The Maximum Matroid for a Graph

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Abstract

The ground set for all matroids in this paper is the set of all edges of a complete graph. The notion of a *maximum matroid for a graph* is introduced. The maximum matroid for K_3 (or 3-cycle) is shown to be the cycle (or graphic) matroid. This result is persued in two directions - to determine the maximum matroid for the m-cycle and to determine the maximum matroid for the complete graph K_m. While the maximum matroid for an m-cycle is determined for all $m \geq 4$, the determination of the maximum matroids for the complete graphs is more complex. The maximum matroid for K_4 is the matroid whose bases are the Laman graphs, related to structural rigidity of frameworks in the plane. The maximum matroid for K_5 is related to a famous 153 year old open problem of J. C. Maxwell. An Algorithm A is provided, whose input is a given graph G and whose output is a matroid $M_A(G)$, defined in terms of its closure operator. The matroid $M_A(G)$ is proved to be the maximum matroid for G, implying that every graph has a unique maximum matroid.

Keywords: graph, matroid, framework

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1 Introduction

Fix n and let K_n be the complete graph on n vertices. A *graph* in this paper is a subset E of the edge set $E(K_n)$ of K_n. The number of edges of a graph E is denoted $|E|$ and graph isomorphism by $\approx$. The vertex set, i.e. the set of vertices incident to the edges of E will be denoted $V(E)$. For a graph $E \subseteq E(K_n)$ and edge $e \in E(K_n)$, the notation $E + e$ is used for $E \cup \{e\}$ and $E - e$ for $E \setminus \{e\}$. All matroids in this paper will be on the ground set $E(K_n)$.

The best known graph matroid is the *cycle matroid*, also called the *graphic matroid*. The independent sets of the cycle matroid are the forests. The bases are the spanning trees. The circuits (minimal dependent sets) are the cycles (no repeated vertices on a cycle). An edge $e = \{x, y\}$ lies in the closure of E if vertices x and y are joined by a

path in $E - e$. The flats (closed sets) are vertex disjoint unions of cliques, and therefore the lattice of flats is isomorphic to the lattice of partitions of an n-element set. There are other matroids on the edge set of a graph that have been extensively studied, for example the bicircular matroid, the transversal matroid, the bond matroid, and the gammoid. For background on matroids see [5, 13].

The circuits of a matroid are the minimal dependent sets. If a matroid on $E(K_n)$ is regarded in terms of its independent sets, then the circuits are, in a sense, the "forbidden subgraphs": a set is independent if and only if it contains no circuit. This paper introduces the notion of a *maximum matroid*, denoted $\widehat{M}(G)$, for a fixed "forbidden" graph G. Basically what we seek in a maximum matroid for G is the matroid, among all the matroids on $E(K_n)$ for which each graph isomorphic to G is a circuit, the one that has, in the strongest sense, the most independent sets. More precisely:

Definition 1. Let G be a subgraph of K_n. The **maximum matroid** $\widehat{M}(G) := \widehat{M}_n(G)$ for G is the matroid on the ground set $E(K_n)$, with the properties that (1) every graph isomorphic to G is a circuit of $\widehat{M}(G)$, and (2) if M is any matroid satisfying property (1), then every independent set in M is independent in $\widehat{M}(G)$.

We say "the" maximum matroid because its existence is proved in Section 6. Uniqueness then follows directly from properties (1) and (2). Note that Definition 1 does not require that the set of graphs isomorphic to G be itself the set of circuits of $\widehat{M}(G)$, only that it be a subset. In the terminology of [6], condition (2) would be expressed as: $\widehat{M}(G)$ *majorizes* M, denoted $\widehat{M}(G) \succeq M$. Also, in matroid terminology, condition (2) in the definition says that the identity map on $E(K_n)$ is a *weak* map from matroid $\widehat{M}(G)$ to matroid M. Technically, for a given graph G, there is a matroid for each n, but the subscript is omitted when no confusion arises. The terminology "maximum" matroid is motivated by item (3) of the next proposition, whose proof follows immediately from basic definitions.

Proposition 1. *For matroids M and M' on the same ground set E, the following are equivalent.*

1. *Every independent set in M is independent in M'.*
2. *Every dependent set in M' is dependent in M.*
3. *$rank_{M'}(A) \geq rank_M(A)$ for all $A \subseteq E$.*

Remark 1. It has come to our attention that, in 1970, H. H. Crapo [3] introduced the notations of truncation and erection of matroids. In 1975 D. Knuth [8] independently came up with the same notions (but not the same names) and gave an algorithm for constructing an erection of a matroid. H. Q. Nguyen, shortly thereafter, without knowledge of Knuth's work, also provided an algorithm. Given two matroids, M of rank r and M' of rank $r+1$, defined on the same set, M is the *truncation* of M', or M' is an *erection* of M, if the flats of M are all the flats of M' whose rank is not r. A matroid

has a unique truncation, but may have many erections. The set of all erections is a lattice and the 1-element is called, by Crapo, the *free erection* and, by Knuth, the *free completion*. The point of this remark is that, although similar is spirit, our concept of maximum matroid for a graph G is **not** the same as a free erection. In particular, we are not starting with a matroid, but with an a single graph (and isomorphic copies). The techniques and algorithms for the free erection do not carry over. The motivation is also different. Crapo and Knuth were likely interested in constructing new matroids or, in Knuth's papar, random matroids, from a given matroid. Our motivation comes from the theory of rigidity of frameworks, as explained in the next paragraph.

Although the results and proofs are graph and matroid theoretic, a main motivation for exploring the maximum matroid $\widehat{M}(G)$ of a graph G comes from the theory of ridigity of frameworks, systems of stiff bars joined at movable joints. In particular, the motivation relates to a 153 year old open problem of J. C. Maxwell [11]. Define a *framework* $(\Gamma, \mathbf{p})$ as a graph Γ together with an embedding $\mathbf{p} : V \to \mathbb{R}^d$ of the vertex set V of Γ into Euclidean space. A framework is *rigid* in $\mathbb{R}^d$ if the only motions of the framework that preserve the distances between adjacent vertices of Γ arise from rigid motions, i.e., orientation preserving isometries of $\mathbb{R}^d$. We refer to [4, 6] for background on structural rigidity and, in particular, the definitions of a generic embedding of a graph in $\mathbb{R}^d$. It suffices for the purposes of this paper to state the fact that, if one generic embedding of Γ is rigid, then all generic embeddings are rigid. So generic rigidity depends only on the graph Γ, not on the embedding.

A well known theorem of Laman [9] provides, in the generic case, a simple combinatorial necessary and sufficient condition for a graph to be rigid (see Theorem 8 in Section 5). In 1864 Maxwell formulated a necessary condition for rigidity in 3-dimensions, a condition analogous to the Laman condition. Maxwell's condition is, however, not sufficient for rigidity (see Section 7). He asked for a characterization of rigidity in 3 dimensions akin to Laman's result in 2-dimensions (see the introductory paragraph of Section 7). No such characterization has been found.

A completely combinatorial algorithm, called Algorithm A is given in Section 6 and is used to prove that every graph G has a maximum matroid. We show in Section 7 that either (1) Algorithm A applied to $G = K_5$ suffices to determine whether or not a given graph is generically rigid in $\mathbb{R}^3$ or (2) a well-known conjecture in rigidity theory, called the *maximal conjecture* in $\mathbb{R}^3$, is false. If (1) is true, then Algorithm A may be the closest one may hope for in the way of an answer to Maxwell's question.

