SELF-SIMILAR GIFS TILINGS OF EUCLIDEAN SPACE

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Abstract. An approach to tiling Euclidean space using a graph iterated function system (GIFS) was introduced by Barnsely and Vince [5]. The main concept in this paper is called a *self-similar GIFS* in which all the similarity mappings in the GIFS have the same scaling ratio. Despite the simplification, the tilings associated with a self-similar GIFS have properties that are expected of a self-similar tiling. Any GIFS based tiling by congruent copies of a finite set of prototiles that may reasonably be considered self-similar must be one of our self-similar GIFS-tiling. Moveover, the space of tilings in this paper is the closure, in an appropriate metric space, of the self-similar tilings of W. Thurston [39], generalized from tiling by translated copies of a set of prototiles to tilings by congruent copies of a set of prototiles.

1. Introduction

The tilings in this paper are tilings of Euclidean space by congruent copies of a finite set of prototiles, in particular those that have self-replicating properties. The mathematical investigation of such tilings has a long history. Consider, for example, the prescient 1619 "monster" tiling of Johannes Kepler in Figure 1. Research has been particularly robust since the discovery of the Penrose tilings [29] in 1974 and quasicrystals in 1984.

⟨fig:mon⟩

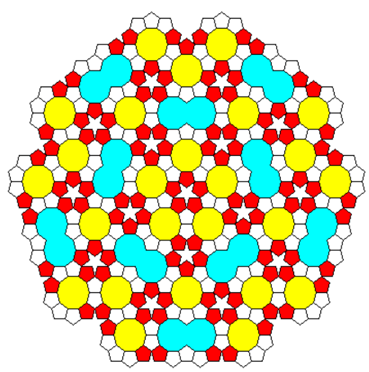

Figure 1. Kelpler's monster

Tilings with self-referential properties has a vast litereature, the subject approached from geometric, topological, analytic, and dynamical points of view. Such tilings, for example, appear early on in work of Mandelbrot [25], Bedford [6], Thurston [39], Kenyon [20] and from a dynamical systems point of view in, for example [30, 33, 37]. Rauzy type tilings appear in [32] and surveys by Berthè and Siegel [8] and by Siegel and Thuswaldner [36], both containing well over 100 references.

The goal of this paper is to provide a unifying overview of this fascinating theory from a graph interated function system (GIFS) point of view. A GIFS is a generealization of an iterated function system (IFS). The concept of a GIFS originated in the construction of deterministic fractals [26] but is implicit in the concept of a Bratteli diagram [10] introduced in 1972 in the

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context of operator algebras. A GIFS is related to the earlier notion of a rep-tile, the term coined by S. Golomb [14] and popularized by Martin Gardner in 1963 in Scientific American [12]. A rep-tile was originally defined as a single polygon that can be tiled by smaller similar and congruent copies of itself. If we generalize from a single polygon to a finite prototile set Q such that each tile in Q is, in turn, tiled by smaller similar copies of prototiles in Q, then we are close to the concept of a GIFS (see Remark 3.2). Systematic use of an IFS in tiling theory gained traction in the early 1990s with the construction of lattice and crysallographic tilings by copies of a single, usually fractal, tile; see for example [4, 13, 15, 21, 22, 38, 40]. The use of a GIFS expands the possibilities to tilings by various shapes and to non-periodic tilings. It makes precise the notion of a geometric substitution tiling; see the survey by Frank [11] and the many references therein.

A GIFS $\mathcal{F} = (G, F)$ consists of a digraph G and a set F of functions assigned to the edges of G. Every GIFS for which the functions are constractions has a unique attractor consisting of a set $(A_1, A_2, \dots, A_n)$ of compact subsets of $\mathbb{R}^d$, one for each vertex of G. The functions in F explicitly indicate how each attractor component is the union of smaller copies of components of $\mathbf{A}$. Formal definitions are provided in Section 3. Figure 2 shows, in the top row, the three attractor components of the last GIFS in Example 9.1. The second row of the figure illustrates that each attractor component is the union of smaller similar copies of the attractor components. In this example, the boundary of each attractor component is a fractal.

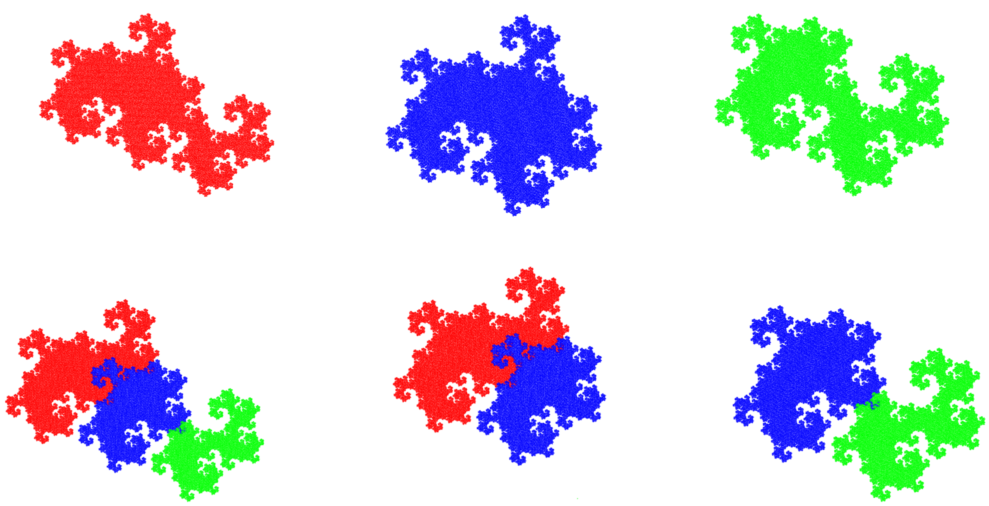

FIGURE 2. Prototile set for the self-similar GIFS of Example 9.1.

⟨fig:r1⟩

The main object in this paper is a *self-similar GIFS* $\mathcal{F}$ and the associated *self-similar GIFS-tilings*. Associated with any self-similar GIFS $\mathcal{F}$ with attractor $(A_1, A_2, \dots, A_n)$ is

- an uncoutable *parameter space* $\mathcal{P} := \mathcal{P}(\mathcal{F})$,
- a *tiling space* $\mathbb{T} := \mathbb{T}(\mathcal{F})$ consisting of a set of tilings of $\mathbb{R}^d$, each with prototile set $\{A_1, A_2, \dots, A_n\}$, and
- a *tiling map* $\mathcal{T} : \mathcal{P} \to \mathbb{T}$ from the parameter space to the tiling space.

Given a self-similar GIFS $\mathcal{F}$, each tiling in $\mathbb{T}(\mathcal{F})$ will be referred to as an *$\mathcal{F}$-tiling*. Both the parameter space and the tiling space are metric spaces, and the tiling map is continuous with respect to these metrics. Via the tiling map there are potentially infinitely many tilings that can be obtained from asingle given self-similar GIFS.

The tilings in Barnsley and Vince [5] use a GIFS, but are very general, including tilings that are somewhat arbitrary. The self-similar GIFS-tilings in this paper are at once a specialization and a simplification - a simplification in that every similarity in the GIFS has the same scaling ratio. Nevertheless, it is shown in Section 6 that any GIFS based tiling by congruent copies of a finite set of prototiles that may reasonably be considered self-similar must be a self-similar GIFS-tiling. Moreover, the self-similar tilings of Thurston and Kenyon [20, 39], generalized from trilings by translation to tilings by congruent copies, are self-similar GIFS-tilings.

A tiling T can be either periodic or non-periodic, periodic if there is a translational symmetry of T, otherwise non-periodic. Figure 3 shows two well known tilings, the twin dragon tiling and the original Rauzy tiling [32]. The twin dragon tiling is periodic and the Rauzy tiling is non-periodic. Both tilings can be obtained from our systematic method for the constuction of self-similar GIFS-tilings. The two additional fractal self-similar GIFS-tilings that appear in Figure 4 are also non-periodic and will be discussed in Section 9.1.

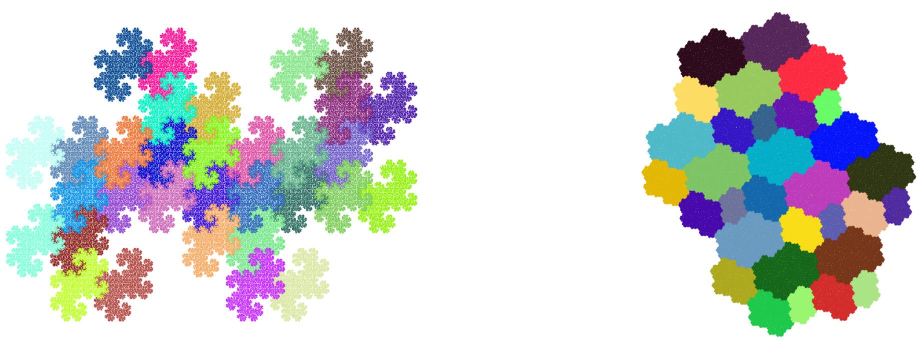

FIGURE 3. Twin Dragon (periodic) and Rauzy (non-periodic) self-similar GIFS-tilings.

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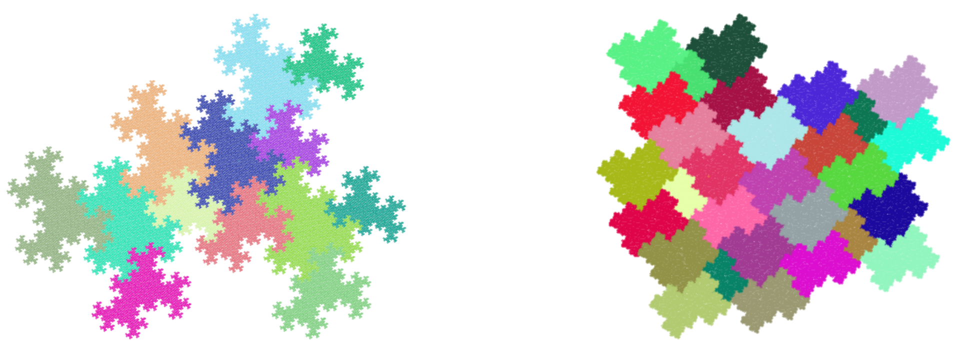

FIGURE 4. self-similar GIFS-tilings from Examples 9.1.

⟨fig:rr1⟩

Since the paper is intended to be self contained and we touch on topics that have appeared previously in some form, not necessarily GIFS related, this paper is, to some extent, expository. In Section 2 we give the organization of the paper and indicate results that are new.

2. Organization and Results

⟨sec:0⟩

Basic notions about tiling Euclidean space, about rep-tiles and rep-sets, and about graph directed iterated function systems are covered in Section 3.

Self-similar GIFS and the associated *self-similar GIFS-tilings* are defined in Section 4. In particular, the parameter space, tiling space, and the tiling map are defined.

Basic properties of self-similar GIFS-tilings are covered in Theorem 5.1 of Section 5. These involve familiar notions in tiling theory like tile frequency, quasiperiodicity, hierarchy, and periodicilty. Proofs that are new in a GIFS setting are included.

Requiring the scaling ratios of all similarity functions in a self-similar GIFS to be equal may seem excessively restrictive. Theorem 6.1 in Section 6, however, essentially states that any GIFS-tiling whose prototile set consists of components of the attractor of the GIFS must be a self-similar GIFS-tiling.

The notion of a self-similar tiling by translates of a set of prototiles was formulated, without reference to a GIFS, by Thurston and Kenyon [20, 39]. Theirs is a global self-similarity. Basically, a tiling T of $\mathbb{R}^d$ is self-similar in their terminogy if there exists a similarity transformation $\phi : \mathbb{R}^d \to \mathbb{R}^d$ with scaling ratio greater than 1 such that, for all $t \in T$, the large tile $\phi(t)$ is tiled, in turn, by tiles in T. In Section 7 Thurston and Kenyon's concept is extended from tilings by translation to tilings by isometric (congruent) copies of a set of prototiles. Calling such a tiling *globally self-similar* we show that:

- Every globally self-similar tiling is a self-similar GIFS-tiling (Theorem 7.1).
- If the parameter of a self-simlilar GIFS-tiling is eventually periodic, then the associated self-similar GIFS-tiling is globally self-similar (Theorem 7.2).
- For a given self-similar GIFS $\mathcal{F}$, the set of globally self-similar tilings in the tiling space $\mathbb{T}(\mathcal{F})$ is dense in $\mathbb{T}(\mathcal{F})$ (Theorem 7.3).

Thus, for a given self-similar GIFS $\mathcal{F}$, the subset of globally self-similar tilings in the set of all self-similar GIFS-tilings play a role analogous to the subset of rational number in the reals.

The subject of Section 8 is a tiling dynamical system $(\mathbb{T}, H)$ of a self-similar GIFS. The function H acts on the tiling space $\mathbb{T}$ by taking a tiling $T \in \mathbb{T}$ one level up, after scaling, in its hierarchy. Theorem 8.1 states that, for a self-similar GIFS $\mathcal{F}$ satisfying natural assumptions, $(\mathbb{T}, H)$ is topologically conjugate to the discrete dynamical system $(\mathcal{P}, S)$, where S is the shift map acting on the parameter space $\mathcal{P}$. The section ends with Theorems 8.2 and 8.3, applying the notions of topological entropy and the Artin-Masur zeta function.

If a GIFS is non-overlapping and has attractor components with nonempty interior, then there are potentially infinitely many associated tilings. In dimensions $d \geq 2$, however, GIFSs with these properties are hard to come by. Section 9 concerns the existence of such self-similar GIFSs. We conclude with a short exposition of an elegant construction, due to Rao, Wen and Yang [31], of Rauzy type tilings using a dual GIFS.

