e \in \mathbb{N} : e \in E\}$ of positive integers associated with each edge $e \in E$ of G such that $\lambda(f_e) = s^{a_e}$ for all $e \in E$. Attach the label a_e to each edge of G. Constuct a new GIFS $\mathcal{F}' = (G', F')$, called the *companion* to $\mathcal{F}$, as follows. To obtain the graph G', consider each edge $e = (u, v)$ of G with $a_e > 1$. Replace e by a path $\sigma(e) := e_1 e_2 \cdots e_{a_e}$ from u to v. Note that no vertex of G has been removed. Also note that G' is strongly connected if and only if G is strongly connected. It is not hard to see that there exist functions $f_{e_1}, f_{e_2}, \dots, f_{e_{a_e}}$ on the respective new edges $e_1, e_2, \dots, e_{a_e}$ such that $\lambda(f_{e_i}) = s$ for $i = 1, 2, \dots, a_e$ and therefore $f_{\sigma(e)} = f_e$. The graph G' and function set F' is the result of the above alterations for all edges e with $a_e > 1$.

Example 5.2 (Companion GIFS)**.** On the left in Figure 5 is the digraph of a GIFS $\mathcal{F} = (G, F)$ where G has one vertex and two loops and $F = \{g_1, g_2\}$, where

$$g_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} s \\ 0 \end{pmatrix}, \qquad g_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s^2 & 0 \\ 0 & -s^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with $s = 1/\sqrt{\tau}$, where τ is the golden ratio. By Proposition 5.1, $\mathcal{F}$ is commensurable, the scaling ratios being $\lambda(g_1) = s$, $\lambda(g_2) = s^2$. The attractor is the Ammann chair tile shown at the left in Figure 6 (either orange polygon). Let $\mathcal{F}' = (G', F)$, where G' is the graph on the right in Figure 5 and $F' = \{f_1, f_2, f_3\}$ is as given in Example 4.1. The scaling ratios are equal: $\lambda(f_1) = \lambda(f_2) = \lambda(f_3) = s$. The two attractor components are the polygons in orange at the left in Figure 6. The TGIFS $\mathcal{F}'$ is the companion of the GIFS $\mathcal{F}$. Both $\mathcal{F}$ and $\mathcal{F}'$ have the same admissible patches.

Proposition 5.2. *The companion GIFS $\mathcal{F}' = (G', F')$ of a commensurable GIFS $\mathcal{F} = (G, F)$ satisfies the following properties:*

1. *the scaling ratios of all functions in F' are equal, and*
2. *every admissible patch of $\mathcal{F}$ is an admissible patch of $\mathcal{F}'$.*

Proof. Concerning statement (1) and referring to Definition 5.3, $\lambda(f_e) = s$ for all edges in G'. Therefore $\lambda(\mathcal{F}') = s$.

Concerning statement (2), denote the set of vertices of G by $\{1, 2, \dots, n\}$ and the set of corresponding attractor components by $\{A_1, A_2, \dots, A_n\}$. Each edge $e = (i, j)$ of G with $a_e > 1$ is replaced in G by a path whose successive vertices we denote by $u = u_0, u_1, u_2, \dots, u_k = v$. Note that the outdegree of u_i is 1 for $i = 0, 1, 2, \dots, k-1$. The sucsessive edges are $e_1 = (u, u_1) = (u_0, u_1), e_2 = (u_1, u_2), \dots, e_k = (u_{k-1}, u_k) = (u_{k-1}, v)$. It is routine to check that in $\mathcal{F}'$, the attractor component of an all vertices from G remain the same, namely $A'_i = A_i$ for $i = 1, 2, \dots, n$. The attractor component A'_{u_i} of each new vertex is defined recursively by $A'_{u_k} = A_{u_k}$ and $A'_{u_i} = f_{e_i}(A'_{u_{i+1}})$ for $i = k-1, k-2, \dots, 1$. Note that the attractor components $A'_{u_1}, A'_{u_2}, \dots, A'_{u_k} = A_v$ are all similar, each scaled down from its successor by a factor s.

Let $W_r(\mathcal{F}, X_r)$ be an admissible patch of $\mathcal{F}$. Define an admissible patch $W_r(\mathcal{F}', X'_r)$ of $\mathcal{F}'$ as follows. For every path $\sigma \in X_r$, let $\sigma' \in X'_r$ be the path obtained by replacing edges e with $a_e > 1$ by paths as in the definition of the companion GIFS. Then $X'_r = \{\sigma' : \sigma \in X_r\}$ is admissible and $W_r(\mathcal{F}', X'_r) = W_r(\mathcal{F}, X_r)$. $\square$

Theorem 5.1 and Proposition 5.2 justify condition (1) in the Defintition 4.1 of a TGIFS - that all functions have the same scaling ratio. Therefore, for the remainder of this section, we assume a TGIFS. To justify, in the Definition 4.2 of a TGIFS-tiling, that all paths in X_k have the same length, first consider the following example.

Example 5.3 (Admissible patches of a GIFS all of whose functions have the same scaling ratio can still be quite irregular)**.** Let $\mathcal{F} = (G, F)$ be the following GIFS on $\mathbb{R}^2$ where G has one vertex and four loops (an ordinary IFS). The functions in F are $f_1(x) = 1/2\,x$, $f_2(x) = 1/2\,x+(1/2, 0)$, $f_3(x) = 1/2\,x+(0, 1/2)$, $f_4(x) = 1/2\,x+(1/2, 1/2)$. Clearly, all the scaling ratios are equal to $1/2$. The attractor of $\mathcal{F}$ is the unit square. Consider patches of the following form. Let k be a power of 2 and let W_k be a patch formed by subdividing the unit square into k^2 squares of side length $1/k$ in the usual way. Then choose a subset of these k^2 small squares *at random* and subdivide each of those into four equal smaller squares. It is routine to check that W_k is an admissible patch of $\mathcal{F}$. Moreover, $W_k = W(\mathcal{F}, X_k)$, where X_k is a set of admissible paths whose lengths are either k or $k + 1$. Therefore $\mathcal{F}$ is a TGIFS that admits quite random admissible patches.

The randomness in Example 5.3 can be avoided by requiring all admissible paths to have the same length. This motivates the following defintion. In the definition, a "scaling" of a patch W means an image of W under a similarity transformation.

Definition 5.4 (GIFS-Based Tiling)**.** An admissible patch $W(\mathcal{F}, X)$ of a TGIFS is *uniform* if all paths in X have the same length. A tiling T of $\mathbb{R}^d$ is *GIFS-based* if there is a GIFS $\mathcal{F}$ all of whose functions have the same scaling ratio and such that T is the nested union of scaled uniform patches.

Theorem 5.2. *A tiling T of $\mathbb{R}^d$ is GIFS-based if and only if there is a TGIFS $\mathcal{F}$ such that T is an $\mathcal{F}$-tiling.*

Proof. Assume that $T = T(\mathcal{F}, \overleftarrow{\theta})$ for some TGIFS $\mathcal{F}$ and some parameter $\overleftarrow{\theta}$. The patch $T(\overleftarrow{\theta}, k)$ is isometric to $\lambda(\mathcal{F})^k\, W(\mathcal{F}, X_k)$, where X_k is as in Definition 4.2 and $W(\mathcal{F}, X_k)$ is the corresponding uniform patch. Therefore, by the definition of the tiling $T(\mathcal{F}, \overleftarrow{\theta})$ in Equation (4.1), T is the nested union of scaled uniform patches.

In the other direction, assume that $\mathcal{F}$ is GIFS-based. Clearly $\mathcal{F}$ is a TGIFS. Assume that T is the nested union of scaled uniform patches. Let $T_0, T_1, T_2 \dots$ be a nested sequence of scaled uniform patches whose union is T. Without loss of generality assume that T_0 consists of a single attractor component, say A_0. Let t be an arbitrary tile in T, and assume that j is the least integer such that $t \in T_j$. Each T_i, $i = 1, 2, \dots$, is a scaled copy of a patch of the form $W_i(\mathcal{F}, X_i)$ in Equation (5.2), where each path in X_i has length k_i for an increasing sequence $\{k_i\}$. Note that this patch $W_i(\mathcal{F}, X_i)$ is a tiling of attractor component A_i. Since the sequence of T_i is nested, it is the case that there is a path ω_i from vertex i to vertex $i-1$ such that $f_{\omega_i}(A_{i-1}) \subset A_i$ for all $i = 1, 2, \dots$. Let $\overleftarrow{\theta} \in \overleftarrow{\Sigma}$ be the concatenated path $\overleftarrow{\theta} = \overleftarrow{\omega}_1 \overleftarrow{\omega}_2 \overleftarrow{\omega}_3 \cdots$. Then

$$f_{\overleftarrow{\theta}|k_j} = f^{-1}_{\omega_1} \circ f^{-1}_{\omega_2} \circ \cdots \circ f^{-1}_{\omega_j} = (f_{\omega_j} \circ f_{\omega_{k-1}} \circ \cdots \circ f_{\omega_1})^{-1}$$

maps the tiling $W_j(\mathcal{F}, X_j)$ of A_j onto T_j. Because each tile in $W_j(\mathcal{F}, X_j)$ is of the form $f_\sigma(A_{\sigma^+})$ where $\sigma^- = j$, $|\sigma| = k_j$, we have

$$t = f_{\overleftarrow{\theta}|k_j} \circ f_\sigma(A_{\sigma^+}) \qquad \text{where} \qquad \sigma^- = \theta^-_{k_j},\ |\sigma| = k_j.$$

But these are exactly the tiles in $T(\mathcal{F}, \overleftarrow{\theta}, k_j)$ as defined in Equation (4.1). □

6. Every Self-Similar Tiling is a TGIFS Tiling

This section concerns self-similar tilings as introduced by Thurston [19]. Basically, a self-similar tiling T is a tiling of $\mathbb{R}^d$ for which there exists a similarity transformation $\phi : \mathbb{R}^d \to \mathbb{R}^d$ with scaling ratio $\lambda(\phi) > 1$ such that, for all $t \in T$, the large tile $\phi(t)$ is tiled, in turn, by tiles in T. The following example shows that an additional requirement is needed in a formal definition of self-similar.