2 Organization and Results

To get a feeling for the concept of the maximum matroid, two fairly simple examples are investigated in Section 3, the maximum matroid $\widehat{M}(C_3)$ for the 3-cycle C_3 (see Theorem 2) and the maximum matroid $\widehat{M}(K_{1,3})$ for the complete bipartite graph $K_{1,3}$ (see Theorem 3). There are two obvious directions in which to generalize Theorem 2, to determine the maximum matroid for K_m, $m \geq 4$, and to determine the maximum

matroid for the m-cycle C_m, $m \geq 4$. In Section 4, the maximum matroid for the m-cycle C_m, for all $m \geq 4$, is determined (Theorems 4 and 5). Section 5 is devoted to the maxmum matroid for K_4 (Theorems 6 and 7) and to its relationship to 2-dimensional framework rigidity. Section 7 is a discussion of the maximum matroid for K_5 and its relation to 3-dimensional framework rigidity and to Maxwell's open problem.

The existence of a maximum matroid for any graph G is proved in Section 6. The proof is via an algorithm, called Algorithm A. The input to Algorithm A is a given graph G and the output is a matroid $M_A(G)$, defined in terms of its closure operator. Although different in its description, Algorithm A was motived by [2, Definition 5.2]; see also the acknowledgement at the end of this paper. Theorem 10 states that $M_A(G)$ satisfies the matroid closure axioms. Theorem 12 states that $M_A(G) = \widehat{M}(G)$, the maximum matroid for the graph G. The proofs of the main results of Section 6, not quite as straightforward as the algorithm itself, use a closure operation. Although it may seem natural to seek a simpler proof based on the circuits, the maximum matroid begin defined in terms a circuit, this proved problematic.

3 First Examples

As examples of the maximum matroid of a graph, we first consider the case where the graph is the complete graph K_3, equivalently the 3-cycle C_3, and the case where the graph is the complete bipartite graph $K_{1,3}$ (star). The proofs of Theorems 2 and 3 use the matroid closure axioms and the matroid circuit axioms . The matroid closure of a set $E \subseteq E(K_n)$ will be denoted $[E]$. The closure axioms for a matroid are as follows, axiom (4) known as the *exchange axiom.*

For all $E, F \subseteq E(K_n)$ and $e \in E(K_n)$:

CL1. $E \subseteq [E]$
CL2. $F \subseteq E \;\Rightarrow\; [F] \subseteq [E]$
CL3. $[[E]] = [E]$
CL4. $f \in [E + e] \setminus [E] \;\Rightarrow\; e \in [E + f]$.

The circuit axioms are as follows. A set $\mathcal{C}$ is the set of circuits of a matroid if:

C1. The empty set is not in $\mathcal{C}$.

C2. No member of $\mathcal{C}$ is a proper subset of another member of $\mathcal{C}$.

C3. If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then $(C_1 \cup C_2) - e$ contains a member of $\mathcal{C}$.

Theorem 2. *The maximum matroid $\widehat{M}(C_3)$ (equivalently $\widehat{M}(K_3)$) is the cycle matroid.*

Proof. Clearly the cycle matroid satisfies property (1) in Definition 1. Let D be any dependent set in the cycle matroid. Via Proposition 1, it is sufficient to prove that D is dependent in any matroid for which each 3-cycle is a circuit. Let C be a cycle contained in D. that C is Let $e_1 = \{v_0, v_1\}, e_2 = \{v_1, v_2\}, \ldots, e_k = \{v_{k-1}, v_0\}$ be the successive edges of C. Let M be any matroid satisfying property (1), and let $[E]$ denote the closure of E in M. It is sufficient to show that C is dependent in M.

Because every triangle is a circuit, we have successively $\{v_0, v_2\} \in [e_1, e_2]$, $\{v_0, v_3\} \in [\{v_0, v_2\}, e_3] \subseteq [e_1, e_2, e_3] \ldots, \{v_0, v_k\} \in [\{v_0, v_{k-1}\}, e_{k-1}] \subseteq [e_1, e_2, \ldots, e_{k-1}]$, $e_k = \{v_0, v_k\} \in [e_1, \ldots, e_{k-1}]$. Therefore cycle C is dependent in M. □

The *k-uniform matroid* is the matroid whose bases are all the subsets of $E(K_n)$ consisting of exactly k edges. If G is a graph with k edges, let $\mathcal{B}(k, G)$ denote the set of all subgraphs of $E(K_n)$, except those isomorphic to G, consisting of exactly k edges. If $\mathcal{B}(k, G)$ comprise the bases of a matroid (it is not a matroid for some choices of G), call it the (k, G)-*uniform matroid*, and denote it by $U(k, G)$. It is routine, for example, to verify a $(3, K_{1,3})$-uniform matroid.

Theorem 3. $\widehat{M}(K_{1,3}) = U(3, K_{1,3})$.

Proof. Clearly $U(3, K_{1,3})$ satisfies property (1) in Definition 1. Let M be any matroid that satisfies property (1) in Definition 1, and let $[E]$ denote the closure of a set E in M. If $D \not\approx K_{1,3}$ is any dependent set in $U(3, K_{1,3})$, then, by Propositioin 1, it suffices to show that D is dependent in M. Hence it suffices to show that if E is any set with exactly four edges, then E is dependent in M.

Assume that there is a path $p = v_0, v_1, v_2$ in E of length 2, and let $e = \{u, v\}$ and $e' = \{u', v'\}$ be the other two edges in E. Also assume that none of u, v, u', v' coincide with any of v_0, v_1, v_2. Let $F = E - e'$. By successively using circuits that are isomorphic to $K_{1,3}$, we obtain $\{v_1, u\}$, $\{v_1, u'\} \in [F]$; then $\{u, u'\} \in [F + \{v_1, u\}]$; then $e' = \{u', v'\} \in [F + \{v_1, u'\} + \{u, u'\}$. Therefore $e' \in [F] = [E - e']$, and E is thus dependent in M. If one of u, v, u', v' coincides with any of v_0, v_1, v_2, then the proof is similar and even shorter.

If there is no path of length 2 in E, then let $E = \{e_1 = \{u_1, v_1\}, e_2 = \{u_2, v_2\}, e_3, e_4\}$ be the set of four pairwise vertex disjoint edges. The sets $C_1 = \{e_1, e_2, \{u_1, v_1\}, e_3\}$ and $C_2 = \{e_1, e_2, \{u_1, v_1\}, e_4\}$ are both dependent in M. If some subset of C_1 or C_2 is dependent, then the proof is even easier than if we assume this is not the case. Hence assume that both C_1 and C_2 are circuits in M. By circuit axiom C3, the set $\{C_1 \cup C_2 - \{u_1, v_1\} = \{e_1, e_2, e_3, e_4\}$ is dependent in M. □

4 The Maximum Matroid for $\mathbf{C_m}$

Let C_m denote the cycle of length m. The easiest case, $m = 3$, was considered in the previous section, the maximum matroid for C_3 being the cycle matroid. The cases $m \geq 4$ are slightly more complicated. For example, let M be any matroid for which every graph isomorphic to C_4 is a circuit. We claim that the graph E on the left in Figure 1 is dependent in M, in particular $f \in [E - f]$. This is shown on the right in Figure 1. Using axiom CL2, first add edge 1 to the closure of $E - f$, then add edge 2, then add edge f.

In the above argument and in some of the proofs in this section, the following algorithm is implicitly used. Let G be a subgraph of K_n and M any matroid for which every graph isomorphic to G is a circuit.