3. Graph Iterated Function System

⟨sec:defs⟩ 3.1. **Tilings, Prototile Sets, and Rep-Sets.** In this paper, a *tile* is a compact subset of R^d, and a *tiling of a set* $X \subseteq \mathbb{R}^d$ is a set of pairwise non-overlapping tiles whose union is X. *Non-overlapping* means that the intersection of any two distinct tiles has measure zero. Two tilings T and T' are *isometric* or *congruent* if there is an isometry of $\mathbb{R}^d$ taking one onto the other, and this is denoted $T \cong T'$. Two tilings T and T' are *equal* denoted $T = T'$ if they are identical.

For a finite set Q of tiles, a *Q-tiling* T is a tiling of $\mathbb{R}^d$ in which each tile is congruent to a tile in Q. The set Q is called a *prototile set* for T. A tile t that is congruent to $q \in Q$ will be referred to as a *type* q tile.

A *rep-set* is a finite multiset Q of tiles in $\mathbb{R}^d$, each tiled by smaller similar copies of tiles in Q. In the recreational literature, these are referred to as irreptiles. We allow multiple copies of tiles to allow for the same shape to be tiled by similar copies of tiles in Q in different ways. For example, a 2×2 square can be tiled by four 1×1 squares or the 2×2 square can be tiled by two 1×1 squares and a 1×2 rectangle. How a GIFS captures the notion of a rep-set is explained in Remark 3.2.

3.2. **Directed Graphs and Adjacency Matrices.** Let $G = (V, E)$ be a finite, strongly connected, directed graph (digraph) with vertex set $V = \{1, 2, \dots, n\}$ and edge set E. A digraph G is *strongly connected* if, for any two vertices i and j, there is a directed path from i to j. In this paper, path always means a directed path, and a path can have repeated vertices and/or edges. The digraph G may have loops and/or multiple edges. A strongly connected digraph G will be called *primitive* if the greatest common divisor of the lengths of all closed paths of G is 1. Equivalently, a strongly connected digraph is primitive if and only if, for k sufficiently large, there is a path of length k joining any two vertices.

For an edge $e = (i, j)$, directed from vertex i to vertex j, the vertex i is denoted e^- and the the vertex j is denoted e^+. Let E_i denote the set of all edges e such that $e^- = i$, i.e., the set of vertices directed out of vertex i.

Associated to a digraph G is its *adjacency matrix* $M := M(G) = (m_{i,j})$, where $m_{i,j}$ is the number of edges from vertex i to vertex j. It is well known that G is strongly connected if and only if the matrix M is irreducible, and G is primitive if and only if M is primitive. A square non-negative matrix M is *primitive* if there is an integer $k \geq 0$ such that all entries of M^k are positive, and M is *irreducible* if for all i, j there is a $k = k(i, j)$ such that $M^k_{i,j} > 0$. Clearly a primitive matrix is irreducible.

The spectral radius of $M(G)$, denoted $\rho(M)$, is an eigenvalue of M called the Perron-Frobenius eigenvalue. The corresponding left and right eigenspaces are 1-dimensional, and $\rho(M)$ has left and right eigenvectors all of whose coordinates are positive.

3.3. **Graph Directed Iterated Function System.**

Definition 3.1 (GIFS)**.** A *graph directed iterated function system (GIFS)* on $\mathbb{R}^d$ is a pair $\mathcal{F} = (G, F)$, where $G = (V, E)$ is a strongly connected digraph and

$$F = \{f_e : e \in E\},$$

where each function $f_e : \mathbb{R}^d \to \mathbb{R}^d$ is a similarity transformation with $\lambda(f_e) < 1.$. The function f_e can be considered as a label on the edge e.

Let $\mathbb{H}$ denote the set of nonempty compact subsets of $\mathbb{R}^d$, and define $\mathbf{F} : \mathbb{H}^n \to \mathbb{H}^n$ as follows. If $\mathbf{X} = (X_1, X_2, \dots, X_n) \in \mathbb{H}^n$, then

$$\mathbf{F}(\mathbf{X}) = \big(F_1(\mathbf{X}), F_2(\mathbf{X}), \dots, F_n(\mathbf{X})\big),$$

where, for $i = 1, 2, \dots, n$,

$$F_i(\mathbf{X}) = \bigcup_{e \in E_i} f_e(X_{e^+}).$$

The following is a well-known result in the theory of graph iterated function systems, which is a generalization of a fundamental result of Hutchinson [17]. In the theorem $\mathbf{F}^k$ denotes the k-fold iteration of $\mathbf{F}$.

⟨thm:john⟩ **Theorem 3.1** ([26])**.** *If (G, F) is a GIFS such that each function in F is a contraction, then there exists a unique $\mathbf{A} = (A_1, A_2, ..., A_n) \in \mathbb{H}^n$ such that*

$$\mathbf{A} = \mathbf{F}(\mathbf{A}) \qquad \text{and} \qquad \mathbf{A} = \lim_{k\to\infty} \mathbf{F}^k(\mathbf{B}) \tag{3.1}$$

eq:aa

independent of $\mathbf{B} \in \mathbb{H}^n$, where convergence is with respect to the Hausdorff metric on $\mathbb{H}^n$.

⟨def:no⟩ **Definition 3.2** (Attractor)**.** The set $\mathbf{A}$ is called the *attractor* of the GIFS, and $\{A_1, A_2, ..., A_n\}$ is its set of *attractor components*, each of which is compact. The first condition in Equation (3.1) can be restated as

$$A_i = \bigcup_{e \in E_i} f_e(A_{e^+}) \tag{3.2}$$

eq:decomp

for $i \in \{1, 2, \dots, n\}$. If, in Equation (3.2), each distinct pair $f_e(A_{e^+}),\ f_{e'}(A_{e'^+})$ is non-overlapping, then A_i is called *non-overlapping*. In this case, $\{f_e(A_{e^+}) : e \in E_i\}$ is a tiling of A_i. If every attractor component is non-overlapping, then the GIFS is called *non-overlapping*.

Although generally not required of a GIFS (G,F), *from here on in this paper all function in F are assumed to be similarity transformations.* The scaling ratio of a similarity $f:\mathbb{R}^d\to\mathbb{R}^d$ is denoted $\lambda(f)$.

Remark 3.1. An ordinary iterated function system (IFS) is the special case of a GIFS whose graph consists of a single vertex with loops and whose attractor consists of a single component.

⟨rem:MG⟩ **Remark 3.2** (Every rep-set is the attractor of a GIFS)**.** FS ven a rep-set Q, let (G,F) be the GIFS where the vertices of G are the elements of Q. Each $p\in Q$ is tiled by smaller similar copies of tiles in Q. If one of those tiles is a smaller similar copy of $p'\in Q$, let $f_{(p,p')}\in F$ be a similarity transformation taking p' onto p. Then

$$p=\bigcup_{e\in E_p} f_e(p_{e^+}),$$

in accordance with Equation (3.2). Conversely, if the attractor components of a GIFS (G,F) have non-empty interior, the GIFS is non-overlapping, and the functions in F are similarities, then the attractor $\mathbf{A}$ of (G,F) is a rep-set.

3.4. **Admissible Patch.** It is assumed in this section that all GIFSs are non-overlapping and that all components of the attractor have nonempty interior. Naturally associated to any GIFS are finite patches of tiles that we call admissible patches. These are natural subdivisions of the attractor components of the GIFS. For a GIFS $\mathcal{F}=(G,F)$, let r be any vertex of G, referred to as a *root*. Referring to Definition (3.2),

$$W_r(\mathcal{F}):=\{f_e(A_{e^+}):e\in E_r\}$$

is a tiling of the attractor component A_r.

Definition 3.3 (Admissible Patch)**.** A set X_r of finite directed paths in G starting at r is called *admissible* if

(1) no proper subpath of a path in X_r lies in X_r, and
(2) if $\sigma\in X_r$ and σ' is any proper subpath of σ starting at vertex r, then $\sigma' e$ is a subpath (not necessarily proper) of a path in X_r for all edges e such that $\sigma'^{+}=e^{-}$.

Applying Equation (3.2) recursively we arrive at the fact that

$$W_r(\mathcal{F},X_r):=\left\{f_\sigma(A_{\sigma^+}):\sigma\in X_r\ \right\}$$

is a tiling of A_r. An *admissible patch* is any patch that is, a possibly scaled, copy of $W_r(\mathcal{F},X_r)$ for some GIFS $\mathcal{F}=(G,F)$, some verrtex r of G, and some admissible set X_r of paths.

4. Self-Similar GIFS-Tiling

⟨sec:GT⟩ 4.1. **Notation.** Denote by $\Sigma^*:=\Sigma^*(G)$ the set of paths of finite length in a digraph $G=(V,E)$ and $\Sigma:=\Sigma(G)$ the set of all infinite paths. An infinite path has a starting vertex but no terminal vertex. A path $\sigma=e_1\,e_2\,\cdots$ will be written as its ordered string of edges $e_i\in E,\ i=1,2\ldots.$ The starting vertex of a path σ will be denoted σ^-, and the terminal vertex of a finite path by σ^+. The length of a finite path σ, i.e., the number of edges, will be denoted $|\sigma|$. A path consisting of a single vertex has length zero.

For $\sigma=e_1e_2\cdots\in\Sigma$ let

$$\sigma|k=e_1e_2\cdots e_k\in\Sigma^*,$$

and $\sigma|0$ the path that is just the vertex σ^-. For any edge e in G, let $\overleftarrow{e}$ be the oppositely directed edge.

Let $\mathcal{F}=(G,F)$ be a GIFS. For any function $f_e\in F$ define

$$f_{\overleftarrow{e}}:=(f_e)^{-1}.$$

Denote by $\overleftarrow{G}$ the digraph obtained from G by reversing the direction on all edges. Define $\overleftarrow{\Sigma}^* := \overleftarrow{\Sigma}^*(G)$ and $\overleftarrow{\Sigma} := \overleftarrow{\Sigma}(G)$ as the set of all finite and infinite paths, respectively, in $\overleftarrow{G}$. For $\sigma = e_1 e_2 e_3 \cdots e_k \in \Sigma^*$, define

$$f_\sigma := f_{e_1} \circ f_{e_2} \circ f_{e_3} \circ \cdots \circ f_{e_k}.$$

For $\overleftarrow{\sigma} = \overleftarrow{e_1}\,\overleftarrow{e_2}\,\overleftarrow{e_3} \cdots \overleftarrow{e_k} \in \overleftarrow{\Sigma}^*$, let

$$f_{\overleftarrow{\sigma}} := f_{\overleftarrow{e_1}} \circ f_{\overleftarrow{e_2}} \circ f_{\overleftarrow{e_3}} \circ \cdots \circ f_{\overleftarrow{e_k}} = f_{e_1}^{-1} \circ f_{e_2}^{-1} \circ f_{e_3}^{-1} \circ \cdots \circ f_{e_k}^{-1}.$$

4.2. **The Parameter Space.** Let $\mathcal{F} = (G, F)$ be a GIFS. Any path $\overleftarrow{\theta} \in \overleftarrow{\Sigma}$ will be referred to as a *parameter* of $\mathcal{F}$. To simplify notation, denote the set of perameters by

$$\mathcal{P} = \mathcal{P}(\mathcal{F}) := \overleftarrow{\Sigma}.$$

Define a metric d on $\mathcal{P}$ by

$$d(\overleftarrow{\sigma}, \overleftarrow{\omega}) = \begin{cases} 0 & \text{if } \overleftarrow{\sigma} = \overleftarrow{\omega} \\ 2^{-k} & \text{otherwise, where } k \text{ is the first integer such that } \overleftarrow{\sigma}_k \neq \overleftarrow{\omega}_k \end{cases}.$$

This makes $(\mathcal{P}, d)$ a compact metric space, which we call the *parameter space* of the self-similar GIFS. A parameter $\overleftarrow{\theta} \in \mathcal{P}$ is *eventually periodic* if there exist $\overleftarrow{\theta}_0, \overleftarrow{\theta}_1 \in \overleftarrow{\Sigma}^*$ such that $\overleftarrow{\theta} = \overleftarrow{\theta}_0\,\overleftarrow{\theta}_1\,\overleftarrow{\theta}_1\,\overleftarrow{\theta}_1 \cdots$.

4.3. **GIFS-Tilings.** According to Theorem 3.1, any GIFS whose functions are contractions has an attractor. An issue, however, is that an attractor component that has empty interior cannot serve as a tile. Even if the attractor components have non-empty interior, it may occur that the GIFS is overlapping (Definition 3.2), resulting in overlap in a tiling obtained from the GIFS. Therefore it is assumed in the following definition that all GIFSs are non-overlapping and that the attractor components have nonempty iinterior.

⟨def:Tgifs⟩ **Definition 4.1** (GIFS-Tiling)**.** Given a GIFS $\mathcal{F} = (G, F)$ and a parameter $\overleftarrow{\theta} = \overleftarrow{\theta}_0\,\overleftarrow{\theta}_1\,\overleftarrow{\theta}_2 \cdots \in \mathcal{P}$, let X_k be an admissible set of paths rooted at θ_k^- such that

(4.1) eq:nest
$$\{\theta_k\, \sigma : \sigma \in X_{k-1}\} \subset X_k \quad \text{for} \quad k = 1, 2, \dots.$$

Let $\mathcal{X} = \mathcal{X}_{\overleftarrow{\theta}} = \{X_k, k = 1, 2, \dots\}$. Define a tiling $T(\mathcal{F}, \overleftarrow{\theta}, \mathcal{X})$ as follows.

(4.2) eq:tiling
$$\begin{aligned} &\text{(patch of tiles)} && T(\overleftarrow{\theta}, k) = \{f_{\overleftarrow{\theta}|k}(t) : t \in W(\mathcal{F}, X_k)\} \\ &\text{(tiling)} && T(\mathcal{F}, \overleftarrow{\theta}, \mathcal{X}) = \bigcup_{k=0}^{\infty} T(\overleftarrow{\theta}, k). \end{aligned}$$

The tiling $T(\mathcal{F}, \overleftarrow{\theta}, \mathcal{X})$ will be referred to as a *GIFS-tiling* or, more generally, an $\mathcal{F}$-tiling.