Example 6.1. Let Q consists of two intervals on the real line, I_1 of length 1 and I_2 of length 2. Let T be *any* tiling of $\mathbb{R}$ with prototile set Q such that

(1) the origin is located at an endpoint of a tile, and
(2) the endpoints of each tile of length 2 are located at even coordinates on $\mathbb{R}$.

Let $\phi(x) = 2x$ for all $x \in \mathbb{R}$. Then for every tile $t \in T$, its image $\phi(t)$ is tiled by tiles in T for all $t \in T$. The issue is that T can be quite random. For example, construct a tiling as follows. Moving along the positive real line starting at the origin, place a random number of tiles of length 2. Follow that by a random even number of tiles of length 1, and repeat in this fashion moving in the positive direction; similarly on the negative real line.

To avoid the randomness in Example 6.1 and similar examples, we will require that, for any tile $t \in T$, the tiling of $\phi(t)$ by tiles in T depends uniquely on the prototile type of t. To impose this requirement formally we introduce the following notions.

Definition 6.1 (Induced Tiling)**.** Let T be a tiling of $\mathbb{R}^d$ with prototile multiset Q, and let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a similarity with scaling ratio $\lambda(\phi) > 1$. We allow Q to be a multiset to allow for multiple ways for $\phi(p)$, $p \in Q$, to be tiled. For each $p \in Q$, let T_p be a tiling of $\phi(p)$ by tiles in Q. Call the set $\{T_p : p \in Q\}$ of patches a *ϕ-tiling rule*. Let $H := \{h_t : t \in T\}$ be a set of isometries, called *tiling isometries*, such that $t = h_t(p)$, where $p \in Q$ is the type of tile $t \in T$. If every $p \in Q$ has trivial symmetry group, then H is uniquely determined. This is the case, for example, in the Definition of self-similar in [7, 19] where the tilings are by copies of the prototiles by translation rather than by isometry as is the case here.

For each $t \in T$, the isometry $\widehat{h}_t := \phi\, h_t\, \phi^{-1}$ maps $\phi(p)$ onto $\phi(t)$. Now

$$\phi(t) = \phi\big(h_t(p)\big) = \widehat{h}_t\, \phi(p) = \bigcup_{q \in T_p} \widehat{h}_t(q).$$

Note that $p := p_t$ in the equation above depends on t. Given the tiling T and tile $t \in T$, we now obtain the *induced tiling* of $\phi(t)$ and the *induced tiling* of T defined by

$$T_{\phi(t)} := \big\{\widehat{h}_t(q) : q \in T_{p_t}\big\} \qquad \text{and} \qquad T_\phi := \bigcup_{t \in T} T_t.$$

The following definition is basically that of Thurston and Kenyon [7], removing the restriction that the isometries in H be translations.

Definition 6.2 (Self-Similar)**.** Given a prototile set Q, a Q-tiling T of $\mathbb{R}^d$ is *self-similar* if there is a similarity ϕ, a ϕ-tiling rule, and a set H of tiling isometries such that $T_\phi = T$. In particular, the image $\phi(t)$ is tiled by tiles in T for all $t \in T$.

Call a similarity transformation ϕ *proper with respect to a tiling* T if the fixed point of ϕ lies in the interior of a tile. As shown in Example 6.2 below, Theorem 6.1 may fail without this condition.

Theorem 6.1. *Every self-similar tiling with proper self-similarity ϕ is an $\mathcal{F}$-tiling for some TGIFS $\mathcal{F}$.*

Proof. Given a self-similar tiling T, define a TGIFS $\mathcal{F}(T) = (G, F)$ as follows. The vertex set of G is $Q := Q(\mathcal{F})$. The edge set E of G is defined by the ϕ-rule as follows. The ϕ-tiling rule can be expressed as

$$\phi(p) = \bigcup_{q \in T_p} h_{p,q}(q) \tag{6.1}$$

for some isometries $h_{p,q}$. In G add an edge e directed from vertex p to vertex q and let $f_e = \phi^{-1} h_{p,q}$. Note that $\lambda(f_e) = 1/\lambda(\phi) < 1$ for all $e \in E$. Equation (6.1) then becomes

$$p = \bigcup_{e \in E_p} f_e(p_{e^+}),$$

corresponding to Equation (3.2) in the definition of a GIFS attractor. Take $F = \{f_e : e \in E\}$. Now $\mathcal{F}(T) = (G, F)$ is a TGIFS with scaling ratio $1/\lambda(\phi)$. Note that the prototiles of Q are isometric copies of the attractor components of $\mathcal{F}(T)$. It remains to shown that there exists a $\overleftarrow{\theta} \in \overleftarrow{\Sigma}$ such that $T = T(\overleftarrow{\theta})$.

Let t_0 be the tile in the self-similar tiling T that contains the fixed point O of ϕ in its interior. If t_0 is of type $p \in Q$, then there is no loss of generality in assuming that $t_0 = p$, and consequently there is an edge (loop) e in G from p to p labeled $f_e = \phi^{-1}$. Consider the parameter $\overleftarrow{\theta} := e\,e\,e \cdots \in \overleftarrow{\Sigma}$ that winds infinitely many times around the loop e. Then $f_{\overleftarrow{\theta}|k} := f_e \circ f_e \circ \cdots \circ f_e = \phi^{-k}$ for all non-negative integers k. We will show that $T = T(\overleftarrow{\theta})$.

Let H be a set of tiling isometries for T such that $T_\theta = T$. We claim that there a set $H' = \{h'_t : t \in T\}$ of tiling isometries such that each $h'_t(p_t) = h_t(p_t)$ for all $t \in T$, and each $h'_t \in H'$ has the form

$$h'_t = \phi^k \circ f_\sigma, \tag{6.2}$$

where $\sigma \in \overleftarrow{\Sigma}^*$ and $|\sigma| = k$. If this is the case, then

$$t = (\phi^k \circ f_\sigma)(p) \in T(\overleftarrow{\theta}, k),$$

where $p = p_{\sigma^+} \in Q$. This is exactly as in Equation (4.1) in Definition (4.2), showing that T is an $\mathcal{F}$-tiling and completing the proof.

It only remains to prove the claim. Since the fixed point O lies interior to t_0, any tile $t \in T$ is contained in $X_k := \phi^k(t_0)$ for some integer k. Note that the sets X_k are nested. The existence of such a set H' of tiling isometries of the form in Equation (6.2) is proved by induction on k. If $t \in X_1 = \phi(t_0) = \phi(p)$, then by Equation (6.1) we have $t = h_{p,p_1}(p_1) = \phi\, f_e(p_1)$ for an edge e directed from p to p_1, where p_1 is the type of tile t. Take $h'_t = h_{p,p_1} = \phi\, f_{e_1}$, which is of the form in Equation (6.2). Assume that h'_t has been defined in the form of Equation (6.2) for all $t \in X_k$, and let $t' \in X_{k+1}$. Then $t' \in \phi(t)$ for some $t \in X_k$, where, by the induction hypothesis, $h'_t = \phi^k \circ f_\sigma$ holds, where $|\sigma| = k$ and σ is a path from p to p_t. By the requirement that $T_\phi = T$ we have $t' = \widehat{h}_t(q)$, where $q \in T_{\phi(p_t)}$ is the type of tile t', i.e., $q = p_{t'}$. By the tiling rule $q = h_{p_t, p_{t'}}(p_{t'}) = (\phi^{-1} \circ f_e)(p_{e^+})$, where e is directed from p_t to $p_{t'}$. Therefore

$$\begin{aligned} t' = \widehat{h}_t(q) = t' &= \big(\phi\, h_t\, (\phi)^{-1}\big)\big(\phi\, f_e(p_{e^+})\big) = \big(\phi\, h'_t\, (\phi)^{-1}\big)\big(\phi\, f_e(p_{e^+})\big) \\ &= \phi\big(\phi^k \circ f_\sigma\big) f_e(p_{e^+}) = \phi^{k+1} \circ f_{\sigma'}(p_{t'}) = \phi^{k+1} \circ f_{\sigma'}(A_{\sigma'^+}), \end{aligned}$$

where $\sigma' = \sigma\, e$. □

Example 6.2 (Counterexample to Theorem 6.1 when the similarity is not assumed to be proper)**.** For the Ammann chair TGIFS (G, F) in Example 4.1, $T := T(\overleftarrow{e}_1 \overleftarrow{e}_1, \cdots)$ tiles the first quandrant of the plane. The union of T and copies of T obtained by reflected in the x and y-axes and by rotation by π about the origin tiles the plane and is self similar but is not a TGIFS-tiling.

Remark 6.1. In [7], it is part of the definition of a self-similar tiling T that T be repetitive (see Definition 4.4). That every self-similar tiling with a proper self-similarity must be repetitive follows from Theorem 6.1 and Theorem 7.2.