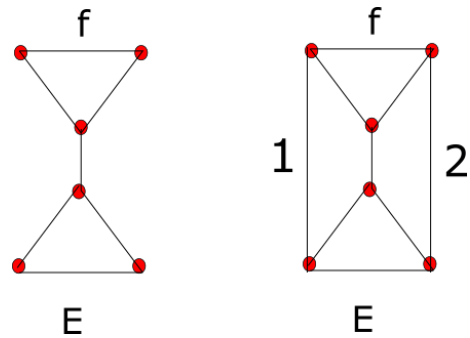

Figure 1: A dependent set.

Input: a set $E \subset E(K_n)$
Output: a set $\overline{E}$ such that $\overline{E} \subseteq [E]$

Initialize: set $\overline{E} = E$ for all $E \subset E(K_n)$

While there is triple (e, F, E) such that $F \subseteq E$, $e \in E(K_n) \setminus E$ and $F + e \approx G$ do $\overline{E} \leftarrow \overline{E} + e$.

It would be convenient if this algorithm was sufficient, without having to invoke axiom CL4. This, unfortunately, is not the case, which is evidenced by complexity of the proof of valididity of Algorithm A in Section 6.

Denote by P_j the path of length $j-1$ with successive vertices $V_j = \{1, 2, \ldots, j\}$, by K_j the complete graph on V_j, and by B_j the complete bipartite graph on V_j with even numbered vertices in one partite set and odd numbered vertices in the other.

Lemma 1. *Let M be any matroid for which every graph isomorphic to C_m is a circuit of M.*

1. *If m is odd, then $[P_j] = K_j$ for all $j \geq m+1$.*

2. *If m is even, then $B_j \subseteq [P_j]$ for all $j \geq m+1$.*

Proof. To prove statements (1) and (2) for $j = m+1$, let $1, 2, \ldots, m+1$ be the successive vertices of P_{m+1}. First note that $\{1, m\} \in [P_{m+1}]$ by considering the m-cycle $1, 2, \ldots, m, 1$. Similarly, $\{2, m+1\} \in [P_{m+1}]$. We next show, by induction on b, that $\{1, b\} \in [P_{m+1}]$ for all even $b \leq m$. This follows from the m-cycle $1, b, b+1, b+2, \ldots, m+1, 2, 3, \ldots, b-2, 1$. If m is odd, then we also have $\{1, m+1\} \in [P_{m+1}]$ by using the m-cycle $1, m+1, 2, 3, 4, \ldots, m-1, 1$. Now, for m odd and for all $1 \leq a < b \leq m+1$, we have the edge $\{a, b\} \in [P_{m+1}]$. This follows, by induction on $b-a$, from the m-cycle $a, b, b-1, b-2, \ldots, a+2, b+1, b+2, b+3. \ldots, a$. This completes the proof of (1). Now assume that m is even. We have $\{2, b\} \in [P_{m+1}]$ for all odd b because of the m-cycle $2, b, b-1, b-2, \ldots, 5, 4, 1, b+1, b+2, \ldots, m+1, 2$. By symmetry we also have $\{m+1, b\} \in [P_{m+1}]$ for all even b and $\{m, b\} \in [P_{m+1}]$ for all odd b. By induction on $b-a$, we have $\{a, b\} \in [P_{m+1}]$ for all $a < b$ of opposite parity. With a even, this is

verified by the m-cycle $a, b, b+1, b+2, \ldots, m+1, a+2, a+3, \ldots, b-1, a+1, 2, 3, 4, \ldots, a$. By symmetry, the same is true with a odd.

We show (1) and (2) in the case $j > m+1$ by induction on j. The statements have been shown to be true for $j = m+2$. If the vertices of P_{j+1} are successively $\{1, 2, \ldots, j, j+1\}$, then we have shown that, in the case m odd, both the complete graph on $\{1, 2, \ldots, j\}$ and the complete graph on $\{2, 3, \ldots, j+1\}$ are contained in $[P_{j+1}]$. Therefore the complete graph on $\{1, 2, \ldots, j+1\}$ and the complete graph on $\{2, 3, \ldots, j+1\}$ are contained in $[P_{j+1}]$. A similar argument suffices in the case m even. □

Denote by $\mathcal{I}$ the set of all subgraphs I of K_n for which every component of I has at most one cycle and that cycle is odd. Let M_O denote the matroid with $\mathcal{I}$ as the set independent sets. If there were no restriction on the parity of the cycle, this would be the bicircular matroid introduced by Simes-Pereira [14]. It is not hard to show that M_O is a matroid. The circuits of M_O are even cycles and pairs of edge disjoint odd cycles joined by a path p (of possibly 0 length) such that the only vertices of p in common with the odd cycles are the two ends. Call the latter type of graph an *odd dumbell*.

Theorem 4. $\widehat{M}(C_4) = M_O$.

Proof. It is required to prove that if M is any matroid for which each graph isomorphic to C_4 is a circuit, then any even cycle or odd dumbell is dependent in M. If C is an even cycle, then C dependent follows from Lemma 1.

If D is an odd dumbell, let D_1 and D_2 be the two odd cycles, an m_1 and m_2-cycle, respectively, and a and b the ends of the path joining D_1 and D_2. Let u_1 and v_1 be the two vertices of D_1 adjacent to a, and let u_2 and v_2 be any two adjacent vertices of D_2, neither adjacent to b. We claim that $e_2 = \{u_2, v_2\} \in [D - e_2]$, which would imply that D is dependent. It follows from Lemma 1 that $e_1 = \{u_1, v_1\} \in [D_1] \subset [D - e_2]$. Let p be the path in D_1 joining u_1 and v_1. In $D - p + e_1$ There exist a unique path from u_1 to u_2 of odd length and a unique path from v_1 to v_2 of odd length. By Lemma 1, $\{u_1, u_2\} \in [D - e_2]$ and $\{v_1, v_2\} \in [D - e_2]$. Since u_2, u_1, v_1, v_2, u_2 is a 4-cycle, $e_2 \in [D - e_2]$. □

Recall that $U(m, C_m)$ denotes the (m, C_m)-uniform matroid as defined in Section 3. Consequently, C is a circuit in $U(m, C_m)$ if and only if $C \not\supseteq C_m$ contains exactly $m+1$ edges or if $C \approx C_m$. It is routine to check that $U(m, C_m)$ is a matroid. Note that the maximum matroids for C_3 and C_4 do not fit the pattern of the maximum matroids for C_m, $m \geq 5$.

Lemma 2. *Assume that $m \geq 5$, $n \geq m+2$, and M is a matroid on ground set $E(K_n)$ for which every graph isomorphic to C_m is a circuit. If H is the union of the m edges of a path p of length m and any additional edge, then H is dependent in M.*

Proof. Let $p = (0, 1, 2, \ldots, m)$ be a path joining successive vertices $0, 1, 2, \ldots, m$, and let f be an additional edge. There are five cases in the proof.

(1) If f is a chord of p (an edge joining any two non-adjacent vertices of p), then H is dependent by Lemma 1 if m is odd. If f is a chord of p joining vertices of different

parity, then, H is dependent by Lemma 1 if m is even. See (5) below for the case where m is even and f is a chord of p joining vertices of the same parity.