Notation aside, the idea is simple. We are just blowing up admissible patches $W(\mathcal{F}, X_k)$ along the vertices of the path $\overleftarrow{\theta}$ that is the parameter. That these patches are nested, i.e, $T(\overleftarrow{\theta}, k) \subset T(\overleftarrow{\theta}, k+1)$ follows from Equation 4.1. Their union gives the tiling $T(\mathcal{F}, \overleftarrow{\theta})$.

4.4. **Self-Similar GIFS-Tilings.** GIFS-tilings are very general and can be quite varied and irregular. Although a GIFS-tiling can have at most finitely many tile shapes up to similarilty, it can have infinitely many tile shapes up to congruence. For example, there are GIFS-tilings of the plane into squares of arbitrarily large size or arbitrarily small size. And there are GIFS-tilings of the plane into squares of two sizes, but the small squares are at arbitrary positions in the tiling. Definition 4.3 below places a strong restrictioin of the kinds of tilings in Definition 4.1.

Definition 4.2 (Self-Similar GIFS)**.** Call a GIFS $\mathcal{F} = (G, F)$ a *self-similar GIFS* if

(1) every function $f \in F$ is a similarity transformation with $\lambda(f)$ independent of $f \in F$, the common value denoted $\lambda(\mathcal{F}) < 1$;

(2) $\mathcal{F}$ is non-overlapping and each component of the attractor of $\mathcal{F}$ has nonempty interior.

The set $\{A_1, \dots, A_n\}$ of attractor components of $\mathcal{F}$ will be called the *prototile set* of $\mathcal{F}$, denoted $Q(\mathcal{F})$.

⟨def:TG⟩ **Definition 4.3** (Self-Similar GIFS-Tiling)**.** Given a self-similar GIFS $\mathcal{F}$, for each parameter $\overleftarrow{\theta} \in \mathcal{P}$, the collection $\mathcal{Y} = \{Y_k : k \geq 1\}$ of admissible sets of paths

$$Y_k = \{\sigma \in \Sigma^* : \sigma^- = \theta_k^-, \, |\sigma| = k\},$$

satisfies inclusion property (4.1). In words, the path σ starts where the path $\overleftarrow{\theta}|k$ ends. Referring to Equations (4.2), let $T(\mathcal{F}, \overleftarrow{\theta}, \mathcal{Y})$ be the tiling for the above particular choice $\mathcal{Y}$ and the corresponding admissible sets $W(\mathcal{F}, Y_k)$, $k = 1, 2, \dots$, of patches. Since $\mathcal{Y}$ is completely determined by the parameter $\overleftarrow{\theta}$, we may write $T(\mathcal{F}, \overleftarrow{\theta})$ instead of $T(\mathcal{F}, \overleftarrow{\theta}, \mathcal{Y})$. The tiling $T(\mathcal{F}, \overleftarrow{\theta})$ will be called a *self-similar GIFS-tiling* and will be referred to as an $\mathcal{F}$-tiling. If the self-similar GIFS $\mathcal{F}$ is understood, we may write $T(\overleftarrow{\theta})$ instead of $T(\mathcal{F}, \overleftarrow{\theta})$.

Definition 4.4 (Tiling Space)**.** The *tiling space* $\mathbb{T} = \mathbb{T}(\mathcal{F})$ of a self-similar GIFS $\mathcal{F}$ is the set of all $\mathcal{F}$-tilings of $\mathbb{R}^d$ endowed with the following metric d:

$$d(T, T') = \inf \big\{\epsilon \, : \, T \text{ and } T' \text{ coincide on a patch covering } B(1/\epsilon)\big\}.$$

The *tiling map*

$$\begin{aligned} \mathcal{T} : \mathcal{P}(\mathcal{F}) &\to \mathbb{T}(\mathcal{F}) \\ \overleftarrow{\theta} &\mapsto T(\mathcal{F}, \overleftarrow{\theta}) \end{aligned}$$

is a continuous map from the parameter space $\mathcal{P}(\mathcal{F})$ onto the tiling space $\mathbb{T}$. For each self-similar GIFS there are potentially uncountably many self-similar GIFS-tilings, one for each parameter, although some, and possible all, may coincide.

Remark 4.1. Almost all self-similar GIFS-tilings fill the whole space $\mathbb{R}^d$. More specifically, for all θ in a dense subset of the parameter space $\mathcal{P}$, the tiling $T(\mathcal{F}, \overleftarrow{\theta})$ covers $\mathbb{R}^d$ [5]. There are cases, however, where $T(\mathcal{F}, \overleftarrow{\theta})$ tiles a subset of $\mathbb{R}^d$. For example, consider the self-similar GIFS (G, F) on $\mathbb{R}$, where G consists of a single vertex and two loops e_1, e_2 and $F = \{f_{e_1}, f_{e_2}\}$, where $f_{e_1}(x) = (1/2)\,x$, $f_{e_2}(x) = (1/2)\,x + 1/2$, and $\overleftarrow{\theta} = \overleftarrow{e}_1 \overleftarrow{e}_1 \cdots$. Then $T(\mathcal{F}, \overleftarrow{\theta})$ tiles the half line $\{x \in \mathbb{R} : x \geq 0\}$. *We assume hereafter, unless stated otherwise, that self-similar GIFS-tiling means a tiling of the whole Euclidean space.* See also [19] for Kellendonk's related notion of "forcing the border of a supertile".

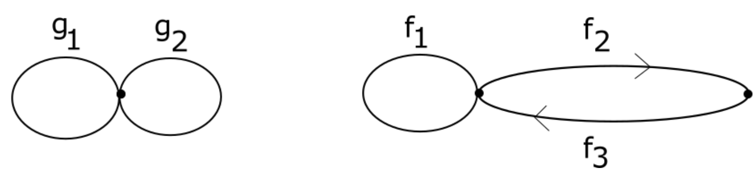

FIGURE 5. A digraph of a GIFS and the digraph of the companion self-similar GIFS.

⟨fig:gg⟩

⟨ex:gb⟩ **Example 4.1** (Ammann Chair Tiling)**.** We illustrate Definition 4.3 with the Ammann chair tiling. The shape, called the A2 tile or sometimes the "golden bee", was discovered by R. Ammann in 1977 and is shown on the left in Figure 6. Figure 5 shows two digraphs. The digraph on the left will be relevant in Section 6. Consider now only the digraph G on the right. With $s = 1/\sqrt{\tau}$, where $\tau = (1 + \sqrt{5})/2$ is the golden ratio, the functions are:

$$f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} s \\ 0 \end{pmatrix}, \quad f_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

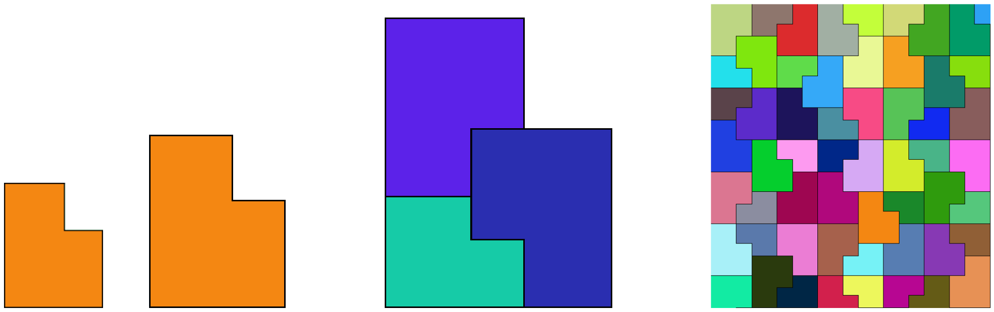

FIGURE 6. Ammann chair tiling: prototiles, second level patch, self-similar GIFS-tiling.

⟨fig:gg2⟩

The respective scaling ratios are $\lambda(f_1) = \lambda(f_2) = \lambda(f_3) = s$. The two attractor components, shown in orange at the left in Figure 6 have nonempty interior. The GIFS (G, F) is non-overlapping. Therefore $\mathcal{F} = (G, \{f_1, f_2, f_3\})$ is a self-similar GIFS. A patch $T(\overleftarrow{\theta}, 2)$ for some $\overleftarrow{\theta} \in \mathcal{P}$ and part of larger patch of a self-similar GIFS-tiling $T(\mathcal{F}, \overleftarrow{\theta})$ are shown at the right in Figure 6.

5. Properties of Self-Similar GIFS Tilings

⟨sec:P⟩ Theorem 5.1 below lists nine properties of self-similar GIFS-tilings. Some of the proofs are omitted because the methods are standard. Statement 3 follows immediatelu from Lemma 5.1. The proofs of statements 4, 5, 6, and 7 are more involved and are included. We start with the definitions of the concepts involved.

Definition 5.1 (Shift Map)**.** The *shift map* $S : \mathcal{P} \to \mathcal{P}$ on the parameter space of a GIFS is defined by $(S\overleftarrow{\theta})_i = \theta_{i+1}$, i.e., $S(e_1e_2e_3\cdots) = e_2e_3\cdots$, and S^k denotes its k^{th} iterate.

That certain tilings have a hierarchical structure has been known at least since Berger's 1966 proof that his set of prototiles is aperiodic [7].

⟨def:h⟩ **Definition 5.2** (Hierarchy)**.** A *hierarchy* for a self-similar GIFS-tiling T with prototile set Q is a sequence $T_0, T_1, T_2, \ldots$ of tilings such that $T_0 = T$ and, for all integers $k \geq 0$, the following properties hold:

(1) T_k is a tiling with prototile set $\{(1/\lambda)^k\, p : p \in Q\}$, up to isometry.
(2) Every tile in T_k is contained in a tile of T_{k+1}.

Call the tiling T_k the k^{th} *level* in the hierarchy of T. A self-similar GIFS $\mathcal{F}$ for which every $\mathcal{F}$-tiling has exactly one hierarchy is called *uniquely hierarchical*. The standard tiling of the plane by squares is an example of a self-similar GIFS-tiling that has many hierarchies.

Definition 5.3 (Tile Frequencies)**.** It has been long known that in any Penrose tiling there are $\tau \approx 1.618$ times as many kites as darts; see [16]. In general, for a prototile $p \in Q(\mathcal{F})$ define $N_{k,\theta}$ and $N_{k,\theta}(p)$ as the number of tiles in the patch $T(\overleftarrow{\theta}, k)$ and the number of tiles of type p in $T(\overleftarrow{\theta}, k)$, respectively. Letting $Q = Q(\mathcal{F}) = \{p_1, p_2, \ldots, p_n\}$, define the *frequency* $\beta_\theta(p)$ of prototile $p \in Q$ in tiling $T(\mathcal{F}, \overleftarrow{\theta})$ and the *frequence vector* $\boldsymbol{\beta}_\theta(\mathcal{F})$ as

$$\beta_\theta(p) := \lim_{k\to\infty} \frac{N_{k,\theta}(p)}{N_{k,\theta}} \qquad \text{and} \qquad \boldsymbol{\beta}_\theta(\mathcal{F}) := \big(\beta_\theta(p_1), \beta_\theta(p_2), \ldots, \beta_\theta(p_n)\big),$$

respectively.

A tiling T is *repetitive*, also called *quasiperiodic*, if, for every patch T_0 of T, there is a real number R such that every ball of radius R contains a patch congruent to T_0.

Call a self-similar GIFS $\mathcal{F}$ *redundant* if $W_i(\mathcal{F}) = W_j(\mathcal{F})$ for some vertices $i \neq j$. Call a self-similar GIFS $\mathcal{F}$ *asymmetric* if, for all vertices i, the only symmetry of A_i that preserves the tiling of the admissible patch $W_i(\mathcal{F})$ is the identity.

⟨lem:sr⟩ **Lemma 5.1.** *Let $\mathcal{F}$ be a GIFS on $\mathbb{R}^d$ such that all of its functions have common scaling ratio and all of its attractor components have nonempty interior. Then $\mathcal{F}$ is non-overlapping if and only if $\lambda(\mathcal{F}) = 1/\sqrt[d]{\rho}$, where ρ is the Perron-Frobenius eigenvalue of $M(G)$.*

Proof. Assume that $\lambda(f) = 1/\sqrt[d]{\rho}$ for all $f \in F$. Denote by x_i the Lebesgue measure of the attractor component A_i, $i = 1, 2, \dots, n$, of $\mathcal{F}$, and let $x = (x_1, \dots, x_n)^t$, where t denotes the transpose. The Lebesgue measure of $f_e(A_{e^+})$ is then $(1/\rho)x_i$. If A_i is overlapping for some i, then $x < (1/\rho)Mx = x$, a contradiction. Here the vector inequality $x < y$ means that $x_i \le y_i$ for all i and $x_i < y_i$ for at least one i.