Proposition 6.1. *For a TGIFS, if parameter $\overleftarrow{\theta}$ is eventually periodic, then $T(\overleftarrow{\theta})$ is self-similar.*

Proof. We must produce a similarity ϕ, a ϕ-tiling rule, and a set H of tiling isometries such that $T_\phi = T$. Since $\overleftarrow{\theta}$ is eventually periodic, there exist $\overleftarrow{\theta}_0, \overleftarrow{\theta}_1 \in \overleftarrow{\Sigma}^*$ such that $\overleftarrow{\theta} = \overleftarrow{\theta}_0\, \overleftarrow{\theta}_1\, \overleftarrow{\theta}_1\, \overleftarrow{\theta}_1 \cdots$. Let $\phi = f_{\overleftarrow{\theta}_0} \circ f_{\overleftarrow{\theta}_1} \circ (f_{\overleftarrow{\theta}_0})^{-1}$, which is a similarity. For $t \in T(\overleftarrow{\theta})$ there is a least integer k such that $t \in T(\overleftarrow{\theta}, |\overleftarrow{\theta}_0| + k|\overleftarrow{\theta}_1|)$, in which case $t = f_{\overleftarrow{\theta}_0} \circ (f_{\overleftarrow{\theta}_1})^k \circ f_\sigma(A^+_\sigma)$ for an appropriate $\sigma \in \Sigma^*$ with $|\sigma| = k+1$. Let $h_t = f_{\overleftarrow{\theta}_0} \circ (f_{\overleftarrow{\theta}_1})^k \circ f_\sigma$ and $H = \{h_t : t \in T\}$ the set of tililng isometries. Since each prototile (attractor component) p is tiled by $\{f_e(p_{e^+}) : e \in E_p\}$, the ϕ-tiling rule is taken to be $T_{\phi(p)} = \{\phi \circ f_e(p_{e^+}) : e \in E_p : e \in E_p\}$. Now an arbitrary tile in the induced tiling T_ϕ is of the form

$$\begin{aligned} t = \widehat{h}_t\, \phi\, f_e(p_{e^+}) &= \phi\, h_t\, \phi^{-1}\, \phi\, f_e(p_{e^+}) = \phi\, f_{\overleftarrow{\theta}_0}\, (f_{\overleftarrow{\theta}_1})^k\, \sigma\, \phi\, f_e(p_{e^+}) \\ &= f_{\overleftarrow{\theta}_0}\, (f_{\overleftarrow{\theta}_1})^{k+1}\, \sigma\, \phi\, f_e(p_{e^+}) = f_{\overleftarrow{\theta}_0}\, (f_{\overleftarrow{\theta}_1})^{k+1}\, \sigma'(p_{\sigma'^+}), \end{aligned}$$

where $\sigma' = \sigma\varepsilon$. The last expression in the equality is again a tile in $T(\overleftarrow{\theta})$. □

7. Tiling Hierarchy

Definition 7.1. Let $\mathcal{F}$ be a TGIFS with prototile set $Q := Q(\mathcal{F})$ and scaling ratio $\lambda := \lambda(\mathcal{F})$. A *hierarchy* for an $\mathcal{F}$-tiling T is a sequence $T_0, T_1, T_2, \dots$ of tilings such that $T_0 = T$ and, for all integers $k \geq 0$, the following properties hold:

(1) T_k is a tiling with prototile set $\{(1/\lambda)^k\, p \,:\, p \in Q\}$.
(2) Every tile in T_k is contained in a tile of T_{k+1}.

Call the tiling T_k the k^{th} *level* in the hierarchy of T. According to Theorem 7.1 below, a hierarchy exists for every TGIFS-tiling. A TGIFS $\mathcal{F}$ for which every $\mathcal{F}$-tiling has exactly one hierarchy is called *uniquely hierarchical.* The standard tiling of the plane by squares is an example of a TGIFS-tiling that is not uniquely hierarchical.

Call two GIFSs $\mathcal{F} = (G, F)$ and $\mathcal{F}' = (G, F')$ *conjugate* if there is a function g such that $F' = gFg^{-1} = \{f_g := gfg^{-1} : f \in F\}$. We will use the notation $\mathcal{F}_g$ for this conjugate of $\mathcal{F}$. The proof of the following lemma is routine.

Lemma 7.1. *If $\mathcal{F} = (G, F)$ is a TGIFS with attractor $A = (A_1, \dots, A_n)$ and g is a similarity transformation, then the conjugate $\mathcal{F}_g$ is a TGIFS with attractor $gA = \big(g(A_1), \dots, g(A_n)\big)$. Moreover, $T(\mathcal{F}_g, \overleftarrow{\theta}) = gT(\mathcal{F}, \overleftarrow{\theta})$ for all $\overleftarrow{\theta} \in \mathcal{P}(\mathcal{F})$.*

Theorem 7.1. *Let $\mathcal{F}$ be a TGIFS. Every $\mathcal{F}$-tiling T has a hierarchy $T = T_0, T_1, T_2, \dots,$ such that each T_k, $k = 0, 1, 2, \dots,$ is a TGIFS tiling. Specifically, for $T = T(\overleftarrow{\theta})$ and $j \geq k \geq 0$, a hierarchy is given by*

$$\begin{aligned} T_k =& T(\mathcal{F}_{f_{\overleftarrow{\theta}|k}}, S^k \overleftarrow{\theta}) \\ =& f_{\overleftarrow{\theta}|k} T(\mathcal{F}, S^k \overleftarrow{\theta}) \\ =& \big\{ (f_{\overleftarrow{\theta}|j} \circ f_\sigma)(A_{\sigma^+}) \,:\, \sigma^- = \theta_j^-,\, |\sigma| = j - k \big\}. \end{aligned} \tag{7.1}$$

Proof. The second equality in Equation (7.1) follows from Lemma 7.1. The third equality follows from the Definition 4.2 of an $\mathcal{F}$-tiling.

It remains to show that $T = T_0, T_1, T_2, \dots$ is indeed a hierarchy. The set of attractor compondents $Q = \{A_1, A_2, \dots, A_n\}$ is a prototile set for T. Every tile $t \in T_k$, as defined in Equation (7.1), has the form

$$t := f_{\overleftarrow{\theta}|k} \circ f^{-1}_{\theta_{k+1}} \circ \cdots \circ f^{-1}_{\theta_{k+j}} \circ f_\sigma(A_{\sigma^+}), \qquad \text{where} \qquad \sigma_1^- = \theta_j^-,\, |\sigma| = j - k.$$

Note that t is isometric to $(1/\lambda)^k (A\sigma^+)$. Therefore condition (1) in Definition 7.1 is satisfied.

It follows from the formula above for t and from Equation (3.2) that

$$\begin{aligned} t &= f_{\overleftarrow{\theta}|k-1} \circ f^{-1}_{\theta_k} \circ \cdots \circ f^{-1}_{\theta_j} \circ f_\sigma \Big(\bigcup_{e \in E_{\sigma^+}} f_e(A_{e^+}) \Big) \\ &= \bigcup \Big\{ f_{\overleftarrow{\theta}|k-1} \circ f^{-1}_{\theta_k} \circ \cdots \circ f^{-1}_{\theta_j} \circ f_\sigma \circ f_e(A_{e^+}) : e \in E_{\sigma^+} \Big\} \\ &\subset \bigcup \Big\{ f_{\overleftarrow{\theta}|k-1} \circ f^{-1}_{\theta_k} \circ \cdots \circ f^{-1}_{\theta_j} \circ f_\omega(A_{\omega^+}) : \omega^- = \theta_{k+j}{}^-,\, |\omega| = j - (k-1) \Big\}. \end{aligned}$$

Because each $f_{\overleftarrow{\theta}|k-1} \circ f^{-1}_{\theta_k} \circ \cdots \circ f^{-1}_{\theta_j} \circ f_\omega(A_{\omega^+})$ in the line above is a tile in T_{k-1}, each tile in T_k is, in turn, tiled by a patch in T_{k-1}. This proves condition (2) in Definition 7.1. $\square$

Remark 7.1. Theorem 7.1 can be extended to show that there is a two-sided hierarchy

$$\dots T_{-2}, T_{-1}, T_0, T_1, T_2, \dots$$

of tilings such that $T_0 = T$ and that satisfies properties (1) and (2) in Definition 7.1 for all $k \in \mathbb{Z}$. To do this let $\cdots \overleftarrow{\theta}_{-2} \overleftarrow{\theta}_{-1} \overleftarrow{\theta}_0 \overleftarrow{\theta}_1 \overleftarrow{\theta}_2 \cdots$ be an arbitrary bi-infinite path that extends the parameter $\overleftarrow{\theta_1\theta_2} \cdots \in \mathcal{P}$. For $k \in \mathbb{Z}$, we extend the notation, which was previously defined for $k \in \mathbb{N}$. For $\overleftarrow{\theta} = \cdots \overleftarrow{\theta}_{-2} \overleftarrow{\theta}_{-1} \overleftarrow{\theta}_0 \overleftarrow{\theta}_1 \overleftarrow{\theta}_2 \cdots$ let

$$S^k(\overleftarrow{\theta}) = \overleftarrow{\theta}_{k+1} \overleftarrow{\theta}_{k+2} \cdots$$

$$f_{\overleftarrow{\theta}|k} = \begin{cases} f^{-1}_{\theta_1} \circ f^{-1}_{\theta_2} \circ \cdots \circ f^{-1}_{\theta_k} & \text{for} \quad k \geq 1 \\ f_{\theta_0} \circ f_{\theta_{-1}} \circ \cdots \circ f_{\theta_{k+1}} & \text{for} \quad k \leq -1. \end{cases}$$

By convention $f_{\overleftarrow{\theta}|0}$ is the identity map. Then Theorem 7.1 and its proof go through without change.

Corollary 7.1. *If TGIFS $\mathcal{F}$ is uniquely hierarchical, then every $\mathcal{F}$-tiling is non-periodic.*

Proof. The proof that a $\mathcal{F}$-tiling is non-periodic is a standard one. A translation of displacement distance D that preserves the tiling T must, due to uniqueness, preserve the level k tiling T_k for all k. If k is sufficiently large, however, the size of the tiles in T_k makes a displacement of fixed distance D impossible. $\square$

Theorem 7.2. *If $\mathcal{F}$ is a TGIFS on $\mathbb{R}^d$, then every $\mathcal{F}$-tiling of $\mathbb{R}^d$ is repetitive.*

Proof. Any patch X of an $\mathcal{F}$-tiling T is contained in $T(\overleftarrow{\theta}, k)$ for some k. Given $n \geq 1$, there exists a real number R such that every ball of radius R contains a tile of $T_n(\overleftarrow{\theta})$, the n^{th} level of the hierarchy of Theorem 7.1. Therefore it suffices to show that there is an n such that every tile of $T_n(\overleftarrow{\theta})$ contains an isometric copy of $T(\overleftarrow{\theta}, k)$.