(2) Let $f = \{m, m+1\}$, where veretex $m+1$ is not a vertex of p. Then H is the set of edges of a path of length $m+1$. Let $q = (0, 1, \ldots, m)$ and $q' = (1, 2, \ldots, m+1)$ be two paths of length m, and let Q and Q' denote the set of edges in q and q', respectively. If e is a chord, say chord $\{1, 4\}$, then $C = Q + e$ and $C' = Q' + e$ are circuits by (1). Therefore $H = C \cup C' - e$ contains a circuit by circuit axiom C3.

(3) Let f be any edge with exactly one vertex, say $j \neq 0, m$, in common with p. If $f = \{j, m+1\}$, then $q = (j+1, j+2, \ldots, m, 0, 1, \ldots, j, m+1)$ and $q' = (m+1, j, j+1, \ldots, m, 0, 1, \ldots, j-1)$ are paths of length $m+1$, and each is dependent in M by (2). We will assume that the corresponding edge sets Q and Q' are circuits; otherwise each contains a circuit, and the subsequent proof becomes easier. Now $H = Q \cup Q' - \{0, m\}$, which contains a circuit by circuit axiom C3.

(4) Let $f = \{m+1, m+2\}$ be any edge with no vertex in common with p. Then $q = (1, \ldots, m, m+1, m+2)$ and $q' = (0, 1, 2, \ldots, m, m+1)$, both paths, are circuits in M by (2). Therefore $H = Q \cup Q' - \{m, m+1\}$ contains a circuit by circuit axiom C3.

(5) The only remaining case is when m is even and f is a chord of $p = (0, 1, 2, \ldots, m)$. Assume that the chord is $f = \{0, j\}$, where $j \neq 1$ is a vertex of p. If $m+1$ is a vertex not on p, then $q = (0, 1, \ldots, m, m+1)$ is a path of length $m+1$, and $q' = (1, 2, \ldots, m, m+1) \cup \{0, j\}$ is the union of a path of length m and an edge with exactly one vertex in common with the path. Both edge sets Q and Q' are circuits in M by (2) and (3), respectively. Therefore $H = Q \cup Q' - \{m, m+1\}$ contains a circuit by circuit axiom C3.

Assume next that the chord is $f = \{1, j\}$, where $j \neq 0, 2$ is a vertex of p. If $m+1$ is a vertex not on p, then $q = (0, 1, \ldots, m, m+1)$ is a path of length $m+1$, and $q' = (1, 2, \ldots, m, m+1) \cup \{1, j\}$ is a graph with $m+1$ edges of the type proved to be dependent in the paragraph above. As before, we assume without loss of generality that corresponding dependent edge sets Q and Q' are circuits. Therefore $H = Q \cup Q' - \{m, m+1\}$ contains a circuit by circuit axiom C3. The case where $f = \{i, j\}$, $1 < i < j$, is an arbitrary chord is proved, just as in the case $i = 1$, by the obvious induction.

An examination of the steps in the above proof reveals that no more than $m+2$ vertices are required. Of course, graph H may itself have more than $m+2$ vertices, in which case n must be at least that number. □

Theorem 5. *If $m \geq 5$ and $n \geq m+2$, then $\widehat{M}_n(C_m) = U(m, C_m)$.*

Proof. Let M be any matroid for which each graph isomorphic to C_m is a circuit and there are no other dependent sets of size m. It suffices to show that every subgraph with $m+1$ edges is dependent in M.

Let H be the union of the set of edges of a path $p = (0, 1, \ldots, m+1-d)$ of length $m+1-d$ and an arbitrary set S of an additional d edges. We will prove, by induction on d, that H is dependent for all $d = 1, \ldots, m+1$. When $d = m+1$, the set S is an arbitrary graph with $m+1$ edges. Therefore, this will complete the proof of Theorem 5.

The statement is true for $d = 1$ by Lemma 2. Assume it is true for d, and let H be the union of the set P edges of the path $p = (0, 1, \ldots, m-d)$ of length $m-d$ and the

set S of an additional $d+1$ edges. Let u be a vertex not in $p \cup V(S)$. If $e, e' \in S$, then $C = P + \{0, u\} - e$ and $C' = P \cup \{0, u\} - e'$ are dependent by the induction hypothesis. As in proof of Lemma 2, there is no loss of generality in assuming that C and C' are circuits. Therefore $H = C \cup C' - \{0, u\}$ contains a circuit by circuit axiom C3.

□

The definition of maximal matroid of a graph G requires that, in the ground set $E(K_n)$, the integer n is at least as large as the number of verices in G. In Theorem 5 it is assumed, to give an additional two vertices wiggle room in the proof, that $n \geq m+2$. We conjecture that Theorem 5 is actually true for $n \geq m$.

5 The Maximum Matroid for K_4

Given non-negative integers a and b, let $\mathcal{I}(a, b)$ denote the set of all subgraphs E of $E(K_n)$ for which $|E'| \leq a|V(E')| - b$ for all $E' \subseteq E$. In [10, 15] these graphs are called (a, b)-*sparse*. The set of all (a, b)-sparse graphs that, in addition, satisfy $|E| = a|V(E)| - b$ are called (a, b)-*tight*. Let $\mathcal{C}(a, b)$ denote the set of all subsets E such that $|E| = a|V(E')| - b + 1$ and $|E'| \leq a|V(E')| - b$ for all $E' \subsetneq E$.

Theorem 6. *Let a and b be integers with $2a > b \geq 0$. The collection $\mathcal{I}(a, b)$ is the set of independent sets of a matroid $M(a, b)$. The collection $\mathcal{C}(a, b)$ is the set of circuits of $M(a, b)$.*

Proof. It will first be shown that $\mathcal{C}(a, b)$ satisfies the circuit axioms of a matroid. It then immediately follows from the definitions of independent and circuit that $\mathcal{I}(a, b)$ is the corresponding family of independent sets.

Since axioms C1 and C2 follow immediately from the definition of $\mathcal{C}(a, b)$, it suffices to check axiom C3. Let E_1 and E_2 be distinct members of $\mathcal{C}(a, b)$ and $e \in E_1 \cap E_2$. If $E = E_1 \cup E_2 - e$, then

$$\begin{aligned} |E| &= |E_1| + |E_2| - 2 \geq (a|V(E_1)| - b + 1) + (a|V(E_2)| - b + 1) - 2 \\ &= a(|V(E_1)| + |V(E_2)|) - 2b \geq a(|V(E)| + 2) - 2b = a|V(E)| + 2(a - b) \\ &> a|V(E)| - b. \end{aligned}$$

Therefore there is a subset $E_0 \subseteq E$ such that $E_0 \in \mathcal{C}(a, b)$. □

The matroid $M(1, 1)$ is the graphic matroid; $M(1, 0)$ is the bicircular matroid. Note that $\mathcal{I}(3, 6)$ is not the set of independent sets of any matroid, i.e., $\mathcal{C}(3, 6)$ is not a set of circuits. In particular, circuit axiom C3 fails when E_1 and E_2 are copies of K_5 that have exactly one edge in common.

Theorem 7. $\widehat{M}(K_4) = M(2, 3)$.