Conversely, assume that $\mathcal{F} = (G, F)$ is non-overlapping. Then $M(\lambda^d x) = x$, where $\lambda := \lambda(\mathcal{F})$ and $M = M(G)$ is the adjacency matrix of G. Equivalently the eigen-equation $Mx = (1/\lambda^d)x$ holds. Since x is positive, it must be an eigenvector corresponding to ρ. Therefore $\lambda(\mathcal{F}) = 1/\rho$. □

⟨thm:pp⟩ **Theorem 5.1.** *For a given self-similar GIFS $\mathcal{F} = (G, F)$ on $\mathbb{R}^d$ with prototile set $Q(\mathcal{F})$ and for every pair $\overleftarrow{\theta}, \overleftarrow{\theta}' \in \mathcal{P}(\mathcal{F})$ the following properties hold for the self-similar GIFS-tiling $T(\mathcal{F}, \overleftarrow{\theta})$.*

1. *Each tile in $T(\mathcal{F}, \overleftarrow{\theta})$ is congruent to a tile in $Q(\mathcal{F})$.*
2. *If $\mathcal{F}$ is primitive, then a congruent copy of each tile in $Q(\mathcal{F})$ occurs in $T(\mathcal{F}, \overleftarrow{\theta})$.*
3. *$\lambda(\mathcal{F}) = 1/\sqrt[d]{\rho(\mathcal{F})}$, where $\rho(\mathcal{F})$ is the Perron-Frobenius eigenvalue of the adjacency matrix of G.*
4. *$\boldsymbol{\beta}_\theta(\mathcal{F})$ is equal to the normalized (unit vector with respect to the 1-norm) positive left eigenvector corresponding to the Perron-Frobenius eigenvalue of $\mathcal{F}$, independent of the parameter $\overleftarrow{\theta} \in \mathcal{P}$.*
5. *Every $\mathcal{F}$-tiling T has a hierarchy $T = T_0, T_1, T_2, \dots$, such that each T_k, $k = 0, 1, 2, \dots$, is a self-similar GIFS-tiling. Moreover, $T_k = f_{\overleftarrow{\theta}|k} T(\mathcal{F}, S^k \overleftarrow{\theta})$ provides such a hierearchy.*
6. *If $S^k(\overleftarrow{\theta}) = S^k(\overleftarrow{\theta}')$ for some k, then $T(\overleftarrow{\theta}) \cong T(\overleftarrow{\theta}')$. Moreover, if $\mathcal{F}$ is uniquely hierarchical, asymmetric and not redundant, then $T(\overleftarrow{\theta}) \cong T(\overleftarrow{\theta}')$ if and only if $S^k(\overleftarrow{\theta}) = S^k(\overleftarrow{\theta}')$ for some k.*
7. *$T(\mathcal{F}, \overleftarrow{\theta})$ is repetitive for all $\overleftarrow{\theta} \in \mathcal{P}$.*
8. *If $\mathcal{F}$ is uniquely hierarchical, tben every $\mathcal{F}$-tiling is non-periodic.*
9. *If $\mathcal{F}$ is uniquely hierarchical, asymmetric, and not redundant, then there are uncountable many $\mathcal{F}$-tilings up to congruence.*

Remark 5.1 (Primitivity is needed in statement 2 of Theorem 5.1)**.** The following is an example of a self-similar GIFS that is not primitive for which a prototile does not appear in a self-similar GIFS-tiling. Let $\mathcal{F} = (G, F)$ be the 1-dimensional self-similar GIFS with digraph given by its adjacency matrix M and $F = \{f_{1,2}, f_{1,3}, f_{2,1}, f_{3,1}\}$ where

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad f_{1,2}(x) = \frac{x}{\sqrt{2}} \qquad f_{1,3}(x) = \frac{x+1}{\sqrt{2}}$$
$$f_{2,1}(x) = f_{3,1}(x) = \frac{x}{\sqrt{2}},$$

and $f_{i,j}$ denotes the function on the edge (i, j). The attractor components are intervals $\{[0, \sqrt{2}], [0, 1], [0, 1]\}$, but $T(\mathcal{F}, \overleftarrow{\theta})$ is a tiling of the real line by intervals of just length $\sqrt{2}$ if $\overleftarrow{\theta} = 1\,2\,1\,2 \cdots$.

Example 5.1 (Tile Frequencies for the Ammann chair tiling of Example 4.1)**.** The adjacency matrix of the self-similar GIFS digraph is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The Perron-Frobenius eigenvalue is $\tau = (1+\sqrt{5})/2$, the golden ratio. The normalized corresponding left eigenvector is $(1/\tau, 1/\tau^2)$. Therefore, asymptotically about 61.80% of the tiles in an Ammann chair tiling are the large prototile, and about 38.20% are the small prototile.

Proof of Properety 4. Denote the (i,j) entry of M^k by $m^{(k)}_{i,j}$, which is the number of paths in the digraph of $\mathcal{F}$ of length k from vertex i t vertex j. For a fixed parameter $\overleftarrow{\theta}$, let $i(k) = \theta^-_k$, i.e., the last vertex in the path $\overleftarrow{\theta}|k$. For ease of notation, let $N_k = N_{k,\theta}$ and $N_k(p) = N_{k,\theta}(p)$. Referring to the Definition 4.3 of an $\mathcal{F}$-tiling we have

$$N_k(p_j) = m^{(k)}_{i(k),j} \qquad \text{and} \qquad N_k = \sum_{j=1}^n m^{(k)}_{i(k),j}.$$

Noting that $N_k = \|\big(N_k(p_1), N_k(p_2), \dots, N_k(p_n)\big)\|_1$ we have

$$\Big(\frac{N_k(p_1)}{N_k}, \frac{N_k(p_2)}{N_k}, \dots, \frac{N_k(p_n)}{N_k}\Big) = \frac{\big(N_k(p_1), N_k(p_2), \dots, N_k(p_n)\big)}{\|\big(N_k(p_1), N_k(p_2), \dots, N_k(p_n)\big)\|_1} = \frac{e_{i(k)} \cdot M^k}{\|e_{i(k)} \cdot M^k\|_1}.$$

A known consequence [27] of the Perron-Frobenius theorem, with Perron-Frobenius eigenvalue ρ, is

$$\lim_{k\to\infty} \frac{M^k}{\rho^k} = v^t \cdot w,$$

where v is the right and w is the left positive eigenvector of M corresponding to eigenvalue ρ and normalized so that $w \cdot v^t = 1$. In fact, we may assume that $\|w\|_1 = 1$. For any matrix B, let B_i denote the i^{th} row of B. Then $e_{i(k)} \cdot (M/\rho)^k = [(M/\rho)^k]_i$. As $k \to \infty$, the sequence $\big(e_{i(k)} \cdot (M/\rho)^k\big)$ of vectors gets close to the sequence of vectors whose terms are

$$e_{i(k)} \cdot (v^t \cdot w) = (e_{i(k)} \cdot v^t) \cdot w = c_k w,$$

where c_k is a constant depending on k that can take on at most n values. Upon normalization we have

$$\boldsymbol{\beta}_\theta(\mathcal{F}) = \lim_{k\to\infty} \frac{e_{i(k)} \cdot M^k}{\|e_{i(k)} \cdot M^k\|_1} = \lim_{k\to\infty} \frac{e_{i(k)} \cdot (\frac{M}{\rho})^k}{\|e_{i(k)} \cdot (\frac{M}{\rho})^k\|_1} = w,$$

independent of the particular standard basis vector independ of the parameter $\overleftarrow{\theta}$. ☐

Proof of Property 5. The set of attractor components $Q = \{A_1, A_2, \dots, A_n\}$ is a prototile set for T. Every tile $t \in T_k$, as defined explicitly in statement 5 of Theorem 5.1 has the form

$$t := f_{\overleftarrow{\theta}|k} \circ f^{-1}_{\theta_{k+1}} \circ \cdots \circ f^{-1}_{\theta_{k+j}} \circ f_\sigma(A_{\sigma^+}), \qquad \text{where} \qquad \sigma^-_1 = \theta^-_j,\ |\sigma| = j-k.$$

Note that t is isometric to $(1/\lambda)^k(A\sigma^+)$. Therefore condition (1) in Definition 5.2 is satisfied.

It follows from the formula above for t and from Equation (3.2) that

$$\begin{aligned} t &= f_{\overleftarrow{\theta}|k-1} \circ f^{-1}_{\theta_k} \circ \cdots \circ f^{-1}_{\theta_j} \circ f_\sigma\Big(\bigcup_{e\in E_{\sigma^+}} f_e(A_{e^+})\Big) \\ &= \bigcup \Big\{ f_{\overleftarrow{\theta}|k-1} \circ f^{-1}_{\theta_k} \circ \cdots \circ f^{-1}_{\theta_j} \circ f_\sigma \circ f_e(A_{e^+}) : e \in E_{\sigma^+} \Big\} \\ &\subset \bigcup \Big\{ f_{\overleftarrow{\theta}|k-1} \circ f^{-1}_{\theta_k} \circ \cdots \circ f^{-1}_{\theta_j} \circ f_\omega(A_{\omega^+}) : \omega^- = \theta_{k+j}{}^-,\ |\omega| = j-(k-1) \Big\}. \end{aligned}$$

Because each $f_{\overleftarrow{\theta}|k-1} \circ f^{-1}_{\theta_k} \circ \cdots \circ f^{-1}_{\theta_j} \circ f_\omega(A_{\omega^+})$ in the line above is a tile in T_{k-1}, each tile in T_k is, in turn, tiled by a patch in T_{k-1}. This proves condition (2) in Definition 5.2. Therefore $T = T_0, T_1, T_2, \dots$ is indeed a hierarchy. ☐

Proof of Property 6. If $S^k(\overleftarrow{\theta}) = S^k(\overleftarrow{\theta}')$ for some k, then an isometry taking $T(\overleftarrow{\theta})$ onto $T(\overleftarrow{\theta}')$ is $f_{\overleftarrow{\theta}'|k} \circ (f_{\overleftarrow{\theta}|k})^{-1}$.

In the other direction, we will show, under the conditions assumed in the statement, that if two tilings $T(\overleftarrow{\theta})$ and $T(\overleftarrow{\theta}')$ are congruent, then there is an integer j such that $S^j(\overleftarrow{\theta}) = S^j(\overleftarrow{\theta}')$. Assume that $T(\overleftarrow{\theta}) \cong T(\overleftarrow{\theta}')$. Let t_0 be an arbitrary tile in $T_0 = T(\overleftarrow{\theta})$ and t_0' the corresponding tile in $T_0' = T(\overleftarrow{\theta}')$ under the isometry; call it $h : \mathbb{R}^d \to \mathbb{R}^d$. In the hierarchy, let $t_k \in T_k(\overleftarrow{\theta})$ be such that $t_k \subset t_{k+1}$ for all $k \geq 0$; define t_k' similarly.

Let $j \geq k$ be sufficiently large that $t_0 \in T(\theta, j)$ and $t_0' \in T(\theta', j)$. Because it is assumed that $\mathcal{F}$ is uniqely hierarchical, the hierarchy must be the one provided in Theorem **??**. Therefore, for all $j \geq k \geq 0$, if $t \in T_k$ and $t' \in T_k$, then

$$t_k = f_{\overleftarrow{\theta}|j} f_\sigma(A_{\sigma^+}) \qquad\qquad t_k' = f_{\overleftarrow{\theta}'|j} f_\omega(A_{\omega^+}),$$

where $|\sigma| = j - k$, $\sigma^- = (\theta_j)^-$ and $|\omega| = j - k$, $\omega^- = (\theta_j')^-$. Let $\sigma = \sigma_{j-k}\,\sigma_{j-k-1}\cdots\sigma_2\,\sigma_1$ and $\omega = \omega_{j-k}\,\omega_{j-k-1}\cdots\omega_2\,\omega_1$. Note that, for $i = 1, 2, \dots, j-k$, the function f_{σ_i} gives the embedding of t_{i-1} into t_i. The same is true for t_k'. If h is an isometry that takes $T(\overleftarrow{\theta})$ onto $T(\overleftarrow{\theta}')$, then h takes t_{k+1} onto t_{k+1}' and the tiling of t_{k+1} by a patch in T_k onto the tiling of t_{k+1}' by a patch in T_k'. By the assumption of non-redundancy and asymmetry, it is the case that $\sigma_i = \omega_i$, $i = 1, 2, \dots, j-k$.

Now take any integer $J > j$, so that

$$t_k = f_{\overleftarrow{\theta}|J} f_\sigma(A_{\sigma^+}) \qquad \text{where} \qquad \sigma = \theta_J\,\theta_{J-1}\cdots\theta_{j+1}\,\sigma_{j-k}\,\sigma_{j-k-1}\cdots\sigma_2\,\sigma_1$$

$$t_k' = f_{\overleftarrow{\theta}'|J} f_\omega(A_{\omega^+}) \qquad \text{where} \qquad \omega = \theta_J'\,\theta_{J-1}'\cdots\theta_{j+1}'\,\omega_{j-k}\,\omega_{j-k-1}\cdots\omega_2\,\omega_1.$$

As above, we have $\theta_i' = \theta_i$ for all $i > j$. □

Proof of Property 7. We use Property 5 of Theorem 5.1 in this proof. Any patch X of an $\mathcal{F}$-tiling T is contained in patch $T(\overleftarrow{\theta}, k)$ for some k. Given $n \geq 1$, there exists a real number R such that every ball of radius R contains a tile of $T_n(\overleftarrow{\theta}) = T_n(\mathcal{F}, \overleftarrow{\theta})$, the n^{th} level of the hierarchy of $T(\overleftarrow{\theta}) = T(\mathcal{F}, \overleftarrow{\theta})$. Therefore it suffices to show that there is an n such that every tile of $T_n(\overleftarrow{\theta})$ contains an isometric copy of $T(\overleftarrow{\theta}, k)$.

Let m be the greatest common divisor of all closed paths in G. We claim that there exists an M such that if $n \geq M$ and $n \equiv 0 \pmod m$ the following holds: for any vertex v of G there is a circuit through v of length n. To prove the claim, let C be a circuit that contains every vertex of G. There are circuits $C_1, C_2, \dots, C_k$ of lengths $mq_1, mq_2, \dots, mq_k$ such that that $\gcd(q_1, q_2, \dots, q_k) = 1$. An elementary result in number theory states there exists an N such that if integer $s \geq N$, then there exists positive integers $a_1, a_2, \dots, a_k$ such that $\sum_{i=1}^k a_i q_i = s$. Therefore, by traversing C with detours around the circuits $C_1, C_2, \dots, C_k$ sufficiently many times, for any vertex v, there is a circuit containing v of length $ms + |C| = m(s + |C|/m)$, an integer multiple of m. Taking $M = mN + |C|$, the claim is proved.