Let m be the greatest common divisor of all closed paths in G. We claim that there exists an M such that if $n \geq M$ and $n \equiv 0 \pmod m$ the following holds: for any vertex v of G there is a circuit through v of length n. To prove the claim, let C be a circuit that contains every vertex of G. There are circuits $C_1, C_2, \dots, C_k$ of lengths $mq_1, mq_2, \dots, mq_k$ such that that $\gcd(q_1, q_2, \dots, q_k) = 1$. An elementary result in number theory states there exists an N such that if integer $s \geq N$, then there exists positive integers $a_1, a_2, \dots, a_k$ such that $\sum_{i=1}^k a_i q_i = s$. Therefore, by traversing C with detours around the circuits $C_1, C_2, \dots, C_k$ sufficiently many times, for any vertex v, there is a circuit containing v of length $ms + |C| = m(s + |C|/m)$, an integer multiple of m. Taking $M = mN + |C|$, the claim is proved.

Given k, we now show that there is an integer n such that every tile of $T_n(\overleftarrow{\theta})$ contains an isometric copy of $T(\overleftarrow{\theta}, k)$. Let D denote the diameter of G, i.e., the greatest directed distance between any two vertices. Let $n = M + k + D$, where M is as in the paragraph above. Let $u = \theta_k^-$, i.e., the last vertex of $\overleftarrow{\theta}|k$ in $\overleftarrow{\Sigma}$. Let $K \geq n$ and let $w = \theta_K^-$. A tile t of $T_n(\overleftarrow{\theta})$ by definition has the form

$$t = f_{\overleftarrow{\theta}|K} \circ f_\sigma(A_{\sigma^+}) \qquad \text{where} \quad |\sigma| = K - n, \, \sigma^- = \theta_K^+.$$

Let $v = \sigma^+$. We claim that there exists a path γ from v to u in G of length $n - k$. Assume that the claim is true, and consider the patch of tiles

$$T = \{ f_{\overleftarrow{\theta}|K} \circ f_\sigma \circ f_\gamma \circ f_\omega(A_{\omega^+}) \, : \, |\omega| = k, \, \omega^- = \gamma^+ \}.$$

Since $|\overleftarrow{\theta}|K|=|K|=(K-n)+(n-k)+k=|\sigma|+|\gamma|+|\omega|$, each tile in T is a tile in $T(\overleftarrow{\theta})$. Moreover, since $\gamma^+=u$, the tiling T is an isometric copy of $T(\overleftarrow{\theta},k)$. It now only remains to prove the claim.

Because G is strongly connected, there is a simple (no crossing) path $\widehat{\gamma}$ from v to u of length at most D, and also a simple path δ from u to w. The concatenation of δ and the path $\theta_K\,\theta_{k-1}\cdots\theta_{k+1}$ form a cycle which implies that $(K-k)+|\delta|\equiv 0 \pmod m$. Similarly, the concatenation of the paths $\widehat{\gamma},\delta$ and σ form a cycle which implies that $(K-n)+|\widehat{\gamma}|+|\delta|\equiv 0 \pmod m$. The two congruences yields $n-k-|\widehat{\gamma}|\equiv 0 \pmod m$. Therefore, by the definition of M, there is a circuit β containing vertex v such that $|\beta|=n-k-|\widehat{\gamma}|$ if $n-k-|\widehat{\gamma}|\geq M=n-k-D$, which is equivalent to $D\geq|\widehat{\gamma}|$, which is clearly true. Taking $\gamma=\beta\,\widehat{\gamma}$ we have

$$|\gamma|=|\beta|+|\widehat{\gamma}|=(n-k-|\widehat{\gamma}|)+|\widehat{\gamma}|=n-k. \qquad \square$$

Given a TGIFS $\mathcal{F}=(G,F)$, consider the TGIFS $\mathcal{F}=(G',F')$ obtained as follows. Let v be a vertex of G. To obtain G' add to G a new vertex; call it v' with an edge (v',i) for ever edge (v,i) of G and an edge (i,v') for ever edge (i,v) of G. To obtain F', let $f_{(v',i)}=f_{(v,i)}$ and $f_{(i,v')}=f_{(i,v)}$ for all i. It is not hard to see that the set of $\mathcal{F}'$-tilings are exactly the set of $\mathcal{F}$-tilings. The new vertex v' is redundant. This motivates the following definition. For a TGIFS (G,F) and a vertex r of G, recall the notation from Equation (5.1) $W_i=W_i(\mathcal{F})=\{f_e(A_{e^+}):e\in E_i\}$ for the patch that tiles attractor component A_i.

Definition 7.2. Call a TGIFS $\mathcal{F}$ *non-redundant* if $W_i(\mathcal{F})=W_j(\mathcal{F})$ if and only if $i=j$. Call a TGIFS $\mathcal{F}$ *asymmetric* if, for all vertices i, the only symmetry of A_i that preserves its tiling $W_i(\mathcal{F})$ is the identity.

Theorem 7.3. *Let $\mathcal{F}$ be a uniquely hierarchical TGIFS that is non-redundant and asymmetric. If $\overleftarrow{\theta},\overleftarrow{\theta}'\in\mathcal{P}(\mathcal{F})$, then*

(1) *$T(\overleftarrow{\theta})\cong T(\overleftarrow{\theta}')$ if and only if $S^j(\overleftarrow{\theta})=S^j(\overleftarrow{\theta}')$ for some integer j, and*
(2) *$T(\overleftarrow{\theta})=T(\overleftarrow{\theta}')$ if and only if $\overleftarrow{\theta}=\overleftarrow{\theta}'$.*

Proof. Proof of statement (1):

One direction is statement (3) of Theorem 4.1.

In the other direction, we will show that if two tilings $T(\overleftarrow{\theta})$ and $T(\overleftarrow{\theta}')$ are congruent, then there is an integer j such that $S^j(\overleftarrow{\theta})=S^j(\overleftarrow{\theta}')$. Assume that $T(\overleftarrow{\theta})\cong T(\overleftarrow{\theta}')$. Let t_0 be an arbitrary tile in $T_0=T(\overleftarrow{\theta})$ and t_0' the corresponding tile in $T_0'=T(\overleftarrow{\theta}')$ under the isometry; call it $h:\mathbb{R}^d\to\mathbb{R}^d$. In the hierarchy, let $t_k\in T_k(\overleftarrow{\theta})$ be such that $t_k\subset t_{k+1}$ for all $k\geq 0$; define t_k' similarly.

Let $j\geq k$ be sufficiently large that $t_0\in T(\theta,j)$ and $t_0'\in T(\theta',j)$. Because it is assumed that $\mathcal{F}$ is uniqely hierarchical, the hierarchy must be the one provided in Theorem 7.1. Therefore, for all $j\geq k\geq 0$, if $t\in T_k$ and $t'\in T_k$, then

$$t_k=f_{\overleftarrow{\theta}|j}f_\sigma(A_{\sigma^+}) \qquad\qquad t_k'=f_{\overleftarrow{\theta}'|j}f_\omega(A_{\omega^+}),$$

where $|\sigma|=j-k$, $\sigma^-=(\theta_j)^-$ and $|\omega|=j-k$, $\omega^-=(\theta_j')^-$. Let $\sigma=\sigma_{j-k}\,\sigma_{j-k-1}\cdots\sigma_2\,\sigma_1$ and $\omega=\omega_{j-k}\,\omega_{j-k-1}\cdots\omega_2\,\omega_1$. Note that, for $i=1,2,\ldots,j-k$, the function f_{σ_i} gives the embedding of t_{i-1} into t_i. The same is true for t_k'. If h is an isometry that takes $T(\overleftarrow{\theta})$ onto $T(\overleftarrow{\theta}')$, then h takes t_{k+1} onto t_{k+1}' and the tiling of t_{k+1} by a patch in T_k onto the tiling of t_{k+1}' by a patch in T_k'. By the assumption of non-redundancy and asymmetry, it is the case that $\sigma_i=\omega_i$, $i=1,2,\ldots,j-k$.

Now take any integer $J>j$, so that

$$t_k=f_{\overleftarrow{\theta}|J}f_\sigma(A_{\sigma^+}) \qquad \text{where} \qquad \sigma=\theta_J\,\theta_{J-1}\cdots\theta_{j+1}\,\sigma_{j-k}\,\sigma_{j-k-1}\cdots\sigma_2\,\sigma_1$$

$$t_k'=f_{\overleftarrow{\theta}'|J}f_\omega(A_{\omega^+}) \qquad \text{where} \qquad \omega=\theta_J'\,\theta_{J-1}'\cdots\theta_{j+1}'\,\omega_{j-k}\,\omega_{j-k-1}\cdots\omega_2\,\omega_1.$$

As above, we have $\theta'_i = \theta_i$ for all $i > j$.