Proof. Clearly every graph isomorphic to K_4 is a circuit of $M(2, 3)$. Let M be any matroid with the property that every graph isomorphic to K_4 is a circuit of M. We

prove by induction on $m := |E|$ that any set E, dependent in $M(2,3)$, is also dependent in M. The statement is clearly true for $m \leq 4$; assume that it is true for any dependent set of size less than m and assume that $|E| = m$. Since E is dependent in $M(2,3)$, there is a circuit E_0 of $M(2,3)$ contained in E. The circuit E_0 cannot contain a vertex of degree 2 (incident, say, to edges $e_1, e_2 \in E_0$); otherwise graph $E_0 - \{e_1, e_2\}$ would be a circuit in $M(2,3)$, contradicting the definition of a circuit as a minimal dependent set. If all vertices of E_0 have degree at least 4, then again E_0 would not be a circuit because $|E_0| \geq 2|V(E_0|$. Therefore, consider a vertex v of degree 3 incident to edges, say, $e_1 = \{v, v_1\}$, $e_2 = \{v, v_2\}$, $e_3 = \{v, v_3\}$ of E_0. If $f_1 = \{v_1, v_2\}, f_2 = \{v_2, v_3\}$, and $f_3 = \{v_3, v_1\}$ are all contained in E_0, then $e_1, e_2, e_3, f_1, f_2, f_3$ forms a K_4, rendering E_0 dependent. Otherwise, assume that $f \in \{f_1, f_2, f_3\} \setminus E_0$. If $E_0 - \{e_1, e_2, e_3\}$ is dependent in $M(2,3)$, then it is dependent in M by the induction hypothesis, and the proof is complete. Otherwise, if $\widehat{E} = E_0 - \{e_1, e_2, e_3\} + f$, then

$$|\widehat{E}| = |E_0| - 2 = 2|V(E_0)| - 2 - 2 = 2(|V(E_0)| - 1) - 2 = 2|V(\widehat{E})| - 2,$$

which, by the induction hypothesis, implies that $\widehat{E}$ is dependent in M, and hence, with respect to the closure operator $[\,\cdot\,]_M$ in M, that $f \in [E_0 - \{e_1, e_2, e_3\}]_M$. Since f was chosen arbitrarily from $\{f_1, f_2, f_3\}$, we have $\{f_1, f_2, f_3\} \subset [E_0 - \{e_1, e_2, e_3\}]_M$. Because he edges, $f_1, f_2, f_3, e_1, e_2, e_3$ form a K_4, we have $e_1 \in [E_0 - e_1]_M$, and therefore that E_0, and hence E, is dependent in M. □

The matroid $M(2,3)$ is of special interest in the theory of rigidity of frameworks, as dicussed in Section 1. A framework is *minimally rigid* in $\mathbb{R}^d$ if it is rigid, but with any edge removed, it is not rigid. A well known theorem of Laman [9] provides, in the generic case, a combinatorial necessary and sufficient condition for a graph to be rigid. For this reason, a $(2,3)$-tight graph is also referred to as a *Laman graph*.

Theorem 8 (Laman 1970)**.** *A graph in the plane is minimally rigid if and only if it is* $(2,3)$*-tight.*

Associated with framework rigidity in $\mathbb{R}^d$ is a *rigidity matroid*, defined as the matroid whose independent sets are the sets of independent row vectors of the rigidity matrix, a matrix derived from the incidence matrix of the graph. The circuits of the 2-dimensional rigidity matroid can be obtained from the tetrahedron by what is called 1-extension and 2-sum [1]. Being somewhat outside the scope of this paper, the exact definitions are omitted. Suffice it to say that a subset $E \subset E(K_n)$, considered as a graph, is rigid in $\mathbb{R}^d$ if and only if the closure of E in the rigidity matroid is the complete graph on $V(E)$. Intuitively, the distances between all pairs of vertices of $V(E)$ are implied by the lengths of edges in E.

Laman's theorem implies that the $\mathbb{R}^2$ rigidity matroid is $M(2,3)$, and therefore that the $\mathbb{R}^2$ rigidity matroid is the maximum matroid $\widehat{M}(K_4)$ (by Theorem 7). Intuitively, (1) the length of any edge of a (generically embedded) K_4 is determined by the lengths of the other five edges, i.e., K_4 is a circuit in the rigidity matroid, and (2) for any graph E in $\mathbb{R}^2$, the set of edges whose lengths are determined by those in E, are implied, in a

sense, purely from these K_4 circuits. Whether results similar to those in $\mathbb{R}^2$ carry over to $\mathbb{R}^3$ is discussed in Section 7

6 Every Graph Has a Unique Maximum Matroid

For each graph G considered in the previous sections, there is a simple description of the maximum matroid for G, for example, for K_3 it is the cycle matroid and for K_4 it is the matroid whose bases are the Laman graphs. This may be misleading. As explained in Section 7, a description of the maximum matroid for even K_5 is problematic. What is proved in this section is that, for any given graph G, there is a maximum matroid for G. This matroid is not described explicitly, but is given by an algorithm that computes the matroid in terms of its closure operation.

For a fixed graph G, Algorithm A below defines a matroid $M_A(G)$, which we call the *A-matroid for G*. That it is indeed a matroid is Theorem 10. The matroid $M_A(G)$ is computed in terms of the closure operator. In the algorithm, for each subset $E \subseteq E(K_n)$, there is a corresponding set $[E]$ of edges of K_n that begins with $[E] = E$ and with edges possibly added to $[E]$ as the algorithm progresses. The set $[E]$ at termination of the algorithm is denoted $[E]^*$ and is called the *closure* of E in $M_A(G)$.

A *step* in Algorithm A is the addition of a single edge e to a set $[E]$ and to $[E']$ for all $E' \supset E$. This is done on the second to last line of Algorithm A. There are two ways that the edge e can be added, by conditions labeled 1 and 2 in the algorithm. These will be referred to as *addition rule 1* and *addition rule 2*. The algorithm proceeds one step at a time, so the steps can be numbered $1, 2, \ldots$. The set $[E]$ just after step i but before step $i+1$ will be referred to as the *closure of E at step i*. The notation $[E]^k$ denotes the closure of E directly after completing loop k of the FOR statement, just before starting loop $k+1$. The number of vertices in a graph G is denoted $n(G)$.

Algorithm A

Input: Graph G

Output: The closure $[E]^*$ for all $E \subset E(K_n)$

Initialize: $[E] = E$ for all $E \subset E(K_n)$

For $k = n(G) - 1$ to $\binom{n}{2}$ do

While there exists a triple (e, E, F) that satisfies all of the following

- $e \in E(K_n) \setminus [E]$ and $F \subseteq E(K_n)$,
- $F \subseteq [E]$,
- $|F| \leq k$,
- one of the following hold:

1. $F + e \approx G$

2. $f \in [F - f + e]$ for some $f \in F$ and $f \notin [F - f]^{k-1}$ for all $f \in F$,

add e to $[E]$.

$[E]^* = [E]$ for all $E \subseteq E(K_n)$

Example 9 (Algorithm A)**.** In this example, $G = C_5$, the 5-cycle. Referring to Figure 2, we explain how Algorithm A adds edge e, in red in panel 2, to the closure of E, the graph shown in panel 1. Consider each of the panels 4-7. During the first k-loop, when $k = 4$, the red edge is added to the closure of each graph in green by addition rule 1. During the $k = 5$ loop, the red edges in panels 4-7 are successively added to the closure of the graph in panel 3, again by addition rule 1. Note that the red edge in panel 7 is edge f in panel 2. Since $f \in [F - f + e]$, addition rule 2 yields $e \in E$. (It is not hard to show that the algorithm could not have, for any $h \in F$, previously added h to the closure of $F - h$, i.e., $h \notin [F - h]$ for all $h \in F$.)