Given k, we now show that there is an integer n such that every tile of $T_n(\overleftarrow{\theta})$ contains an isometric copy of $T(\overleftarrow{\theta}, k)$. Let D denote the diameter of G, i.e., the greatest directed distance between any two vertices. Let $n = M + k + D$, where M is as in the paragraph above. Let $u = \theta_k^-$, i.e., the last vertex of $\overleftarrow{\theta}|k$ in $\overleftarrow{\Sigma}$. Let $K \geq n$ and let $w = \theta_K^-$. A tile t of $T_n(\overleftarrow{\theta})$ by definition has the form

$$t = f_{\overleftarrow{\theta}|K} \circ f_\sigma(A_{\sigma^+}) \qquad \text{where} \quad |\sigma| = K - n,\ \sigma^- = \theta_K^+.$$

Let $v = \sigma^+$. We claim that there exists a path γ from v to u in G of length $n - k$. Assume that the claim is true, and consider the patch of tiles

$$T = \{f_{\overleftarrow{\theta}|K} \circ f_\sigma \circ f_\gamma \circ f_\omega(A_{\omega^+}) \,:\, |\omega| = k,\ \omega^- = \gamma^+\}.$$

Since $|\overleftarrow{\theta}|K|| = |K| = (K-n)+(n-k)+k = |\sigma|+|\gamma|+|\omega|$, each tile in T is a tile in $T(\overleftarrow{\theta})$. Moreover, since $\gamma^+ = u$, the tiling T is an isometric copy of $T(\overleftarrow{\theta}, k)$. It now only remains to prove the claim.

Because G is strongly connected, there is a simple (no crossing) path $\widehat{\gamma}$ from v to u of length at most D, and also a simple path δ from u to w. The concatenation of δ and the path $\theta_K\,\theta_{k-1}\cdots\theta_{k+1}$ form a cycle which implies that $(K-k)+|\delta| \equiv 0 \pmod m$. Similarly, the concatenation of the paths $\widehat{\gamma}, \delta$ and σ form a cycle which implies that $(K-n)+|\widehat{\gamma}|+|\delta| \equiv 0 \pmod m$. The two congruences yields $n-k-|\widehat{\gamma}| \equiv 0 \pmod m$. Therefore, by the definition of M, there is a circuit β containing vertex v such that $|\beta| = n-k-|\widehat{\gamma}|$ if $n-k-|\widehat{\gamma}| \geq M = n-k-D$, which is equivalent to $D \geq |\widehat{\gamma}|$, which is clearly true. Taking $\gamma = \beta\,\widehat{\gamma}$ we have

$$|\gamma| = |\beta| + |\widehat{\gamma}| = (n-k-|\widehat{\gamma}|) + |\widehat{\gamma}| = n-k. \qquad \square$$

6. Every GIFS-Based Tiling is a Self-Similar GIFS-Tiling

⟨sec:J⟩ The main result of this section is as follows.

⟨thm:F⟩ **Theorem 6.1.** *Let $\mathcal{F}$ be a primitive GIFS with no two attractor components that are similar but not congruent. Any $\mathcal{F}$-tiling T (Definition 4.1) whose prototile set is the set of components of the attractor of $\mathcal{F}$ must be a self-similar $\mathcal{F}$-tiling (Definition 4.3).*

Remark 6.1. The necessity of assumeing $\mathcal{F}$ primitive is explained in Remark 6.2 below. The reason for the assumption on the components of the attractor of $\mathcal{F}$ is as follows. Assume that there exist attractor components p, p' of self-similar GIFS $\mathcal{F}$ that are similar but not congruent with $\lambda(p) > \lambda(p')$. Assume further that T is a self-similar $\mathcal{F}$-tiling as in Definition 4.3. In T it is possible that there exists a (possibly infinite) set T' of tiles congruent to p in the first level hierarcy of T. Take any *arbitrary* subset T'' of tiles in T' and remove from each tile in T'' the tiles in T that it contains. What results is still an $\mathcal{F}$-tiling, but a type of tiling that we eliminate from consideration due to its arbitrary nature.

The proof of Theorem 6.1 hinges on the concept of commensurability and on several lemmas. The notion of commensurability in the context of tiling arose as early as 1976 in a paper of Kakutani [18] and in Sadun's 1998 generalization of the pinwheel tiling [34].

Definition 6.1 (Commensurable GIFS)**.** Let F be a set of similarity transformations from $\mathbb{R}^d$ onto $\mathbb{R}^d$. For $f, g \in F$, call f and g *commensurable* if

$$\frac{\log\big(\lambda(f)\big)}{\log\big(\lambda(g)\big)} \in \mathbb{Q}.$$

Call F *commensurable* if every pair of functions in F is commensurable, and call a GIFS (G, F) *commensurable* if F is commensurable.

⟨prop:C⟩ **Proposition 6.1.** *A set F of similarities is commensurable if and only if there is a real number $s > 0$ and a set $\{b_f \in \mathbb{N} : f \in F\}$ of positive integers such that $\lambda(f) = s^{b_f}$ for $f \in F$.*

Proof. The existence of a real $s > 0$ and a set $\{b_f \in \mathbb{N} : f \in F\}$ of positive integers such that $\lambda(f) = s^{b_f}$ for all $f \in F$ clearly implies that F is commensurable.

In the other direction, let

$$\alpha_f = \log_s\big(\lambda(f)\big) \qquad \text{so that} \qquad \lambda(f) = s^{\alpha_f}.$$

Let $f_0 \in F$. By the assumption that F is commensurable, there is a $d \in \mathbb{N}$ and $b_f \in \mathbb{N}$ for all $f \in F$ such that $\alpha_f/\alpha_{f_0} = b_f/d$. Let $s' = s^{\frac{\alpha_{f_0}}{d}}$. Then, for all $f \in F$,

$$\lambda(f) = s^{\alpha_f} = \Big(s^{\frac{\alpha_{f_0}}{d}}\Big)^{b_f} = (s')^{b_f}. \qquad \square$$

Let G be a strongly connected digraph whose edges are colored using q colors, $q \geq 2$. For an admissible set X_r of paths rooted at a vertex r of G, call two paths *equivalent* if they contain the same number of edges of each color, and let $|X_r|_\equiv$ denote the number of equivalence classes.

⟨lem:inf⟩ **Lemma 6.1.** *Let G be a strongly connected, primitive digraph whose edges are colored using q colors, $q \geq 2$. For every integer N there exists an M such that, if X_r is an admissible set of paths rooted at vertex r with $|X_r| \geq M$, then $|X_r|_\equiv \geq N$.*

Proof. If the lemma holds for every 2-coloring, then it holds for every q-coloring. To see this, let the colors be $\{1, 2, \dots, q\}$. For all edges colored $3, 4, \dots q$, change the colors to color 2. The number $|X_r|$ does not change, and $|X_r|_\equiv$ cannot increase.

We now prove the result for every 2-coloring (say red and blue) of its edge set. By way of contradiction assume that there is a 2-coloring, a vertex r, a natural number N, and a sequence $(X_k)_{k\geq 1}$ of admissible paths rooted at r such that

(1) $|X_k| \to \infty$ as $k \to \infty$, and
(2) $|X_k|_\equiv \leq N$.

In particular, from (2) it follows that there must be a bound m, independent of k, such that the set of lengths satisfies $|\{|\sigma| \, : \, \sigma \in X_k\}| \leq m$. We claim that there is a sequence (Y_k) of admissible sets of paths satisfying properties (1) and (2) above and such that, for each k, all paths in Y_k have the same length. To prove the claim, denote the lengths of paths in X_k by $l_1(k) > l_2(k) > \cdots > l_m(k)$. We now prove the claim by induction on m, the number of distinct lengths of paths in the X_k. The claim is triviially true for $m = 1$. Assume it true for $m - 1$ and let the sequence $(X_k)_{k\geq 1}$ have paths of m different lengths.

Let Y_k be the set of paths obtained from X_k by replacing (pruning) each path σ of length $l_1(k)$ by its subpath σ' rooted at r and having length l_2. Note that Y_k remains a set of admissible paths. We will show that the sequence (Y_k) of sets of admissible paths satisfies conditions (1) and (2). This will complete the induction argument because Y_k has one less path length than X_k.

Concerning condition (1), it follows from the definition of an admissible set of paths that, if v is the second to last vertex of σ and v' is the last vertex of σ', then the outdegree in G of all vertices on the path from v' to v, including v' but not including v, have outdegree 1. This implies that $|Y_k| \leq |X_k|/\Delta$, where Δ is the maximum outdegree of vertices in G. Therefore $|Y_k| \to \infty$ as $k \to \infty$.

Concerning condition (2), first note that there cannot exist a cycle C in G such that the outdegree of every vertex on C is 1. Otherwise there would exist an attractor component with empty interior, which we are assuming in this section is not the case. Therefore, the longest path γ such that every vertex, except the last, has outdegree 1 is less than n, the order of G. Assume that, in going from X_k to Y_k we prune just one prune just one length at a time, obtaining a sequence $X + k = X_k^0, X_k^1, \dots, X_k^q = Y_k$, where $q = l_1(k) - l_1(k) < n$. Since adjoining a single edge to a path can change the number of red edges (or blue edges) by at most 1, we have $|X_k^{i+1}|_\equiv \leq 2|X_k^i|_\equiv$. Therefore $|Y_k|_\equiv \leq 2^n|Y_k|_\equiv \leq 2^n N$.

It remains to prove a contradiction in the case that, for each k, all paths in Y_k have the same length $n(k)$. If all paths in Y_k have the same length, then by the definition of admissible path, Y_k is the set of all paths of length $n(k)$ rooted at vertex r in G. There exists a closed path c_1 in G containing r that is not monochromatic (recall that both red and blue are used in the coloring). Let L_1 denote the length of c_1. By primitivity of G, there exists another closed path c_2 in G containing r whose length L_2 is relatively prime to L_1. Any non-negative integer solution x, y to the equation

$$xL_1 + yL_2 = n(k)$$

provides a path $c_k(x, y)$ in Y_k obtained by winding x times around c_1 followed by winding y times around c_2. Call such a path a (c_1, c_2)-path. For clarity we omit the index k in what follows, i.e., k fixed and, for example, $n = n(k)$. If n is sufficiently large, then from elementary number theory there are positive integers x_0, y_0 such that $x_0 L_1 + y_0 L_2 = n$. It follows that

$x = x_0 - jL_2$, $y = y_0 + jL_1$ is also a solution for any $j \in \mathbb{Z}$. Since we seek non-negative solutions, the condition

$$\frac{x_0}{L_2} \geq j \geq -\frac{y_0}{L_1}$$

must be satisfied, which implies that there are

$$\left\lfloor \frac{x_0}{L_2} + \frac{y_0}{L_1} \right\rfloor = \left\lfloor \frac{n}{L_1 L_2} \right\rfloor \xrightarrow[n\to\infty]{} \infty$$

solutions. In other words, the number of (c_1, c_2)-paths in Y_k goes to infinity with k.

Denote by a_1, a_2 the number of red edges on L_1 and L_2, respectively. Two of the (c_1, c_2)-paths in Y_k are in the same color equivalence class if and only if they contain the same number of red edges. Counting the number of red edges on the path corresponding to solution x, y, i.e., to each valid j, we obtain $xa_1 + ya_2 = (x_0 - ja_2)a_1 + (y_0 + ja_1)a_2$ red edges. Therefore, two (c_1, c_2)-paths, which we denote by $c(i)$ and $d(j)$, are in the same equivalence class if and only if

$$(x_0 - ja_2)a_1 + (y_0 + ja_1)a_2 = (x_0 - ia_2)a_1 + (y_0 + ia_1)a_2$$

which simplifies to $(i - j)(L_1a_2 - L_2a_1) = 0$. If $i = j$, then $c(i) = c(j)$. That $L_1a_2 = L_2a_1$ is impossible since L_1 and L_2 are relatively prime and $0 < a_1 < L_1$. We have shown that (c_1, c_2)-path in Y_k is in its own equivalence class. Since the number of (c_1, c_2)-path in Y_k goes to infinity with k, we have the desired contradiction to condition (2). □

⟨lem:nc⟩ **Lemma 6.2.** *Let $\mathcal{F}$ be a primitive GIFS. If there exists a tiling T of $\mathbb{R}^d$ (not necessarily a GIFS-tiling) having a finite prototile set and containing admissible $\mathcal{F}$-patches of arbitrary large cardinality, then $\mathcal{F}$ must be commensurable.*

Proof. For a set W of tiles, let $|W|$ denote the cardinality of W, and let $|W|_\equiv$ denote the number of tiles up to congruence. Because it is assumed that T has a finite prototile set and contains admissible (scaled) $\mathcal{F}$-patches of arbitrary large cardinality, there must be a sequence $\{W_r(X_k)\}$ of (unscaled) admissible patches such that $|W_r(X_k)|_\equiv$ is bounded but $\lim_{k\to\infty} |W_r(X_k)| = \infty$.

By way of contradiction, assume that $\mathcal{F}$ is not commensurable. We will show that, for every N there exists a k_0 such that $|W_r(X_{k_0})|_\equiv \geq N$, a contradiction.

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the scaling ratios of the functions in F. Let $s = \lambda_1$, and define α_i, $i = 1, 2, \ldots, m$, by $\lambda_i = s^{\alpha_i}$. Note that $\alpha_1 = 1$. The commensurable relation is an equivalence relation. Partition the set $\{\alpha_1, \ldots, \alpha_m\}$ into equivalence classes, and call two edges of G equivalent if the corresponding $\alpha' s$ are equivalent. Let the number of equivalence classes be q, which is at least 2 by Proposition 6.1. Color the edges of G in q colors according to their equivalence class. For an edge e, denote the color by $\alpha(e)$. For a path $\sigma \in \Sigma^*(G)$, define $\alpha(\sigma) := \sum_{e\in\sigma} \alpha(e)$.