Proof of statement (2):

Given parameters $\overleftarrow{\theta}, \overleftarrow{\theta}' \in \widehat{\mathcal{P}}$, we will show that $T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$ implies $\overleftarrow{\theta} = \overleftarrow{\theta}'$. Assume the contrary, that $\overleftarrow{\theta} \neq \overleftarrow{\theta}'$ and $T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$. Let k be the greatest integer such that $\overleftarrow{\theta}_i = \overleftarrow{\theta}_i'$ for $i \leq k$. By the assumption that $\mathcal{F}$ is uniquely hierarchical and by Theorem 7.1, $T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$ implies that $T(S^k\overleftarrow{\theta}) = T(S^k\overleftarrow{\theta}')$. Therefore we may assume, without loss of generality, that $\overleftarrow{\theta}_1 \neq \overleftarrow{\theta}'_1$, i.e., $\theta_1 \neq \theta'_1$. Let $v_1 = \theta_1^+, v_2 = \theta_1^-, v'_1 = \theta_1'^+, v'_2 = \theta_1'^-$. Since A_{v_1} and $A_{v'_1}$ are both tiles in $T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$ containing 0, they must be equal (see Definition 4.1); therefore $v_1 = v'_1$. Now

$$0 \in A_{v_1} = A_{\theta_1^+} \subset f_{\theta_1}^{-1}(A_{v_2}) \in T_1(\overleftarrow{\theta}).$$

Because the same equation holds for θ' we have

$$0 \in f_{\theta'_1}^{-1}(A_{v'_2}) \in T_1(\overleftarrow{\theta}').$$

However, because of the uniqueness of the hierarchy, $T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$ implies $T_1(\overleftarrow{\theta}) = T_1(\overleftarrow{\theta}')$. Therefore we have

$$f_{\theta_1}^{-1}(A_{v_2}) = f_{\theta'_1}^{-1}(A_{v'_2}).$$

By Theorem 7.1, $f_{\theta_1}^{-1}(A_{v_2}) \in T_1(\overleftarrow{\theta})$ and $f_{\theta'_1}^{-1}(A_{v'_2}) \in T_1(\overleftarrow{\theta}')$. By the uniqueness of the hierarcy $T_1(\overleftarrow{\theta}) = T_1(\overleftarrow{\theta}')$. But $f_{\theta_1}^{-1}(A_{v_2})$ is tiled by patch $W_{v_2} = \{f_{(\theta_1}^{-1} \circ f_e)(A_{e^+}) : e \in E_{v_2}\}$ in $T(\overleftarrow{\theta})$ and similarly $f_{\theta'_1}^{-1}(A_{v'_2})$ is tiled by patch $W_{v'_2} = \{f_{(\theta'_1}^{-1} \circ f_e)(A_{e^+}) : e \in E_{v_2}\}$ in $T(\overleftarrow{\theta}')$. Since $T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$ and it is assumed that $\mathcal{F}$ is non-redundant, it must be the case that $v_2 = v'_2$.

Now the isometry $f_{\theta'_1}^{-1} f_{\theta_1}$ is a symmetry of A_{v_2} taking the patch W_{v_2} onto $W_{v'_2}$. By the assumption of asymmetry, $f_{\theta_1} = f_{\theta'_1}$. This implies that $\theta_1 = \theta'_1$; otherwise the patch $W_{v_2} := \big\{f_e(A_{e^+}) : e \in E_{v_2}\big\}$ is overlapping. This contradicts $\overleftarrow{\theta}_1 \neq \overleftarrow{\theta}'_1$. □

Corollary 7.2. *If $\mathcal{F}$ is uniquely hierarchical, non-redundant and asymmetric, then there are an uncountable number of $\mathcal{F}$-tilings.*

Proof. The corollary follows from Theorem 7.3 because there are uncountably many parameters such that $S^j(\overleftarrow{\theta}) \neq S^j(\overleftarrow{\theta}')$ for all $j \geq 0$ for any distinct pair $\overleftarrow{\theta}, \overleftarrow{\theta}'$ of them. □

8. A Tiling Dynamical System

Definition 8.1. Let $B(r)$ denote a ball of radius r centered at the origin. The *tiling space* $\mathbb{T} = \mathbb{T}(\mathcal{F})$ of a TGIFS $\mathcal{F}$ is the set of all $\mathcal{F}$-tilings of $\mathbb{R}^d$ endowed with the following metric d:

$$d(T, T') = \inf\big\{\epsilon \,:\, T \text{ and } T' \text{ coincide on a patch covering } B(1/\epsilon)\big\}.$$

The *tiling map*

$$\begin{aligned} \mathcal{T} : \mathcal{P}(\mathcal{F}) &\to \mathbb{T}(\mathcal{F}) \\ \overleftarrow{\theta} &\mapsto T(\overleftarrow{\theta}) \end{aligned}$$

is a continuous map from the parameter space $\mathcal{P}(\mathcal{F})$ onto the tiling space $\mathbb{T}$.

Proposition 8.1. *For a TGIFS $\mathcal{F}$, the set of self-similar $\mathcal{F}$-tilings in $\mathbb{T}(\mathcal{F})$ is dense in the tiling space $\mathbb{T}(\mathcal{F})$.*

Proof. Let $T = T(\overleftarrow{\theta}) \in \mathbb{T}(\mathcal{F})$ be given. Let $\overleftarrow{\theta}(k) \in \mathcal{P}$ be chosen to have the property that $\theta(k)$ is eventually periodic and such that the first k edges of $\overleftarrow{\theta}(k)$ and $\overleftarrow{\theta}$ are the same. By the continuity of $\mathcal{T}$, it follows from $\lim_{k\to\infty} \overleftarrow{\theta}(k) = \overleftarrow{\theta}$ that $\lim_{k\to\infty} T\big(\overleftarrow{\theta}(k)\big) = T(\overleftarrow{\theta}) = T$.

Because $T\big(\overleftarrow{\theta}(k)\big)$ is self-similar by Theorem 4.1, the tiling T is in the closure of the set of self-similar $\mathcal{F}$-tilings. $\square$

Definition 8.2 ($\mathcal{F}$-dynamical system)**.** For a TGIFS $\mathcal{F}$, define the map $H : \mathbb{T} \to \mathbb{T}$ by

$$H(T) = f_{\theta_1}(T_1)$$

for all $T \in \mathbb{T}$. In other words, H is a map that takes a TGIFS tiling, up to a scaling down, to the next level in its hierarchy.

Theorem 8.1. *Let $\mathcal{F}$ be a non-redundant, asymmetric, uniquely hierarchical TGIFS. The two dynamical systems $(\mathcal{P}, S)$ and $(\mathbb{T}, H)$ are topologically conjugate discrete dynamical systems, the topological conjugation being the tiling map $\mathcal{T}$. As a commuting diagram we have*

$$\begin{tikzcd} \mathcal{P} \arrow[r, "S"] \arrow[d, "\mathcal{T}"'] & \mathcal{P} \arrow[d, "\mathcal{T}"] \\ \mathbb{T} \arrow[r, "H"'] & \mathbb{T} \end{tikzcd}$$

Proof. Statement (2) of Theorem 7.3 states that the tiling map $\mathbb{T}$ is bijective. A bijective continuous map on a metric space is a homeomorphism. Theorems 7.1 implies that

$$H\big(T(\overleftarrow{\theta})\big) = f_{\theta_1}T(\overleftarrow{\theta}) = f_{\theta_1}f_{\theta_1}^{-1}T_1(\overleftarrow{\theta}) = T(S\overleftarrow{\theta}). \qquad \square$$

Remark 8.1. A tiling space $\widetilde{\mathbb{T}} = \widetilde{\mathbb{T}}(\mathcal{F})$ of a TGIFS $\mathcal{F}$ can alternatively be defined as the set of all $\mathcal{F}$-tilings of $\mathbb{R}^d$ up to isometry endowed with the metric

$$\widetilde{d}(T, T') = \inf\big\{\epsilon : T \text{ and } h(T') \text{ coincide on a patch covering } B(1/\epsilon) \text{ for some isometry } h\big\}.$$

Call two parameters equivalent, denoted by $\overleftarrow{\theta} \sim \overleftarrow{\theta'}$, if $S^j\overleftarrow{\theta} = S^j\overleftarrow{\theta'}$ for some $j \geq 0$. If $\widetilde{\mathcal{P}} = \mathcal{P}/\sim$, then Theorem 7.3 implies that the tiling map

$$\begin{aligned} \widetilde{\mathcal{T}} : \widetilde{\mathcal{P}}(\mathcal{F}) &\to \widetilde{\mathbb{T}}(\mathcal{F}) \\ \langle \overleftarrow{\theta} \rangle &\mapsto \langle T(\overleftarrow{\theta}) \rangle, \end{aligned}$$

is well defined, where $\langle \cdot \rangle$ denotes the equivalence class. The tiling map $\widetilde{\mathbb{T}}$ is continuous with respect to a metric on $\widetilde{\mathcal{P}}$ defined so that $\langle \overleftarrow{\theta} \rangle$ and $\langle \overleftarrow{\theta}' \rangle$ are close if, for some j there is a large k such that $S^j(\overleftarrow{\theta}|k) = S^j(\overleftarrow{\theta}'|k)$, then $\widetilde{\mathcal{T}}$. Define $\widetilde{H} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$ by $\widetilde{H}(T) = \lambda(\mathcal{F})\, T$. The topological conjugacy is given by the following commuting diagram.

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\widetilde{S}"] \arrow[d, two heads, "\widetilde{\mathcal{T}}"'] & \mathcal{P} \arrow[d, two heads, "\widetilde{\mathcal{T}}"] \\ \widetilde{\mathbb{T}} \arrow[r, "\widetilde{H}"'] & \widetilde{\mathbb{T}} \end{tikzcd}$$

8.1. **Results using invariants.** Let $\mathcal{F}$ be a TGIFS. Throughout this section, it is assumed that $\mathcal{F}$ is non-redundant, asymmetric and uniquely hierarchical, or at least that the tiling map $\mathcal{T}$ induces a topological conjugacy as in the diagram above.

Definition 8.3. Let $p_k := p_k(\mathcal{F})$ denote the number of $\mathcal{F}$-tilings T such that $H^k(T) = T$. In other words, p_k is the number of $\mathcal{F}$-tilings such that its n^{th}-level hierarchical tiling T_k, scaled down by a factor $(f_{\overleftarrow{\theta}|k})^{-1}$, is equal to the original tiling T. In this sense, p_k counts the number of $\mathcal{F}$-tilings whose hierarchy $T = T_0, T_1, T_2 \ldots$ cycles with period k.