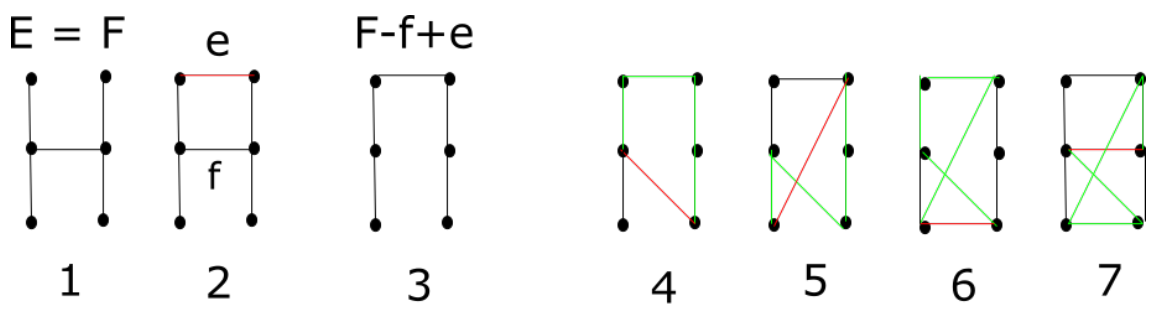

Figure 2: See Example 9.

The proofs of the main results of this section procced as follows. Lemma 9 states that the closure operator defined by Algorithm A is independent of the order in which the steps in the algorithm are performed. Theorem 10 states that the closure operator defined by Algorithm A satisfies closure axioms CL1-CL4 of a matroid as given in Section 3. Theorem 11 states, in particular, that each subgraph of K_n isomorphic to G is a circuit. Theorem 12 states that $M_A(G) = \widehat{M}(G)$, i.e., that $M_A(G)$ is the maximum matroid for G. The proofs of these results depend on the technical Lemmas 3-8.

Lemma 3. *If $E \subseteq H$, then $[E]^k \subseteq [H]^k$ for all k.*

Proof. Assume that e is added to $[E]$ during FOR loop $j \leq k$, and that e is the first such edge added with the property that $e \notin [H]^j$. Then there is an $F \subseteq [E]$, $|F| \leq j$, such that the conditions in edge addition rule 1 or 2 hold. By the minimality of e, also $F \subseteq [H]^j$. Therefore e is added to $[H]$ by the end of FOR loop j. □

Lemma 4. *For all $E \subseteq E(K_n)$ and for all k we have $[[E]^k]^k = [E]^k$.*

Proof. The containment $[E]^k \subseteq [[E]^k]^k$ follows from Lemma 3. To prove the opposite containment, let $H = [E]^k$ and, by way of contradiction, assume that $e \in [H]^k \setminus H$. Assume further that e is the first such edge added, say at step i during the FOR loop $j \leq k$. Since it is the first such edge, we have $[H] = H$ at step $i-1$ of the algorithm. But this means that there is an $F \subseteq [H] = H = [E]^k$, $|F| \leq j \leq k$, such that one of the edge addition rules (1) or (2) in the algorithm holds. If this were the case, however, then e would be added to $[E]$ by the end of FOR loop k, because the same conditions (1) or (2) would still hold at that stage (note that the requirement $f \notin [F-f]^{k-1}$ for all $f \in F$ in edge addition (2) is determined by the end of FOR loop $k-1$). But this contradicts $e \notin H = [E]^k$. □

Corollary 1. *For all $E \subseteq E(K_n)$ and for all $j \leq k$ we have $[[E]^j]^k = [E]^k$.*

Proof. Using Lemma 3 we have $[E]^k \subseteq [[E]^j]^k \subseteq [[E]^k]^k$, the first inclusion because $E \subseteq [E]^j$ and the second inclusion because $[E]^j \subseteq [E]^k$. The result then follows from Lemma 4. □

Lemma 5. *For any $H \subset E(K_n)$, there is a subset $H' \subseteq H$ such that $[H']^k = [H]^k$ and $h \notin [H'-h]^k$ for all $h \in H'$.*

Proof. Let $h_1 \in H$ be such that $h_1 \in [H-h_1]^k$. If no such h_1 exists, then $h \notin [H-h]^k$ for all $h \in H$. Continuing in this was, there is a sequence $(h_1, h_2, \ldots, h_m)$ of elements of H and a sequence $(H_0, H_1, H_2, \ldots, H_m)$ of distinct subsets of H such that

$$\begin{aligned} &H_0 := H, \\ &H_i := H_{i-1} - h_i, \qquad \text{and} \qquad h_i \in H_{i-1} \cap [H_i]^k \qquad \text{for} \quad i = 1, 2 \ldots, m, \\ &\quad h \notin [H_m - h]^k \quad \text{for all} \;\; h \in H_m. \end{aligned}$$

Taking $H' = H_m$ in the statement of the lemma, it now suffices to prove that $[H_i]^k = [H_{i-1}]^k$ for $i = 1, 2, \ldots, m$. By Lemma 3, $[H_i]^k = [H_{i-1} - h_i]^k \subseteq [H_{i-1}]^k$. For the opposite inclusion, use Lemma 4 to obtain

$$[H_{i-1}]^k = [H_i + h_i]^k \subseteq [[H_i]^k]^k = [H_i]^k.$$

□

Lemma 6. *For all $e, f \in E(K_n)$, $H \subseteq E(K_n)$, if $|H| \leq k$ and $f \in [H+e]^{k+1} \setminus [H]^k$, then $e \in [H+f]^{k+1}$.*

Proof. Assume that $f \in [H+e]^{k+1} \setminus [H]^k$. By Lemma 5 there is an H' such that $[H']^k = [H]^k$ and $h \notin [H'-h]^k$ for all $h \in H'$. With notation as in the proof of Lemma 5, we have

$$[H_{i-1} + e]^{k+1} = [H_i + h_i + e]^{k+1} \subseteq [[H_i]^k + e]^{k+1} \subseteq [[H_i + e]^k]^{k+1} = [H_i + e]^{k+1},$$

for $i = 1, 2, \ldots, m$, the last equality by Maximum 1. Therefore $f \in [H+e]^{k+1} \subseteq [H'+e]^{k+1}$. Now $f \in [H'+e]^{k+1} \setminus [H']^k$ and $h \notin [H'-h]^k$ for all $h \in H'$. Letting

$E = F = H' + f$, that is equivalent to: $|F| \le k+1$, $f \in [F - f + e]^{k+1} \setminus [F - f]^k$, and $h \notin [F - f - h]^k$ for all $h \in F - f$. That $h \notin [F - f - h]^k \subseteq [F - h]^k$ for all $h \in F - f$ implies that $h' \notin [F - h']^k$ for all $h' \in F$ because $f \notin [F - f]^k$. Trivially $F \subseteq [E]^k$. Therefore, by an edge addition of type (2) in the algorithm, if not already there by then, e would be added to the closure of E by at the end of the $k+1$ FOR loop. Hence $e \in [E]^{k+1} = [H' + f]^{k+1} \subseteq [H + f]^{k+1}$. □

Lemma 7. *If*

1. $|E| = k$,
2. $|F| = k+1$, *and*
3. $F \subseteq [E]^k$,

then $f \in [F - f]^k$ *for some* $f \in F$.