Let $\sigma, \omega \in X_k$. Because a set of pairwise incommensurable numbers are linearly independent over $\mathbb{Q}$, we have that $\alpha(\sigma) = \alpha(\omega)$ if and only if, for every color, the number of occurences of that color in σ equals the the number of occurences of that color in ω. Now $\lambda(f_\sigma) = \lambda(f_\omega)$ if and only if $\alpha(\sigma) = \alpha(\omega)$ if and only if σ and ω are in the same color equivalence class. Because $|X_k| = |W_r(X_k)|$ we have $\lim_{k\to\infty} |X_k| = \infty$. Call $\lambda(f_\sigma)$ the scaling ratio of the path σ. By Lemma 6.1, for every N there exists a k_0 such that if $k \geq k_0$, then $|X_k|_\equiv \geq N$. Therefore, for every N there exists a k_0 such that if $k \geq k_0$, then there exists at least N paths in X_k with pairwise different scaling ratios λ. For $\sigma \in X_k$, there are at most n (order of digraph G) possibilities for σ^+, which implies that, for $k \geq k_0$, there are at least N/n distinct tiles $f_\sigma(A_{\sigma^+})$ in $W_k(X_k)$, i.e., $|W_r(X_{k_0})|_\equiv \geq N/n$. □

⟨rem:prim⟩ **Remark 6.2** (Primitivity in Theorem 6.2 is necessary)**.** The following is a counterexample to Lemma 6.2 if the assumption of primitivity is removed. Consider the GIFS $\mathcal{F} = (G, F)$ on $\mathbb{R}$, where G is the graph consisting of two vertices r, r', with edges e_1, e_2 from r to r' and edges e_3, e_4 from r' to r. Note that G is not primitive; the lengths of all closed paths are divisible by 2. Let $F = \{f_{e_1}, f_{e_2}, f_{e_3}, f_{e_4}\}$ where $f_{e_1}(x) = 3/4\, x$, $f_{e_2}(x) = 3/4\, x + 1/2$, $f_{e_3}(x) = 1/3\, x$, $f_{e_4}(x) = 1/3\, x + 1/3$. The attractor components of $\mathcal{F}$ are the intervals $A_r = [0, 1]$, $A_{r'} = [0, 2/3]$. Note that the scaling ratios 3/4 and 1/3 are not commensurable; thus $\mathcal{F}$ is not commensurable. Let $X_r(k)$

be the set of all paths in G rooted at r of length $2k$; this set of paths is admissible. It is routine to check that the admissible patch $W(\mathcal{F}, X_r(k))$ consists of the interval $[0, 4^k]$ subdivided into 4^k unit intervals. Let T be the tiling of the line by unit intervals. Thus T contains admissible patches of arbitrary large cardinality.

⟨def:companion⟩ **Definition 6.2** (Companion GIFS)**.** Let $\mathcal{F} = (G, F)$ be a commensurable GIFS. By Proposition 6.1 there is an $s > 0$ and a set $\{a_e \in \mathbb{N} : e \in E\}$ of positive integers associated with each edge $e \in E$ of G such that $\lambda(f_e) = s^{a_e}$ for all $e \in E$. Attach the label a_e to each edge of G. Constuct a new GIFS $\mathcal{F}' = (G', F')$, called the *companion* of $\mathcal{F}$, as follows. To obtain the graph G', consider each edge $e = (u, v)$ of G with $a_e > 1$. Replace e by a path $\sigma(e) := e_1 e_2 \cdots e_{a_e}$ from u to v. Note that no vertex of G has been removed. Also note that G' is strongly connected if and only if G is strongly connected. It is not hard to see that there exist functions $f_{e_1}, f_{e_2}, \dots, f_{e_{a_e}}$ on the respective new edges $e_1, e_2, \dots, e_{a_e}$ such that $\lambda(f_{e_i}) = s$ for $i = 1, 2, \dots, a_e$ and therefore $f_{\sigma(e)} = f_e$. The graph G' and function set F' is the result of the above alterations for all edges e with $a_e > 1$.

Example 6.1 (Companion GIFS)**.** On the left in Figure 5 is the digraph of a GIFS $\mathcal{F} = (G, F)$ where G has one vertex and two loops and $F = \{g_1, g_2\}$, where

$$g_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} s \\ 0 \end{pmatrix}, \qquad g_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s^2 & 0 \\ 0 & -s^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with $s = 1/\sqrt{\tau}$, where τ is the golden ratio. By Proposition 6.1, $\mathcal{F}$ is commensurable, the scaling ratios being $\lambda(g_1) = s$, $\lambda(g_2) = s^2$. The attractor is the Ammann chair tile shown at the left in Figure 6 (either orange polygon). Let $\mathcal{F}' = (G', F)$, where G' is the graph on the right in Figure 5 and $F' = \{f_1, f_2, f_3\}$ is as given in Example 4.1. This is the companion of $\mathcal{F}$. The scaling ratios are equal: $\lambda(f_1) = \lambda(f_2) = \lambda(f_3) = s$; $\mathcal{F}$ is a self-similar GIFS. The two attractor components of $\mathcal{F}'$ are the polygons in orange at the left in Figure 6. The self-similar GIFS $\mathcal{F}'$ is the companion of the GIFS $\mathcal{F}$. Every $\mathcal{F}$-tiling $T(\overleftarrow{\theta})$ with prototile set consisting of the two orange polygons on the left in Figure 6 is also a self-similar $\mathcal{F}'$-tiling $T(\overleftarrow{\theta}')$ for an appropriate parameter $\overleftarrow{\theta}'$ as constructed in the proof of Lemma 6.3.

⟨lem:adm⟩ **Lemma 6.3.** *The companion GIFS $\mathcal{F}' = (G', F')$ of a commensurable GIFS $\mathcal{F} = (G, F)$ satisfies the following properties:*

(1) *$\mathcal{F}'$ is a self-similar GIFS, and*
(2) *every $\mathcal{F}$-tiling is a $\mathcal{F}'$-tiling.*

Proof. Concerning statement (1) and referring to Definition 6.2, $\lambda(f_e) = s$ for all edges in G'. Therefore $\lambda(\mathcal{F}') = s$.

Concerning statement (2), denote the set of vertices of G by $\{1, 2, \dots, n\}$ and the set of corresponding attractor components by $\{A_1, A_2, \dots, A_n\}$. Each edge $e = (i, j)$ of G with $a_e > 1$ is replaced in G' by a path whose successive vertices we denote by $u = u_0, u_1, u_2, \dots, u_k = v$. Note that the outdegree of u_i is 1 for $i = 0, 1, 2, \dots, k-1$. The sucsessive edges are $e_1 = (u, u_1) = (u_0, u_1), e_2 = (u_1, u_2), \dots, e_k = (u_{k-1}, u_k) = (u_{k-1}, v)$. It is routine to check that in $\mathcal{F}'$, the attractor component of all vertices from G remain the same, namely $A_i' = A_i$ for $i = 1, 2, \dots, n$. The attractor component A'_{u_i} of each new vertex is defined recursively by $A'_{u_k} = A_{u_k}$ and $A'_{u_i} = f_{e_i}(A'_{u_{i+1}})$ for $i = k-1, k-2, \dots, 1$. Note that the attractor components $A'_{u_1}, A'_{u_2}, \dots, A'_{u_k} = A_v$ are all similar, each scaled down from its successor by a factor s.

Let $\overleftarrow{\theta} \in \mathcal{P}(\mathcal{F})$ and $\mathcal{X} = \{X_1, X_2, \dots\}$ a corresponding collection of admissible sest of paths for $k = 1, 2, \dots$. Let $T(\mathcal{F}, \overleftarrow{\theta}, \mathcal{X})$ be the associated GIFS-tiling. Define $\overleftarrow{\theta}' \in \mathcal{P}(\mathcal{F}')$ by replacing edges e in $\overleftarrow{\theta}$ with $a_e > 1$ by paths as in the definition of the companion GIFS. On vertices u of $\overleftarrow{\theta}'$ that correspond to vertices of $\overleftarrow{\theta}$ retain the admissible set X_u of paths. On each new vertex u_i, $1 \le i < k$, define the admissible set X_{u_i} of paths by $e_i\, e_{i-1}, \dots e_1$ followed by the admissible set X_u of paths at u. It is now routine that statement (2) of Lemma 6.3 holds. $\square$

Proof of Theorem 6.1. Theorem 6.1 follows from Lemma 6.2 and Lemma 6.3 and the following. For a GIFS-tiling to be a self-similar GIFS-tiling it is required in Definition 4.3 that if $\sigma \in Y(k)$, then $|\sigma| = k$, i.e., the length of the admissible path σ must equal the length k of the initial path of the parameter $\overleftarrow{\theta}$. This is insursed by the assumption that no two attractor components of the original GIFS are similar but not congruent. Those attractor components introduced in a companion GIFS are not relevant since the introduced vertices of the graph G of the GIFS have outdegree 1. □

7. Every Globally Self-Similar Tiling is a Self-Similar GIFS-Tiling

⟨sec:GSS⟩ This section concerns the self-similar tilings introduced by Thurston [39] in the context of tilings by translation of a set of prototiles. Basically, a self-similar tiling T is a tiling of $\mathbb{R}^d$ for which there exists a similarity transformation $\phi : \mathbb{R}^d \to \mathbb{R}^d$ with scaling ratio $\lambda(\phi) > 1$ such that, for all $t \in T$, the large tile $\phi(t)$ is tiled, in turn, by tiles in T. The following example shows that an additional requirement is needed in a formal definition of self-similar.

⟨ex:ss⟩ **Example 7.1.** Let Q consists of two intervals on the real line, I_1 of length 1 and I_2 of length 2. Let T be *any* tiling of $\mathbb{R}$ with prototile set Q such that

(1) the origin is located at an endpoint of a tile, and
(2) the endpoints of each tile of length 2 are located at even coordinates on $\mathbb{R}$.

Let $\phi(x) = 2x$ for all $x \in \mathbb{R}$. Then for every tile $t \in T$, its image $\phi(t)$ is tiled by tiles in T for all $t \in T$. The issue is that T can be quite random. For example, construct a tiling as follows. Moving along the positive real line starting at the origin, place a random number of tiles of length 2. Follow that by a random even number of tiles of length 1, and repeat in this fashion moving in the positive direction; similarly on the negative real line.

To avoid the randomness in Example 7.1 and similar examples and to extend from tilings by translation to tilings by isometric images of the prototiles, we will require that, for any tile $t \in T$, the tiling of $\phi(t)$ by tiles in T depends uniquely on the prototile type of t. To impose this requirement rigorously requires additional notation.

Definition 7.1 (Induced Tiling)**.** Let T be a tiling of $\mathbb{R}^d$ with prototile multiset Q, and let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a similarity with scaling ratio $\lambda(\phi) > 1$. We allow Q to be a multiset to allow for multiple ways for $\phi(p)$, $p \in Q$, to be tiled. For each $p \in Q$, let T_p be a tiling of $\phi(p)$ by tiles in Q. Call the set $\{T_p : p \in Q\}$ of patches a *ϕ-tiling rule*. Let $H := \{h_t : t \in T\}$ be a set of isometries, called *tiling isometries*, such that $t = h_t(p)$, where $p \in Q$ is the type of tile $t \in T$. If every $p \in Q$ has trivial symmetry group, then H is uniquely determined. This is the case, for example, in the definition of self-similar in [20, 39] where the tilings are by copies of the prototiles by translation rather than by isometry as is the case here.

For each $t \in T$, the isometry $\widehat{h}_t := \phi\, h_t\, \phi^{-1}$ maps $\phi(p)$ onto $\phi(t)$. Now

$$\phi(t) = \phi\big(h_t(p)\big) = \widehat{h}_t\, \phi(p) = \bigcup_{q \in T_p} \widehat{h}_t(q).$$

Note that $p := p_t$ in the equation above depends on t. Given the tiling T and tile $t \in T$, we now obtain the *induced tiling* of $\phi(t)$ and the *induced tiling* of T defined by

$$T_{\phi(t)} := \big\{\widehat{h}_t(q) : q \in T_{p_t}\big\} \qquad \text{and} \qquad T_\phi := \bigcup_{t \in T} T_t.$$

The following definition is basically that of Thurston and Kenyon [20], removing the restriction that the isometries in H be translations.

Definition 7.2 (Global Self-Similarity)**.** Given a prototile set Q, a Q-tiling T of $\mathbb{R}^d$ is *globally self-similar* if there is a similarity ϕ, a ϕ-tiling rule, and a set H of tiling isometries such that $T_\phi = T$. In particular, the image $\phi(t)$ is tiled by tiles in T for all $t \in T$.

Call a similarity transformation ϕ *proper with respect to a tiling* T if the fixed point of ϕ lies in the interior of a tile. As shown in Example 7.1 below, Theorem 7.1 may fail without this condition.