Theorem 8.2. *For a TGIFS $\mathcal{F}$ as above, a generating function for the sequence $\{p_k\}_{n=1}^{\infty}$ is given by*

$$\sum_{k=1}^{\infty} \frac{p_k}{k}\, x^k = \log\Big(\frac{1}{\det(I - xM)}\Big),$$

where M is the adjacency matrix of the digraph of $\mathcal{F}$.

Proof. An element in the parameter space $\mathcal{P}$ of $\mathcal{F}$ can be viewed as a one sided word in the alphabet V, where V is the set of vertices of the digraph $\overleftarrow{G}$. The parameter space $\mathcal{P}$ is clearly shift invariant. In the terminology of symbolic dynamics, the dynamical system $(\mathcal{P}, S)$ is a 1-step shift of finite type. This means that there is a finite set W of ordered pairs of elements of V, i.e. a set of edges in the complete digraph on V, such that $\mathcal{P}$ consists of all words (paths in $\overleftarrow{G}$) that do not contain an ordered pair (edge) in W.

The Artin-Mazur zeta function of a dynamical system (X, g) is defined by

$$\zeta(x) = \exp\Big(\sum_{k=1}^{\infty} \frac{q_k}{k}\, x^k\Big),$$

where q_k is the number of points of period n of X under the action of g. (Note that a point of period n is also a point of period any multiple of n.) For our shift of finite type $(\mathcal{P}, S)$, the number q_k is thus the number of parameters $\overleftarrow{\theta}$ such that $S^k \overleftarrow{\theta} = \overleftarrow{\theta}$, equivalently the number of closed paths in $\overleftarrow{G}$ of length k. The zeta function is a well-known invariant of topological conjugacy and, for a shift of finite type, can be computed by the Bowen-Lanford formula:

$$\zeta(x) = \frac{1}{\det(I - xM^t)} = \frac{1}{\det(I - xM)},$$

where M is the adjacency matrix of the digraph G and its transpose M^t is the adjacency matrix of the digraph $\overleftarrow{G}$. For our shift of finite type $(\mathbb{T}, H)$, the number q_k is thus the number of tilings $T \in \mathbb{T}$ such that $H^k(T) = T$, equivalently $q_k = p_k$.

According to Theorem 8.1, the two dynamical systems $(\mathbb{T}, H)$ and $(\mathcal{P}, S)$ are topologically conjugate and therefore have the same zeta function. Hence

$$\exp\Big(\sum_{k=1}^{\infty} \frac{p_k}{k}\, x^k\Big) = \frac{1}{\det(I - xM)}$$

and Theorem 8.2 follows. □

Example 8.1. A simple computer calculation gives the following series for the Ammann chair TGIFS of Example 4.1:

$$\sum_{k=1}^{\infty} \frac{p_k}{n}\, x^k = x + \frac{3x^2}{2} + \frac{4x^3}{3} + \frac{7x^4}{4} + \frac{11x^5}{5} + \frac{18x^6}{6} + \frac{29x^7}{7} + \frac{47x^8}{8} + \frac{76x^9}{9} + \cdots .$$

For example, there are 4 Ammann chair tilings for which the hierarchy cylces with period 3. Referring to the graph in the right panel of Figure 5, these correspond in $(\mathcal{P}, S)$, via topological conjugacy, to the 4 parameters $\overline{111}, \overline{132}, \overline{213}, \overline{321}$, where the individual digits are shorthand for the edges in $\overleftarrow{G}$ with those function subscripts, and the bar over the numbers means that the three numbers repeat.

Lemma 8.1. *Let $\mathcal{F}$ be a uniquely hierarchical TGIFS, and let $\overleftarrow{\theta}, \overleftarrow{\theta}' \in \mathcal{P}$. Then $T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$ if and only if $T(\theta, k) = T(\theta', k)$ for $k = 0, 1, 2, \ldots$.*

Proof. Since $T(\overleftarrow{\theta})$ is the nested union of the $T(\theta, k)$, clearly $T(\theta, k) = T(\theta', k)$ for $k = 0, 1, 2, \ldots$ implies that $T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$.

In the other direction, by Theorem 7.1, the union of the tiles in $T(\theta, k)$ is itself a tile $t_k \in T_k$; similarly $T(\theta', k)$ is a tile $t'_k \in T'_k$. Since $0 \in t_k \cap t'_k$ and $T := T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$ has a unique hierarchy, it follows that $T(\theta, k) = T(\theta', k)$. □

Definition 8.4. For a uniquely hierarchical TGIFS $\mathcal{F}$, let $N_k := N_k(\mathcal{F})$ denote the number of distinct (pairwise unequal) patches $T(\overleftarrow{\theta}, k)$ over all $\overleftarrow{\theta} \in \mathcal{P}$. In view of Lemma 8.1, the growth of the sequence $\{N_k\}_{k=1}^{\infty}$ is a measure of how fast $\mathcal{F}$-tilings can be distinguished by looking at these increasingly large finite patches.

Theorem 8.3. *For a TGIFS $\mathcal{F}$, let $\rho := \rho(\mathcal{F})$ denote the Perron-Frobenius eigenvalue of the adjacency matrix of the digraph of $\mathcal{F}$. Then asymptotically*

$$N_k \simeq e^{k \log \rho}, \qquad \textit{i.e.},$$

$$\lim_{k\to\infty} \sqrt[k]{N_k} = \rho.$$

Proof. Given a TGIFS $\mathcal{F} = (G, F)$, the topological entropy of the shift of finite type $(\mathcal{P}, S)$ is defined by

$$h(\mathcal{P}) = \lim_{k\to\infty} \frac{1}{k} \log \widehat{N}_k,$$

where $\widehat{N}_k$ is the number of paths of length k in the digraph $\overleftarrow{G}$, which equals the number of paths of length k in the digraph G. The same method used in the proof of statement (2) of Theorem 7.3 shows that $T(\overleftarrow{\theta}, k) = T(\overleftarrow{\theta}', k)$ if and only if $\overleftarrow{\theta}|k = \overleftarrow{\theta}'|k$. Therefore $N_k = \widehat{N}_k$ for all k.

For $(\mathcal{P}, S)$ (and more generally for any shift of finite type) it is well known that $h(\mathcal{P}) = \rho(\mathcal{F})$, the Perron-Frobenius eigenvalue of M. Therefore

$$\lim_{k\to\infty} \frac{1}{k} \log N_k = \lim_{k\to\infty} \frac{1}{k} \log \widehat{N}_k = \rho,$$

which is equivalent to $\lim_{k\to\infty} \sqrt[k]{N_k} = \rho$. □

Example 8.2. Applying Theorem 8.3 to the Ammann chair TGIFS $\mathcal{F}$ of Example 4.1 yields

$$N_k \simeq \tau^k \approx e^{.4812\, n},$$

where τ is the golden ratio.

9. The Existence of TGIFSs

TGIFSs are plentiful and easy to construct in dimension 1; see Theorem 9.1 below. Polygonal rep-sets in dimension 2, as defined in Section 3.3, have appeared in recreational websites as well as mathematical journals. These were mostly discovered in an ad hoc manner, some very clever. They include the Ammann chair tile in Example 4.1, Robinson's triangle variant of the Penrose tiles, and the pinwheel tile (the tile due to J. Conway, the tiling due to C. Radin). All of the above examples give rise to a TGIFS and the associated tilings obtained by the method provided in Section 4. According to Theorem 9.2 below, for every dimension d and for every Perron number ρ, there exists a TGIFS on $\mathbb{R}^d$ whose scaling ratio is $1/\rho$; the attractor components are boxes.

In general, however, families of non-polyhedral tilings in dimension $d \geq 2$ with self-replicating properties are hard to come by. As long as the functions in F are constractions, a GIFS (G, F) has an attractor. The difficulty in finding GIFSs in dimensions $d \geq 2$ whose attractor components may serve as tiles may be explained by the following conjecture.

Conjecture 9.1. Given a strongly connected digraph G and dimension d, let $\mathcal{G}$ denote the set of all GIFS (G, F), where the functions in F are similarities taking $\mathbb{R}^d$ onto $\mathbb{R}^d$. The topology of compact convergence can be put on the set of all similarities $f : \mathbb{R}^d \to \mathbb{R}^d$, and hence a topology of compact convergence can be assigned to $\mathcal{G}$ via the product topology on F. Given a connected digraph G, we conjecture that the set of GIFSs in $\mathcal{G}$ whose attractor components are non-overlapping and have nonempty interior is nowhere dense in $\mathcal{G}$ with respect to the topology of compact convergence.

Two known algebraic constructions that do lead to TGIFS-tilings use digit sets and Rauzy fractals. Examples based on these techniques appear in Section 9.2. Tilings of $\mathbb{R}^d$ by copies of a single digit tile are typically periodic. Digit tiles received considerable attention starting the 1980's; see [21] and references therein. Tilings using Rauzy fractals are typically non-periodic. The Rauzy fractal and its generalizations, often called central tiles, are usually obtained from symbolic dynamics or from numeration systems (β-expansions) and are also related to model sets obtained by the cut-and-project method. Although a general discussion of Rauzy tilings does not, the elegant GIFS approach due to Rao, Wen and Yang [15] does fall within the scope of this paper.

9.1. **Existence of TGIFs in terms of the scaling ratio and digraph.** A real number $\rho > 1$ is a *Perron number* if it is a real algebraic integer such that the moduli of all other Galois conjugates (roots of the minimal polynomial of ρ) are less than $|\rho|$. Call a real algebraic integer $\rho > 1$ a *weak-Perron number* if the moduli of all of its Galois conjugates are less than or equal to $|\rho|$. A real number $\rho > 1$ is a *Pisot number* if it is a real algebraic integer such that the moduli of all of its other Galois conjugates are less than 1. It is a *unit pisot number* if, in addition, the constant term in its minimal polynomial is ± 1.