Proof. For any $f_1 \in F \setminus E$, there is a set $E' \subseteq E$ such that

$$\begin{aligned} &f_1 \in [E']^k \\ &f_1 \notin [E' - e_1]^k \quad \text{for all } \ e_1 \in E'. \end{aligned}$$

Choose such an $e_1 \notin F$. This is possible unless $E' \subset F$, in which case $f_1 \in [E']^k \subseteq [F - f_1]^k$, completing the proof of this lemma. Now use Lemma 6 with $H = E' - e_1$, $e = e_1$, $f = f_1$ to conclude that $e_1 \in [E' - e_1 + f_1]^k \subseteq [E - e_1 + f_1]^k$. Let $E_1 = E - e_1 + f_1$, and note that

$$\begin{aligned} (0) \quad & [E]^k = [E - e_1 + e_1]^k \subseteq [(E - e_1) \cup [E - e_1 + f_1]^k]^k = [[E - e_1 + f_1]^k]^k \\ & \qquad = [E - e_1 + f_1]^k = [E_1]^k, \\ (1') \quad & |E_1| = k, \\ (3') \quad & F \subseteq [E_1]^k, \end{aligned}$$

the second to last equality of (0) by Lemma 4 and statement (3′) by (0).

Now repeat the above procedure to obtain a sequence $E_1, E_2, \ldots, E_k$ of sets, a sequence $e_1, e_2, \ldots, e_k$ of edges, and a sequence $f_1, f_2, \ldots, f_k$ of distinct edges in F such that, for $i = 1, 2, \ldots, k$,

1. $e_i \in E_{i-1} \setminus \{f_1, f_2, \ldots, f_{i-1}\}$,
2. $E_i = E_{i-1} - e_i + f_i$,
3. $|E_i| = k$,
4. $F \subseteq [E_i]^k$.

Denoting its $k+1$ elements by $F = \{f_0, f_1, f_2, \ldots, f_k\}$, at the last stage we have

$$f_0 \in F \subseteq [E_k]^k = [E - e_1 - e_2 - \cdots - e_k + f_1 + f_2 + \cdots + f_k]^k = [F - f_0]^k.$$

□

Lemma 8. *If $[E]^*$ denotes closure of set E at the termination of Algorithm A, then $[E]^* = [E]^{|E|}$ for every $E \subseteq E(K_n)$.*

Proof. Assume, by way of contradiction, that $|E| = k$ and $e \in [E]^* \setminus [E]^k$. Assume further that e is the first edge added to $[E]$ after FOR loop k, say during FOR loop $j > k$. Since $j \geq k+1 \geq n(G)$, it could not have been a type (1) edge addition. For an edge addition of type (2), let F' be any subset of F with $|F'| = k+1$. By Lemma 7 there is an $f \in F'$ such that $f \in [F' - f]^k$, and hence an $f \in F$ such that $f \in [F - f]^{j-1}$. But this contradicts the requirement on F in the algorithm for a type (2) edge addition. □

Lemma 9. *For any $E \in E(K_n)$, the closure $[E]^*$ for Algorithm A at termination is independent of the order that edges are added.*

Proof. To show the independence of order, let A_1 and A_2 be two runs of the algorithm, with different orders. Let $[E]_1$ denote the closure of E at some specific step in A_1; similarly for $[E]_2$. Let $[E]_1^k$ and $[E]_2^k$ be the respective closures just before FOR loop $k+1$ begins, and let $[E]_1^*$ and $[E]_2^*$ denote the closures of E at termination, using A_1 and A_2, respectively. Assume, by way of contradiction, that $[E]_1^k = [E]_2^k$ for some k and for all $E \subseteq E(K_n)$, and that e is the first edge added during the $k+1$ loop of A_1, say to $[E]_1$, such that $e \notin [E]_2^*$. Assume that k is minimal for which this is true.

If the edge addition is of type (1), then clearly e will, at some point in algorithm A_2, be added to the closure of E. Therefore it must be a type (2) edge addition. Thus there is an $F \subseteq [E]_1$ with $|F| \leq k+1$ such that $f \notin [F-f]_1^k$ for all $f \in F$ and $f \in [F-f+e]_1$ for some $f \in F$. Since $|F - f| \leq k$, by Lemma 8 and the minimality of k, we have

$$[F-f]_1^k = [F-f]_2^k = [F-f]_2^*.$$

Since $f \notin [F-f]_1^k$ for all $f \in F$, we have

$$f \notin [F-f]_2^* \quad \text{for all} \quad f \in F. \tag{1}$$

Moreover, by the minimality condition - that e is the first edge addition with $e \in [E]_1 \setminus [E]_2^*$ during FOR loop $k+1$ of A_1 - and since $f \in [F-f+e]_1$ prior to the addition of edge e to $[E]_1$ - we have

$$f \in [F-f+e]_2^*. \tag{2}$$

By (1) and (2) above, at some step before termination in algorithm A_2, edge e will be added to the current closure of of E. Therefore $e \in [E]_2^*$, a contradiction. □

Theorem 10. *The closure operator constructed by Algorithm A satisfies the closure axioms of a matroid.*

Proof. Recall, in the notation of Algorithm A, the closure axioms for a matroid. For all $E, F \subseteq E(K_n)$:

1. $E \subseteq [E]^*$,
2. $F \subseteq E$ implies $[F]^* \subseteq [E]^*$,

3. $[[E]^*]^* = [E]^*$,

4. For any $e, f \in E(K_n)$ and $E \subseteq E(K_n)$ if $f \in [E+e]^* \setminus [E]^*$, then $e \in [E+f]^*$.

Axiom (1) is clear, and axiom (2) follows from Lemma 3. Axiom (3) follows from Lemma 4 by taking $k = \binom{n}{2}$.

Concerning axiom (4), assume $f \in [E+e]^* \setminus [E]^*$. If $|E| = k$, then by Lemma 8 we have $f \in [E+e]^{k+1} \setminus [E]^k$. Taking $H = E$ in Lemma 6 gives $e \in [E+f]^{k+1} = [E+f]^*$, again using Lemma 8. □

Theorem 11. *Given a graph G, Algorithm A produces a matroid $M_A(G)$ for which every isomorphic copy of G is a circuit of the matroid, and for which there are no circuits with fewer edges than G.*

Proof. Denote the number of edges in G by $|G|$. We first show, by way of contradiction, that there are no dependent sets with less than $|G|$ edges. Suppose that there is a circuit C with less than $|G|$ edges. An edge $e \in C$ must have been added to $[C-e]$ at step, say i, in Algorithm A. Assume that the pair (e, C) is minimum in that there is no pair (e', C') with C' satisfying $|C'| \leq |C|$ such that $e' \notin [C'-e']$ is added to $[C'-e']$ in Algorithm A at a step $j < i$. Let $H = C - e$. By the minimality of the step number $[H] = H$ prior to step i (before e is added to $[H]$). Since $|C| < |G|$, e could not have been added to $[H]$ by rule 1. By addition rule 2, there is a set $F \subseteq [H] = H$ and an edge $f \in F$ such that f is added to $[F - f + e]$, at some step $j < i$. Letting C' denote the circuit $F + e$, the pair (f, C') contradicts the minimality of (e, C) because $j < i$.

Theorem 10 immediately implies that the matroid $M_A(G)$ is well defined and produces a closure operator satisfying the closure axioms of a matroid. Clearly, edge addition of type (1) implies that each copy of G is dependent. By the paragraph above, each copy of G is a circuit. □

Theorem 12. *The maximum matroid for a given graph G is the A-matroid $M_A(G)$.*

Proof. By Theorem 11, every graph isomorphic to G is a circuit of $M_A(G)$, thus verifying condition 1 in Definition 1. To prove condition 2, let M be any matroid for which every graph isomorphic to G is a circuit. Denote the closure of a set E at an arbitrary step in Algorithm A by $[E]$ and the closure at termination by $[E]^*$. Let $\mathcal{C}$ denote the set of all circuits of M. We claim that, whenever an edge e is added to an independent set E in the A-matroid, say at step i in Algorithm A, then there is a a $C \in \mathcal{C}$ such that $C \subseteq E + e$. The claim would imply the theorem.