⟨thm:Gtiling⟩ **Theorem 7.1.** *Every globally self-similar tiling with proper self-similarity ϕ is a sefl-simlar GIFS-tiling.*

Proof. Given a self-similar tiling T, define a self-similar GIFS $\mathcal{F}(T) = (G, F)$ as follows. The vertex set of G is $Q := Q(\mathcal{F})$. The edge set E of G is defined by the ϕ-rule as follows. The ϕ-tiling rule can be expressed as

$$\phi(p) = \bigcup_{q \in T_p} h_{p,q}(q) \tag{7.1}$$ eq:c1

for some isometries $h_{p,q}$. In G add an edge e directed from vertex p to vertex q and let $f_e = \phi^{-1} h_{p,q}$. Note that $\lambda(f_e) = 1/\lambda(\phi) < 1$ for all $e \in E$. Equation (7.1) then becomes

$$p = \bigcup_{e \in E_p} f_e(p_{e^+}),$$

corresponding to Equation (3.2) in the definition of a GIFS attractor. Take $F = \{f_e : e \in E\}$. Now $\mathcal{F}(T) = (G, F)$ is a self-similar GIFS with scaling ratio $1/\lambda(\phi)$. Note that the prototiles of Q are isometric copies of the attractor components of $\mathcal{F}(T)$. It remains to shown that there exists a $\overleftarrow{\theta} \in \overleftarrow{\Sigma}$ such that $T = T(\overleftarrow{\theta})$.

Let t_0 be the tile in the self-similar tiling T that contains the fixed point O of ϕ in its interior. If t_0 is of type $p \in Q$, then there is no loss of generality in assuming that $t_0 = p$, and consequently there is an edge (loop) e in G from p to p labeled $f_e = \phi^{-1}$. Consider the parameter $\overleftarrow{\theta} := e\,e\,e \cdots \in \overleftarrow{\Sigma}$ that winds infinitely many times around the loop e. Then $f_{\overleftarrow{\theta}|k} := f_e \circ f_e \circ \cdots \circ f_e = \phi^{-k}$ for all non-negative integers k. We will show that $T = T(\overleftarrow{\theta})$.

Let H be a set of tiling isometries for T such that $T_\theta = T$. We claim that there a set $H' = \{h'_t : t \in T\}$ of tiling isometries such that each $h'_t(p_t) = h_t(p_t)$ for all $t \in T$, and each $h'_t \in H'$ has the form

$$h'_t = \phi^k \circ f_\sigma, \tag{7.2}$$ eq:proof0

where $\sigma \in \overleftarrow{\Sigma}^*$ and $|\sigma| = k$. If this is the case, then

$$t = (\phi^k \circ f_\sigma)(p) \in T(\overleftarrow{\theta}, k),$$

where $p = p_{\sigma^+} \in Q$. This is exactly as in Equation (4.2) in Definition (4.1), showing that T is an $\mathcal{F}$-tiling and completing the proof.

It only remains to prove the claim. Since the fixed point O lies interior to t_0, any tile $t \in T$ is contained in $X_k := \phi^k(t_0)$ for some integer k. Note that the sets X_k are nested. The existence of such a set H' of tiling isometries of the form in Equation (7.2) is proved by induction on k. If $t \in X_1 = \phi(t_0) = \phi(p)$, then by Equation (7.1) we have $t = h_{p,p_1}(p_1) = \phi\, f_e(p_1)$ for an edge e directed from p to p_1, where p_1 is the type of tile t. Take $h'_t = h_{p,p_1} = \phi\, f_{e_1}$, which is of the form in Equation (7.2). Assume that h'_t has been defined in the form of Equation (7.2) for all $t \in X_k$, and let $t' \in X_{k+1}$. Then $t' \in \phi(t)$ for some $t \in X_k$, where, by the induction hypothesis, $h'_t = \phi^k \circ f_\sigma$ holds, where $|\sigma| = k$ and σ is a path from p to p_t. By the requirement that $T_\phi = T$ we have $t' = \widehat{h}_t(q)$, where $q \in T_{\phi(p_t)}$ is the type of tile t', i.e., $q = p_{t'}$. By the tiling rule $q = h_{p_t,p_{t'}}(p_{t'}) = (\phi^{-1} \circ f_e)(p_{e^+})$, where e is directed from p_t to $p_{t'}$. Therefore

$$\begin{aligned} t' = \widehat{h}_t(q) = t' &= \big(\phi\, h_t\, (\phi)^{-1}\big)\big(\phi\, f_e(p_{e^+})\big) = \big(\phi\, h'_t\, (\phi)^{-1}\big)\big(\phi\, f_e(p_{e^+})\big) \\ &= \phi\big(\phi^k \circ f_\sigma\big) f_e(p_{e^+}) = \phi^{k+1} \circ f_{\sigma'}(p_{t'}) = \phi^{k+1} \circ f_{\sigma'}(A_{\sigma'^+}), \end{aligned}$$

where $\sigma' = \sigma\, e$. □

⟨ex:quarters⟩ **Remark 7.1** (Theorem 7.1 may not hold when the self-similarity is not proper)**.** For the Ammann chair self-similar GIFS (G, F) in Example 4.1, $T := T(\overleftarrow{e}_1\, \overleftarrow{e}_1, \cdots)$ tiles the first quandrant of the plane. The union of T and copies of T obtained by reflected in the x and y-axes and by rotation by π about the origin tiles the plane and is self-similar but is not a self-similar GIFS-tiling.

Remark 7.2. In [20], it is part of the definition of a self-similar tiling T that T be repetitive. That every self-similar tiling with a proper self-similarity must be repetitive follows from Theorem 7.1 and statement (7) of Theorem 5.1.

⟨thm:R⟩ **Theorem 7.2.** *For a self-simlar GIFS-tiling $T(\mathcal{F}, \overleftarrow{\theta})$, if the parameter $\overleftarrow{\theta}$ is eventually periodic, then $T(\mathcal{F}, \overleftarrow{\theta})$ is globally self-similar.*

Proof. We must produce a similarity ϕ, a ϕ-tiling rule, and a set H of tiling isometries such that $T_\phi = T$. Since $\overleftarrow{\theta}$ is eventually periodic, there exist $\overleftarrow{\theta}_0, \overleftarrow{\theta}_1 \in \overleftarrow{\Sigma}^*$ such that $\overleftarrow{\theta} = \overleftarrow{\theta}_0\, \overleftarrow{\theta}_1\, \overleftarrow{\theta}_1\, \overleftarrow{\theta}_1 \cdots$. Let $\phi = f_{\overleftarrow{\theta}_0} \circ f_{\overleftarrow{\theta}_1} \circ (f_{\overleftarrow{\theta}_0})^{-1}$, which is a similarity. For $t \in T(\mathcal{F}, \overleftarrow{\theta})$ there is a least integer k such that $t \in T(\overleftarrow{\theta}, |\overleftarrow{\theta}_0| + k|\overleftarrow{\theta}_1|)$, in which case $t = f_{\overleftarrow{\theta}_0} \circ (f_{\overleftarrow{\theta}_1})^k \circ f_\sigma(A^+_\sigma)$ for an appropriate $\sigma \in \Sigma^*$ with $|\sigma| = k+1$. Let $h_t = f_{\overleftarrow{\theta}_0} \circ (f_{\overleftarrow{\theta}_1})^k \circ f_\sigma$ and $H = \{h_t : t \in T\}$ the set of tiling isometries. Since each prototile (attractor component) p is tiled by $\{f_e(p_{e^+}) : e \in E_p\}$, the ϕ-tiling rule is taken to be $T_{\phi(p)} = \{\phi \circ f_e(p_{e^+}) : e \in E_p : e \in E_p\}$. Now an arbitrary tile in the induced tiling T_ϕ is of the form

$$\begin{aligned} t &= \widehat{h}_t\, \phi\, f_e(p_{e^+}) = \phi\, h_t\, \phi^{-1}\, \phi\, f_e(p_{e^+}) = \phi\, f_{\overleftarrow{\theta}_0}\, (f_{\overleftarrow{\theta}_1})^k\, \sigma\, \phi\, f_e(p_{e^+}) \\ &= f_{\overleftarrow{\theta}_0}\, (f_{\overleftarrow{\theta}_1})^{k+1}\, \sigma\, \phi\, f_e(p_{e^+}) = f_{\overleftarrow{\theta}_0}\, (f_{\overleftarrow{\theta}_1})^{k+1}\, \sigma'(p_{\sigma'^+}), \end{aligned}$$

where $\sigma' = \sigma\varepsilon$. The last expression in the equality is again a tile in $T(\mathcal{F}, \overleftarrow{\theta})$. □

⟨thm:closure⟩ **Theorem 7.3.** *For a given self-similalr GIFS $\mathcal{F}$, the set of globally self-similar $\mathcal{F}$-tilings is dense in the tiling space $\mathbb{T}(\mathcal{F})$.*

Proof. Let $T = T(\mathcal{F}, \overleftarrow{\theta}) \in \mathbb{T}(\mathcal{F})$ be given. Let $\overleftarrow{\theta}(k) \in \mathcal{P}$ be chosen to have the property that $\theta(k)$ is eventually periodic and such that the first k edges of $\overleftarrow{\theta}(k)$ and $\overleftarrow{\theta}$ are the same. By the continuity of the tiling map $\mathcal{T}$, it follows from $\lim_{k\to\infty} \overleftarrow{\theta}(k) = \overleftarrow{\theta}$ that $\lim_{k\to\infty} T\big(\mathcal{F}, \overleftarrow{\theta}(k)\big) = T(\mathcal{F}, \overleftarrow{\theta}) = T$. Because $T\big(\mathcal{F}, \overleftarrow{\theta}(k)\big)$ is self-similar by Theorem 7.2, the tiling T is in the closure of the set of self-similar $\mathcal{F}$-tilings. □

8. A GIFS Tiling Dynamical System

⟨sec:DS⟩ For a self-similar GIFS $\mathcal{F}$, two dynamical systems are considered in this section. One is $(\mathbb{T}, H)$, the action $H : \mathbb{T}(\mathcal{F}) \to \mathbb{T}(\mathcal{F})$ on the tiling space $\mathcal{P}$ of $\mathcal{F}$ defined as follows. For a self-similar $\mathcal{F}$-tiling $T = T(\mathcal{F}, \overleftarrow{\theta}) \in \mathbb{T}$ with parameter $\overleftarrow{\theta} = \overleftarrow{\theta}_1\, \overleftarrow{\theta}_2 \cdots \in \mathcal{P}(\mathcal{F})$, let $T = T_0, T_1, T_2, \dots$ denote its hierarchy and let

$$H(T) = f_{\theta_1}(T_1)$$

for all $T \in \mathbb{T}$. In other words, H is a map that takes a self-similar GIFS-tiling to a scaled down tiling at the next level in its hierarchy.

Throughout this section it will be assumed that all self-similar GIFSs are non-redundant, asymmetric and uniquely hierarchical. It will also be assumed, without loss of generality, that the origin is contained in the interior of each attractor component and, if two distinct attractor components are congruent, then they are placed so as not to coincide. In this case it can be shown, similarly to Property 6 in Theorem 5.1, that if $\mathcal{F}$ is a uniquely hierarchical, asymmetric and not redundant, self-similar GIFS and $\overleftarrow{\theta}$ and $\overleftarrow{\theta}'$ are two parameters, then $T(\mathcal{F}, \overleftarrow{\theta}) = T(\mathcal{F}, \overleftarrow{\theta}')$ if and only if $\overleftarrow{\theta} = \overleftarrow{\theta}'$.

⟨thm:TC⟩ **Theorem 8.1.** *For a self-similar GIFS the two dynamical systems* $(\mathcal{P}, S)$ *and* $(\mathbb{T}, H)$ *are topologically conjugate discrete dynamical systems, the topological conjugation being the tiling map* $\mathcal{T}$. *As a commuting diagram we have*

$$\begin{array}{ccccc} & & S & & \\ & \mathcal{P} & \rightarrow & \mathcal{P} & \\ \mathcal{T} & \downarrow & & \downarrow & \mathcal{T} \\ & \mathbb{T} & \rightarrow & \mathbb{T} & \\ & & H & & \end{array}$$

Proof. By the comments prior to the statement of the theorem, the tiling map $\mathbb{T}$ is bijective. A bijective continuous map on a metric space is a homeomorphism. Property 5 of Theorem 5.1 implies that the diagram commutes:

$$H\big(T(\mathcal{F}, \overleftarrow{\theta})\big) = f_{\theta_1} T_1(\mathcal{F}, \overleftarrow{\theta}) = f_{\theta_1} f_{\theta_1}^{-1} T(\mathcal{F}, S\overleftarrow{\theta}) = T(\mathcal{F}, S\overleftarrow{\theta}). \qquad \square$$

8.1. **Application 1.** Let $\mathcal{F}$ be a self-similar GIFS. Throughout this section, it is assumed that $\mathcal{F}$ is non-redundant, asymmetric and uniquely hierarchical, or at least that the tiling map $\mathcal{T}$ induces a topological conjugacy as in the diagram above.

Let $p_k := p_k(\mathcal{F})$ denote the number of $\mathcal{F}$-tilings T such that $H^k(T) = T$. In other words, p_k is the number of $\mathcal{F}$-tilings such that its n^{th}-level hierarchical tiling T_k, scaled down by a factor $(f_{\overleftarrow{\theta}|k})^{-1}$, is equal to the original tiling T. In this sense, p_k counts the number of $\mathcal{F}$-tilings whose hierarchy $T = T_0, T_1, T_2 \dots$ cycles with period k.

⟨thm:zeta⟩ **Theorem 8.2.** *For a self-similar GIFS* $\mathcal{F}$ *as above, a generating function for the sequence* $\{p_k\}_{n=1}^{\infty}$ *is given by*

$$\sum_{k=1}^{\infty} \frac{p_k}{k}\, x^k = \log\Big(\frac{1}{\det(I - xM)}\Big),$$

where M *is the adjacency matrix of the digraph of* $\mathcal{F}$.

Proof. An element in the parameter space $\mathcal{P}$ of $\mathcal{F}$ can be viewed as a one sided word in the alphabet V, where V is the set of vertices of the digraph $\overleftarrow{G}$. The parameter space $\mathcal{P}$ is clearly shift invariant. In the terminology of symbolic dynamics, the dynamical system $(\mathcal{P}, S)$ is a 1-step shift of finite type. This means that there is a finite set W of ordered pairs of elements of V, i.e. a set of edges in the complete digraph on V, such that $\mathcal{P}$ consists of all words (paths in $\overleftarrow{G}$) that do not contain an ordered pair (edge) in W.