Proposition 9.1. *If $\mathcal{F}$ is a TGIFS on $\mathbb{R}^d$ with scaling ratio λ, then $1/\lambda$ is a weak-Perron number.*

Proof. Let $\rho = 1/\lambda$. By condition (2) in Definition 4.1, we know that ρ^d is the Perron-Frobenius eigenvalue of $M(G)$. If the characteristic polynomial of $M(G)$ is $p(x)$, then ρ is a root of $\widehat{p}(z) = p(z^d)$, all of whose roots are less than or equal to ρ. □

Note that there exists no TGIFS whose digraph is a directed cycle.

Theorem 9.1. (1) *For any $0 < \lambda < 1$ such that $1/\lambda$ is a Perron number, there exists a primitive TGIFS on $\mathbb{R}$ whose scaling ratio is λ.*

(2) *For every strongly connected digraph $G = (V, E)$, not a directed cycle, there exists a primitive TGIFS $\mathcal{F} = (G, F)$ on $\mathbb{R}$ such that $f_e(x) = (1/\rho)\,(x + d_e)$ for all $e \in E$, where ρ is the Perron-Frobenius eigenvalue of the adjacency matrix $M(G)$ of G and $d_e \in \mathbb{Q}(\rho)$.*

Proof. Concerning statement (1), let $0 < \lambda < 1$ and let $\rho = 1/\lambda$. The result [10, Theorem 1] states that if ρ is a Perron number, then there is a primitive non-negative integral matrix $M = (m_{i,j})$, $i, j \in \{1, 2, \dots, n\}$ whose spectral radius is ρ. A TGIFS on $\mathbb{R}$, whose adjacency matrix is M, can be obtained as follows. Let the vertex set of G be $V = \{1, 2, \dots, n\}$. For each pair (i, j) of vertices, add $m_{i,j}$ edges from vertex i to vertex j, and label edge e with a function of the form $f_e(x) = (1/\rho)(x + d_e)$, where d_e is defined as follows. Let $(x_1, x_2, \dots, x_n)$ be a positive right eigenvector corresponding to the eigenvalue ρ, which exists by the Perron-Frobenius theorem. The set $\{x_1, x_2, \dots, x_n\}$ of coordinates is a solution to the linear system

$$x_j = \sum_{i=1}^{n} \lambda\, m_{j,i}\, x_i, \qquad j = 1, 2, \dots, n,$$

and hence can be chosen to lie in $\mathbb{Q}(\rho)$. Consider the intervals $I_j = [0, x_j]$, $j = 1, 2, \dots, n$, on the real line. From the system of linear equations above, the interval I_j can be tiled (often in many different ways) by translated copies of the intervals $[0, \lambda\, x_i]$, $i = 1, 2, \dots, n$. The d_e are the translation distances multiplied by ρ, which are sums of the x_is. Therefore $d_e \in \mathbb{Q}(\rho)$ for all $e \in E$.

The adjacency matrix $M(G)$ determines the digraph G, so, as long as the Perron-Frobenius eigenvalue of $M(G)$ does not equal 1, the proof of statement (2) is as in the paragraph above. Since $M(G)$ is a non-negative, integral matrix, the Perron-Frobenius eigenvalue equals 1 if and only if M is orthogonal which can occur if and only if $M(G)$ is a permuatation matrix if and only if G is a directed cycle. □

Theorem 9.2. *For any $0 < \lambda < 1$ such that $1/\lambda$ is a Perron number, there exists a primitive TGIFS on $\mathbb{R}^d$ whose scaling ratio is λ.*

Proof. The case $d = 1$ is Theorem 9.1. So let $\mathcal{F} = (G, F)$ be such a 1-dimensional TGIFS with scaling ratio $\lambda = 1/\rho$ and with attractor components $I_j = [0, x_j]$, $j = 1, 2, \dots, n$, intervals on the real line. Denote the vertex set of G by $V = \{1, 2, \dots n\}$ and the edge set by E. In dimension d, let $\mathcal{F}_d = (G_d, F_d)$ be the GIFS, where $G_d = (V_d, E_d)$ is the digraph with vertex set $V_d = V^d$ and edge set $E_d = E^d$. The edge $\mathbf{e} := (e_1, e_2, \dots, e_d) \in E_d$ joins vertex $\mathbf{i} := (i_1, i_2, \dots, i_d)$ to vertex $\mathbf{j} := (j_1, j_2, \dots, j_d)$ where $e_k = (i_k, j_k)$ for $k = 1, 2, \dots d$. Label edge $\mathbf{e}$ with the function $f_{\mathbf{e}}(y_1, y_2, \dots y_d) = \big(f_{e_1}(y_1), f_{e_2}(y_2), \dots, f_{e_d}(y_d)\big)$ to obtain F_d.

It is routine to check that $\mathcal{F}$ and $\mathcal{F}_d$ have the same scaling ratio. The GIFS $\mathcal{F}_d$ is primitive for the following reason. Since G is primitive, there exists a positive integer m such that, for any two vertices of G there is a directed path of length m from one to the other. Let $\mathbf{i} := (i_1, i_2, \dots, i_d)$ and $\mathbf{j} := (j_1, j_2, \dots, j_d)$ be two arbitrary vertices of G_d. For $k = 1, 2, \dots, d$, let $e_{k,1}\, e_{k,2} \cdots e_{k,m}$ be a path of length m in G from vertex i_k to vertex j_k. For $q = 1, 2, \dots, m$, let $\mathbf{e_q} = (e_{1,q}, e_{2,q}, \dots, e_{d,q})$. Then $\mathbf{e_1}\, \mathbf{e_2} \cdots \mathbf{e_m}$ is a path of length m from vertex $\mathbf{i}$ to vertex $\mathbf{j}$ in G_d.

For each $\mathbf{i} := (i_1, i_2, \dots, i_d) \in V_d$, let $B_{\mathbf{i}} = I_{i_1} \times I_{i_2} \times \cdots \times I_{i_d}$, which is a box in $\mathbb{R}^d$. We leave it as an exercise to verify that

$$B_{\mathbf{i}} = \bigcup_{\mathbf{e} \in (E_d)_{\mathbf{i}}} f_{\mathbf{e}}(B_{\mathbf{e}^+})$$

for all $\mathbf{i} \in V_d$ and that the union is non-overlapping. This shows that $\{B_{\mathbf{i}} : \mathbf{i} \in V_d\}$ is the set of attractor components of $\mathcal{F}_d$. □

Theorem 9.1 suggests the following questions.

Question 9.1. For which strongly connected digraphs G and integers $d \geq 2$, does there exist a TGIFS on $\mathbb{R}^d$ whose digraph is G?

Question 9.2. For the digraph G at the right in Figure 5, there exists two TGIFs on $\mathbb{R}^2$ whose attractor components are polygons. In each case, the two attractor components are related by a similarity transformation. In the first case, one of the attractor components is a right triangles with side lengths 1, $\sqrt{\tau}$, τ, where τ is the golden ratio. In the second case, the attractor components are the Ammann chair tiles shown on the left in Figure 6. It is a consequence of a result of Schmerl [18] that these are the only two TGIFSs with digraph G whose attractor components are polygons. Does there exist a TGIF on $\mathbb{R}^2$ with non-polygonal attractor? On $\mathbb{R}$, the well-known Fibonacci tilings are the $\mathcal{F}$-tilings for a TGIFS $\mathcal{F}$ with digraph G. In that sense the Fibonacci tilings and the Ammann chair tilings are "siblings", the TGIFS that generete them having the same digraph G. Do there exist such siblings for G in all dimensions?

9.2. **Two algebraic constructions of TGIFSs.**

Digit Tiling

Let L be a $d \times d$ integer matrix such that all eigenvalues of L are greater than 1 in modulus. Such a matrix L is called an *expanding matrix.* Let D denote a complete set of coset representatives of the quotient $\mathbb{Z}^d / L(\mathbb{Z}^d)$ with $0 \in D$. Such a set D is called a *digit set.* For each $d \in D$, let $f_d(x) = L^{-1}(x + d)$ and let $F = \{f_d : d \in D\}$. Then F is an IFS. Note that it is also the GIFS $\mathcal{F}(L, D) = (G, F)$, where digraph G consists of a single vertex with $|D|$ loops. For a proof of the following theorem see, for example, [8, 20].

Theorem 9.3. *If L is an expanding integer matrix and D is a digit set, then the attractor of $\mathcal{F}(L, D)$ has nonempty interior and is non-overlapping. In particular, if L is a similarity, then $\mathcal{F}(L, D)$ is a TGIFS.*

The attractor of $\mathcal{F}(L,D)$ is called a *digit tile* and the associated TGIFS-tilings are *digit tilings*. The tiling on the left in Figure 3 is a digit tiling where $L = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $D = \{(0,0),(1,0)\}$. The digit tile in this case is often referred to as the "twin dragon". Digit tilings have natural generalizations: (1) from the lattice $\mathbb{Z}^d$ to any d-dimensional lattice and (2) from the d-dimensional crystallographic group generated by d linearly independent translations, i.e., a lattice, to any d-dimensional crystallographic group.

Rauzy Tiling

Definition 9.1 (Dual GIFS)**.** A slightly more general definition of the following appers in [15]. Let $G = (V,E)$ be a strongly connected digraph of order $d+1$, not a cycle, and let ρ be the Perron-Frobenius eignevalue of the adjacency matrix of G. According to Theorem 9.1, there is a TGIFS $\mathcal{F} = (G,F)$ on $\mathbb{R}$ (often many such TGIFSs) such that

$$f_e(x) = \frac{1}{\rho}\,(x + d_e),$$

where $d_e \in \mathbb{Q}(\rho)$ for all $e \in E$. Such 1-dimensional TGIFSs can be obtained explicitly and easily using the method provided in the proof of Theorem 9.1. Call any such TGIFS a 1-dimensional TGIFS or simply a *d1-TGIFS*.