The claim is proved by induction on the step number i in Algorithm A when e is added to E. Assume that edge e is added at step i to a set E that is independent in the A-matroid. Note that, since E is independent in the A-matroid, the closure $[E]$ is just E prior to step i. If e is added by addition rule 1, then there is an $F \subseteq [E] = E$ such that $F + e \approx G$. Since $F + e$ is a circuit in M, clearly there is a $C \in \mathcal{C}$ such that $C \subseteq F+e \subseteq E+e$. Note that the first edge addition in Algorithm A must be by addition rule 1; therefore the claim is true for $i = 1$. If e is added to $[E] = E$ by addition rule 2, then there is a pair (f, F) such that $f \in F \subseteq [E] = E$ and edge f is added to $[F - f + e]$

at some step $j < i$. Note that F is independent in the A-matroid because, by addition rule 2, we have $f \notin [F-f]^{k-1} = [F-f]^*$, the last equality by Lemma 8. By the induction hypothesis, since $j < i$, there a $C \in \mathcal{C}$ such that $C \subseteq F + f = F + e \subseteq E + e$. $\square$

7 The Maximum Matroid for K_5

Although Theorems 11 and 12 guarantee that it exists, we have no explicit description of the maximum matroid for K_5. There does exist a matroid for which every copy of K_5 is a circuit, namely the 3-dimensional rigidity matroid discussed in Section 5. This relates to a 153 year old open problem dating back to J. C. Maxwell [11]. Laman's theorem (Theorem 8 in Section 5), provides a simple combinatorial characterization of rigidity in 2 dimensions. Maxwell asked for a such a characterization in 3 dimensions, but no such characterization has been found. The natural analog of Theorem 8 for $\mathbb{R}^3$ is to replace the condition that the graph be $(2,3)$-tight by the condition that it be $(3,6)$-tight. The resulting statement, however, is false, as shown by the graph, called the "double banana," in Figure 3. This graph is $(3,6)$-tight, but is clearly not rigid.

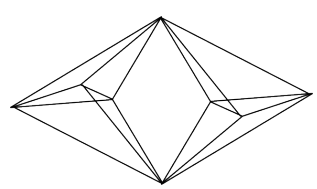

Figure 3: Double banana.

In dimension 2, the rigidity matroid is the maximum matroid $\widehat{M}(K_4)$, as explained in Section 5. Therefore it is natural to ask:

Question 1. Is the 3-dimensaional rigidity matroid equal to the maximum matroid $\widehat{M}(K_5)$ for K_5?

If the answer to the above question is "yes", then Algorithm A, which is purely combinatorial, may be the closest one may hope for in the way of an answer to Maxwell's question. In this case, to determine whether or not a graph E is rigid in $\mathbb{R}^3$, it would be sufficient to compute the closure $[E]^*$ using Algorithm A. Then E would be rigid if and only if $[E]^*$ is the complete graph on the vertex set $V(E)$. (It should be admitted that Algorithm A, in its present form, is not computationally efficient.)

Independent of the answer to Question 1, Algorithm A may provide an upper bound on the rank function of the 3-dimensional rigidity matroid, a topic of importance in combinatorial rigidity theory [7]. By Proposition 1, the rank $\text{rank}(E)$ of a graph E in the 3D rigidity matroid is bounded above by the rank $\text{rank}_A(E)$ of E in the A-matroid (the maximum matroid for K_5), i.e., $\text{rank}(E) \leq \text{rank}_A(E)$.

Question 1 is also related to what is referred to as the *maximal conjecture*. An *abstract rigidity matroid* is defined by six axioms, the closure axioms CL1-CL4 and an additional two closure axioms. Since it is peripheral to this paper, we omit the

defintion and a proof that $\widehat{M}(K_5)$ is an abstract rigidity matroid. Denote by $M \succeq M'$ the statement that M majorizes M', i.e., every independent set in matroid M' is independent in matroid M. Denote by $R(d)$ the d-dimensional rigidity matroid. The maximal conjecture states: $R(d) \succeq M$ for every abstract rigidity matroid M. The maximal conjecture is known to be true for $d = 2$ and false for $d \geq 4$. The conjecture is open for $d = 3$. We now know that

1. $\widehat{M}(K_5) \succeq R(3)$ (because each K_5 is a circuit in $R(3)$, and $\widehat{M}(K_5)$ is the maximal matroid for K_5), and

2. $R(3) \succeq \widehat{M}(K_5)$ if the maximal conjecture is true for $d = 3$.

Therefore, either $\widehat{M}(K_5) = R(3)$, providing an affirmative answer to Question 1 (and a partial solution to the question of Maxwell), or else the maximal conjecture is false in $\mathbb{R}^3$.

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References

[1] A. R. Berg and T. Jordán, A proof of Connelly's conjecture on 3-connected circuits of the rigidity matroid, *J. Combin. Theory Ser. B*, **88** (2003), 77-97.

[2] J. Cheng, Towards Combinatorial Chcaracterizations and Algorithms for Bar-and-Joint Independence and Rigidity in 3D and Higher Dimensions, Ph.D. Thesis, University of Florida, 2013.

[3] H. H. Crapo, Erecting Geometries, In Proc. Second Chapel Hill Conf. on Combinatorial Mathematics and its Applications (Univ. North Carolina, Chapel Hill, N.C., 1970), pages 74-99.

[4] H. Crapo and W. Whiteley, The Geometry of Rigid Structures, Encyclopedia of Math., Cambridge University Press, (1993).

[5] G. Gordon and J. McNulty, Matroids, a Geometric Introduction, Cambridge University Press, Cambridge, UK, 2012.

[6] J.E. Graver, B. Servatius and H. Servatius, Combinatorial Rigidity, *Graduate Studies in Mathematics*, Amer. Math. Soc., 1993.

[7] B. Jackson and T. Jordán, On the rank function of the 3-dimensional rigidity matroid, *Internat. J. Comput. Geom. Appl.*, **16** (2006), 415-429.

[8] D. Knuth, Random matroids, *Discrete Math.* **12** (1975), 341-358.

[9] G. Laman, On graphs and rigidity of plane skeletal structures, *J. Engrg. Math.* **4** (1970), 331-340.

[10] A. Lee and I. Streinu, Pebble game algorithms and sparse graphs, *Disc. Math.*, **308** (2008), 1425-1437.

[11] J. C. Maxwell, On the calculation of the equilibrium and stiffness of frames, *Philos. Mag.* **27** (1864), 294-299.

[12] H. Q. Nguyen, Hien Q. Constructing the free erection of a geometry, *J. Combin. Theory Ser. B*, **27** (1979), 216-224.

[13] J. Oxley, Matroid Theory, Graduate Texts in Mathematics, Oxford University Press, Oxford, 1992.

[14] J. M. S. Simes-Pereira, On subgraphs as matroid cells, *Math. Zeitschrift*, **127** (1972), 315-322.

[15] I. Streinu and L. Theran, Sparse hypergraphs and pebble game algorithms, *Eur. Jour. of Combinatorics*, **30** (2009), 1944-1964.