Let $\mathcal{F} = (G,F)$ be a d1-TGIFS. If ρ is a Pisot unit, then the *dual GIFS* $\mathcal{F}^* = (\overleftarrow{G}, F^*)$ of $\mathcal{F}$ is defined as follows. The reverse digraph $\overleftarrow{G}$ is exactly as defined in Section 4.2. Let $\rho = \rho_0, \rho_1, \dots, \rho_d$ be the Galois conjugates of ρ ordered as follows:

$$\rho > 1 > |\rho_1| \geq |\rho_2| \geq \cdots \geq |\rho_d|,$$

where complex conjugates appear consecutively. Let

$$B' = diag(\rho_1, \rho_2, \dots, \rho_d),$$

and replace each pair $z, \overline{z}$ of complex conjugates in B' by the 2×2 real block $\begin{pmatrix} Re\, z & -Im\, z \\ Im\, z & Re\, z \end{pmatrix}$ to obtain the matrix B. Since $1, \rho, \rho^2, \dots, \rho^d$ is a basis for $\mathbb{Q}(\rho)$, we have $x = \sum_{i=0}^d x_i \rho^i$ for any $x \in \mathbb{Q}(\rho)$, where the $x_i \in \mathbb{Q}$. The *dual* of x is defined as

$$x^* = \sum_{i=0}^{d} x_i (\rho_1^i, \dots, \rho_d^i)^t \in \mathbb{C}^d,$$

where pair $z, \overline{z}$ of complex conjugates is replaced with $Re\, z, Im\, z$. Equivalently, the star operator is the unique linear map $* : \mathbb{Q}(\rho) \to \mathbb{R}^d$ such that $(\rho\, x)^* = Bx^*$ for all $x \in \mathbb{Q}(\rho)$. Now set $F^* = \{f_{\overleftarrow{e}}(x) = Bx + d_e^* : e \in E\}$. The *dual* of $\mathcal{F}$ is $\mathcal{F}^* := (\overleftarrow{G}, F^*)$, which is a GIFS on $\mathbb{R}^d$.

The following theorem is essentially [15, Theorem 1.2].

Theorem 9.4. *The dual of a d1-TGIFS $\mathcal{F}$ for which $1/\lambda(\mathcal{F})$ is a Pisot unit is also a TGIFS.*

Remark 9.1 (on the Proof of Theorem 9.4)**.** A GIFS satisfies the OSC if there exists open sets $U_1, \dots, U_n$ such that

$$\bigcup_{e \in E_i} f_e(U_{e+}) \subset U_i$$

for $i = 1, 2, \dots, n$, where n is the order of the digraph of the GIFS. Three results lead to the proof of Theorem 9.4.

First, necessary and sufficient conditions for a GIFS to satisfy the OSC, in terms of certain "digits" is given in [11].

Second, a result in [15], using the above necessary and sufficient conditions, immediately implies that if d1-TGIFS $\mathcal{F}$ satisfies (1) the open set condition (OSC) and (2) that $1/\lambda(\mathcal{F})$ is a Pisot number, then $\mathcal{F}^*$ also satisfies the OSC. Concerning condition (1), the attractor components of a d1-TGIFS are closed intervals; hence the OSC is satisfied by taking the U_i as the set of corresponding open intervals. Condition (2) is, by assumption, satisfied. Therefore $\mathcal{F}^*$ satisfies the OSC.

Third, a result in [9] immediately implies that if $\mathcal{F}^* = (\overleftarrow{G}, F^*)$ is such that (1) the functions in F^* are affine of the form $f_e(x) = B(x) + d_e$ for all $e \in E$, (2) $\mathcal{F}^*$ satisfies the OSC, and (3) $1/\rho^* = |\det(B)|$, where ρ^* is the Perron-Frobenius eigenvalue of $M(\overleftarrow{G})$, then the attactor components of $\mathcal{F}^*$ have nonempty interior. Condition (1) is clearly true, and condition (2) follows immediatly from the previous result from [11]. Concerning condition (3), we have

$$|\det(B)| = |\rho_1 \cdot \rho_2 \cdots \rho_d| = \frac{1}{\rho} = \frac{1}{\rho^*}.$$

The second equality follows from the fact that ρ is a unit and the last equality from the fact that $M(\overleftarrow{G})$ is the transpose of $M(G)$.

Therefore the attractor components of $\mathcal{F}^*$ have nonempty interior. Moreover, because $\lambda(\mathcal{F}^*) = \sqrt{|\det(B)|} = 1/\sqrt{\rho^*}$, Proposition 4.1 implies that $\mathcal{F}^*$ is a TGIFS.

Example 9.1 (Rauzy tilings from a TGIFS)**.** GIFS duality is used here to obtain fractal, non-periodic TGIFS-tilings of $\mathbb{R}^2$. We start with an initial d1-TGIFS $\mathcal{F} = (G, F)$ on $\mathbb{R}$, where G is an order 3 strongly connected digraph and the Perron-Frobenius eigenvalue ρ of the adjacency matrix $M(G)$ is a Pisot unit. It is not hard to find many such d1-TGIFSs as follows. Regarding the 9 integer entries of $M(G)$ as variables, it is easy to solve for values of these variables so that (1) $\det(M(G)) = \pm 1$, (2) the degree 3 characteristic polynomial of $M(G)$ has a pair of complex roots, and (3) G is strongly connected. That $\rho > 1$ together with condition (2) immediately implies that ρ is a Pisot number. Condition (1) further implies that ρ is a unit. Because $M(G)$ is a 3×3 matrix, the dual $\mathcal{F}^*$ is a TGIFS on $\mathbb{R}^2$. With the digraph G in hand, we use Theorem 9.1 (and its proof) to obtain possibilities for F and hence possibilities for the d1-TGIFS $\mathcal{F}$. We then use Theorem 9.4 to obtain F^* and hence $\mathcal{F}^* = (\overleftarrow{G}, F^*)$. The tiling method of this paper is then used to produce the $\mathcal{F}^*$-tiling.

For each of the following examples we provide an adjacency matrix $M = M(G)$, which determines the digraph G, the characteristic polynomial $p(x)$ of M, whose Perron-Frobenius eigenvalue ρ is a Pisot unit and equals $1/\lambda(\mathcal{F})$, and the functions in F.

(1) The original Rauzy fractal is the non-overlapping union of three smaller similar copies of itself. A tiling based on the Rauzy fractal appears on the right in Figure 3. This TGIFS-tiling can be constructed from the dual of this d1-TGIFS:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad p(x) = x^3 - x^2 - x - 1 \qquad \begin{array}{l} f_{2,1}(x) = f_{31}(x) = f_{1,1}(x) = (1/\rho)x \\ f_{1,2}(x) = f_{2,3}(x) = (x-1)/\rho, \end{array}$$

where $f_{i,j}$ is the function in F that is the label of edge (i, j).

(2) A tiling from the the dual of the data below appears on the left in Figure 7.

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad p(x) = x^3 - x^2 - 1 \qquad \begin{array}{l} f_{2,1}(x) = f_{3,2}(x) = f_{1,1}(x) = (1/\rho)x \\ f_{1,3}(x) = (x-1)/\rho \end{array}$$

(3) The tilings on the left and right, respectively, in Figure 4 are from the duals of these d1-TGIFSs:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad p(x) = x^3 - x^2 - 2x - 1 \qquad \begin{array}{l} f_{3,1}(x) = f_{2,1}(x) = f_{1,1}(x) = (1/\rho)x \\ f_{1,3}(x) = f_{2,3}(x) = (x+\rho)/\rho \\ f_{1,2}(x) = (x+\rho+1)/\rho \end{array}$$

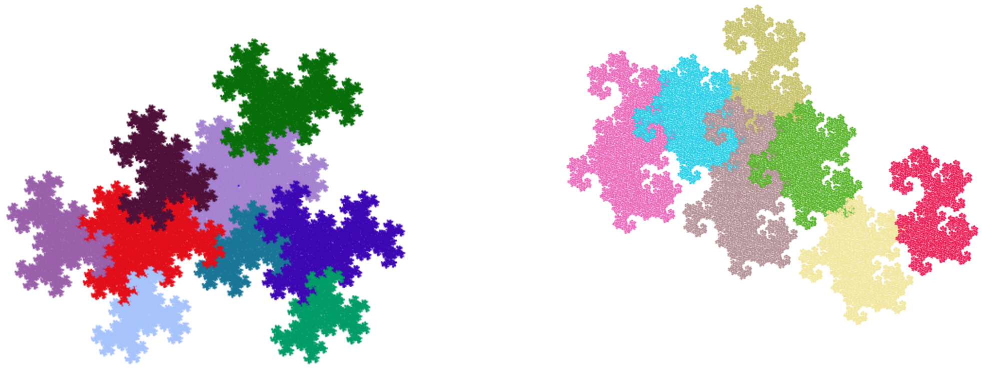

FIGURE 7. Tilings from duals given in Example 9.1.

(4)

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad p(x) = x^3 - 2x^2 - x - 1 \qquad \begin{aligned} f_{2,3}(x) &= f_{3,1}(x) = f_{1,1}(x) = (1/\rho)x \\ f_{2,1}(x) &= (x-1)/\rho \\ f_{1,2}(x) &= (x+\rho)/\rho \\ f_{2,2}(x) &= (x+\rho+1)/\rho \end{aligned}$$

(5) The tiling on the right in Figure 7 is from the dual of the d1-TGIFS below. The three prototiles also appear in Figure 2.

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad p(x) = x^3 - 3x^2 + 2x - 1 \qquad \begin{aligned} f_{2,3}(x) &= f_{3,3}(x) = f_{1,1}(x) = (1/\rho)x \\ f_{3,1}(x) &= f_{2,1}(x) = (x-1)\rho \\ f_{2,2}(x) &= (x+\rho)/\rho \\ f_{1,2}(x) &= (x+\rho-1)/\rho \end{aligned}$$

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