The Artin-Mazur zeta function of a dynamical system (X, g) is defined by

$$\zeta(x) = \exp\Big(\sum_{k=1}^{\infty} \frac{q_k}{k}\, x^k\Big),$$

where q_k is the number of points of period n of X under the action of g. (Note that a point of period n is also a point of period any multiple of n.) For our shift of finite type $(\mathcal{P}, S)$, the number q_k is thus the number of parameters $\overleftarrow{\theta}$ such that $S^k\overleftarrow{\theta} = \overleftarrow{\theta}$, equivalently the number of closed paths in $\overleftarrow{G}$ of length k. The zeta function is a well-known invariant of topological conjugacy and, for a shift of finite type, can be computed by the Bowen-Lanford formula:

$$\zeta(x) = \frac{1}{\det(I - xM^t)} = \frac{1}{\det(I - xM)},$$

where M is the adjacency matrix of the digraph G and its transpose M^t is the adjacency matrix of the digraph $\overleftarrow{G}$. For our shift of finite type $(\mathbb{T}, H)$, the number q_k is thus the number of tilings $T \in \mathbb{T}$ such that $H^k(T) = T$, equivalently $q_k = p_k$.

According to Theorem 8.1, the two dynamical systems $(\mathbb{T}, H)$ and $(\mathcal{P}, S)$ are topologically conjugate and therefore have the same zeta function. Hence

$$\exp\Big(\sum_{k=1}^{\infty} \frac{p_k}{k}\, x^k\Big) = \frac{1}{\det(I - xM)}$$

and Theorem 8.2 follows. □

We note that the Artin-Mazur zeta invariant used in the prove above is also investigated by Anderson and Putnam [2] for substitution tiling spaces.

Example 8.1. A simple computer calculation gives the following series for the Ammann chair tilings of Example 4.1:

$$\sum_{k=1}^{\infty} \frac{p_k}{n}\, x^k = x + \frac{3x^2}{2} + \frac{4x^3}{3} + \frac{7x^4}{4} + \frac{11x^5}{5} + \frac{18x^6}{6} + \frac{29x^7}{7} + \frac{47x^8}{8} + \frac{76x^9}{9} + \cdots .$$

For example, there are 4 Ammann chair tilings for which the hierarchy cycles with period 3. Referring to the graph in the right panel of Figure 5, these correspond in $(\mathcal{P}, S)$, via topological conjugacy, to the 4 parameters $\overline{111}, \overline{132}, \overline{213}, \overline{321}$, where the individual digits are shorthand for the edges in $\overleftarrow{G}$ with those function subscripts, and the bar over the numbers means that the three numbers repeat.

8.2. **Application 2.**

⟨lem:ds⟩ **Lemma 8.1.** *Let $\mathcal{F}$ be a self-similar GIFS, and let $\overleftarrow{\theta}, \overleftarrow{\theta}' \in \mathcal{P}$. Then $T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$ if and only if $T(\theta, k) = T(\theta', k)$ for $k = 0, 1, 2, \ldots$.*

Proof. Since $T(\overleftarrow{\theta})$ is the nested union of the $T(\theta, k)$, clearly $T(\theta, k) = T(\theta', k)$ for $k = 0, 1, 2, \ldots$ implies that $T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$.

In the other direction, it follows from the proof of of Property 5 of Theorem 5.1 that the union of the tiles in $T(\theta, k)$ is itself a tile $t_k \in T_k$; similarly the union of the tiles in $T(\theta', k)$ is a tile $t_k' \in T_k'$. Since $0 \in t_k \cap t_k'$ and $T := T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$ has a unique hierarchy, it follows that $T(\theta, k) = T(\theta', k)$. □

For a self-similar GIFS $\mathcal{F}$, let $N_k := N_k(\mathcal{F})$ denote the number of distinct (pairwise unequal) patches $T(\overleftarrow{\theta}, k)$ over all $\overleftarrow{\theta} \in \mathcal{P}$. In view of Lemma 8.1, if two $\mathcal{F}$-tilings $T(\overleftarrow{\theta})$ and $T(\overleftarrow{\theta}')$ are not equal, then, for some k, the patches $T(\overleftarrow{\theta}, k)$ and $T(\overleftarrow{\theta}', k)$ will not be equal. Therefore, the growth of the sequence $\{N_k\}_{k=1}^{\infty}$ is a measure of how fast $\mathcal{F}$-tilings can be distinguished by looking at these increasingly large finite patches.

⟨thm:ent⟩ **Theorem 8.3.** *For a self-similar GIFS $\mathcal{F}$, let $\rho := \rho(\mathcal{F})$ denote the Perron-Frobenius eigenvalue of the adjacency matrix of the digraph of $\mathcal{F}$. Then asymptotically*

$$N_k \simeq e^{k \log \rho}, \qquad \textit{i.e.},$$

$$\lim_{k\to\infty} \sqrt[k]{N_k} = \rho.$$

Proof. Given a self-similar GIFS $\mathcal{F} = (G, F)$, the topological entropy of the shift of finite type $(\mathcal{P}, S)$ is defined by

$$h(\mathcal{P}) = \lim_{k\to\infty} \frac{1}{k} \log \widehat{N}_k,$$

where $\widehat{N}_k$ is the number of paths of length k in the digraph $\overleftarrow{G}$, which equals the number of paths of length k in the digraph G. In the same way that $T(\overleftarrow{\theta}) = T(\overleftarrow{\theta}')$ if and only if $\overleftarrow{\theta} = \overleftarrow{\theta}'$, we have $T(\overleftarrow{\theta}, k) = T(\overleftarrow{\theta}', k)$ if and only if $\overleftarrow{\theta}|k = \overleftarrow{\theta}'|k$. Therefore $N_k = \widehat{N}_k$ for all k.

For $(\mathcal{P}, S)$ (and more generally for any shift of finite type) it is well known that $h(\mathcal{P}) = \rho(\mathcal{F})$, the Perron-Frobenius eigenvalue of M. Therefore

$$\lim_{k\to\infty} \frac{1}{k} \log N_k = \lim_{k\to\infty} \frac{1}{k} \log \widehat{N}_k = \rho,$$

which is equivalent to $\lim_{k\to\infty} \sqrt[k]{N_k} = \rho$. □

Example 8.2. Applying Theorem 8.3 to the Ammann chair self-similar GIFS $\mathcal{F}$ of Example 4.1 yields

$$N_k \simeq \tau^k \approx e^{.4812\,k},$$

where τ is the golden ratio.

9. The Existence of Self-Similar GIFSs

⟨sec:E⟩ According to Theorem 9.1 below, self-similar GIFSs are easy to construct in dimension 1. Polygonal rep-sets in dimension 2 may lead to self-similar GIFSs. These have appeared in both recreational websites and mathematical journals, mostly discovered in an ad hoc manner, some very clever. They include the Ammann chair tile in Example 4.1, Robinson's triangle variant of the Penrose tiles, and the pinwheel tile (the tile due to J. Conway, the tiling due to C. Radin). According to Theorem 9.2 below, for every dimension d and for every Perron number ρ, there exists a self-similar GIFS on $\mathbb{R}^d$ whose scaling ratio is $1/\rho$; the attractor components are boxes. All of the above examples give self-similar GIFS tilings by the method in Definitions 4.1 and 4.3. The digit IFSs referred to in the introduction [41] are self-similar IFSs and produce lattice tilings by translates of a single prototile.

Non-polyhedral, aperiodic tilings in dimension $d \geq 2$ with self-replicating properties are hard to come by. The difficulty in producing such a self-similar GIFS is that the attractor is required to be non-overlapping and the attractor components to have non-empthy interior. Rauzy type fractals, often called central tiles, do admit non-period tilings. They are are usually obtained from symbolic dynamics [9] or from numeration systems (β-expansions) [1] and are also related to model sets obtained by the cut-and-project method [28]. Although a general discussion of Rauzy tilings does not fall within the scope of this paper, we give a shot exposition in Section 9.1 of the elegant GIFS approach due to Rao, Wen and Yang [31].

A real number $\rho > 1$ is a *Perron number* if it is a real algebraic integer such that the moduli of all other Galois conjugates (roots of the minimal polynomial of ρ) are less than $|\rho|$. Call a real algebraic integer $\rho > 1$ a *weak-Perron number* if the moduli of all of its Galois conjugates are less than or equal to $|\rho|$. A real number $\rho > 1$ is a *Pisot number* if it is a real algebraic integer such that the moduli of all of its other Galois conjugates are less than 1. It is a *unit pisot number* if, in addition, the constant term in its minimal polynomial is ± 1.

Proposition 9.1. *If $\mathcal{F}$ is a self-similar GIFS on $\mathbb{R}^d$ with scaling ratio λ, then $1/\lambda$ is a weak-Perron number.*

Proof. Let $\rho = 1/\lambda$. By condition (2) in Statement 3 of Theoem 5.1 we know that ρ^d is the Perron-Frobenius eigenvalue of $M(G)$. If the characteristic polynomial of $M(G)$ is $p(x)$, then ρ is a root of $\widehat{p}(z) = p(z^d)$, all of whose roots are less than or equal to ρ. □

Note that there exists no self-similar GIFS whose digraph is a directed cycle.

⟨thm:P2⟩ **Theorem 9.1.** (1) *For any $0 < \lambda < 1$ such that $1/\lambda$ is a Perron number, there exists a primitive self-similar GIFS on $\mathbb{R}$ whose scaling ratio is λ.*

(2) *For every strongly connected digraph $G = (V, E)$, not a directed cycle, there exists a primitive self-similar GIFS $\mathcal{F} = (G, F)$ on $\mathbb{R}$ such that $f_e(x) = (1/\rho)\,(x + d_e)$ for all $e \in E$, where ρ is the Perron-Frobenius eigenvalue of the adjacency matrix $M(G)$ of G and $d_e \in \mathbb{Q}(\rho)$.*

Proof. Concerning statement (1), let $0 < \lambda < 1$ and let $\rho = 1/\lambda$. The result [23, Theorem 1] states that if ρ is a Perron number, then there is a primitive non-negative integral matrix $M = (m_{i,j})$, $i, j \in \{1, 2, \dots, n\}$ whose spectral radius is ρ. A self-similar GIFS on $\mathbb{R}$, whose adjacency matrix is M, can be obtained as follows. Let the vertex set of G be $V = \{1, 2, \dots, n\}$. For each pair (i, j) of vertices, add $m_{i,j}$ edges from vertex i to vertex j, and label edge e with a function of the form $f_e(x) = (1/\rho)(x + d_e)$, where d_e is defined as follows. Let $(x_1, x_2, \dots, x_n)$ be a positive right eigenvector corresponding to the eigenvalue ρ, which exists by the Perron-Frobenius theorem. The set $\{x_1, x_2, \dots, x_n\}$ of coordinates is a solution to the linear system

$$x_j = \sum_{i=1}^{n} \lambda\, m_{j,i}\, x_i, \qquad j = 1, 2, \dots, n,$$

and hence can be chosen to lie in $\mathbb{Q}(\rho)$. Consider the intervals $I_j = [0, x_j]$, $j = 1, 2, \dots, n$, on the real line. From the system of linear equations above, the interval I_j can be tiled (often in many different ways) by translated copies of the intervals $[0, \lambda\, x_i]$, $i = 1, 2, \dots, n$. The d_e are the translation distances multiplied by ρ, which are sums of the x_is. Therefore $d_e \in \mathbb{Q}(\rho)$ for all $e \in E$.

The adjacency matrix $M(G)$ determines the digraph G, so, as long as the Perron-Frobenius eigenvalue of $M(G)$ does not equal 1, the proof of statement (2) is as in the paragraph above. Since $M(G)$ is a non-negative, integral matrix, the Perron-Frobenius eigenvalue equals 1 if and only if M is orthogonal which can occur if and only if $M(G)$ is a permuatation matrix if and only if G is a directed cycle. □

⟨thm:P⟩ **Theorem 9.2.** *For any $0 < \lambda < 1$ such that $1/\lambda$ is a Perron number, there exists a primitive self-similar GIFS on $\mathbb{R}^d$ whose scaling ratio is λ. The components of the attractor are boxes.*

Proof. The case $d = 1$ is Theorem 9.1. So let $\mathcal{F} = (G, F)$ be such a 1-dimensional self-similar GIFS with scaling ratio $\lambda = 1/\rho$ and with attractor components $I_j = [0, x_j]$, $j = 1, 2, \dots, n$, intervals on the real line. Denote the vertex set of G by $V = \{1, 2, \dots n\}$ and the edge set by E.

The result is extended to arbitrary dimension by a product construction. In dimension d, let $\mathcal{F}_d = (G_d, F_d)$ be the GIFS, where $G_d = (V_d, E_d)$ is the digraph with vertex set $V_d = V^d$ and edge set $E_d = E^d$. The edge $\mathbf{e} := (e_1, e_2, \dots, e_d) \in E_d$ joins vertex $\mathbf{i} := (i_1, i_2, \dots, i_d)$ to vertex $\mathbf{j} := (j_1, j_2, \dots, j_d)$ where $e_k = (i_k, j_k)$ for $k = 1, 2, \dots d$. Label edge $\mathbf{e}$ with the function $f_{\mathbf{e}}(y_1, y_2, \dots y_d) = \big(f_{e_1}(y_1), f_{e_2}(y_2), \dots, f_{e_d}(y_d)\big)$ to obtain F_d.

It is routine to check that $\mathcal{F}$ and $\mathcal{F}_d$ have the same scaling ratio. The GIFS $\mathcal{F}_d$ is primitive for the following reason. Since G is primitive, there exists a positive integer m such that, for any two vertices of G there is a directed path of length m from one to the other. Let $\mathbf{i} := (i_1, i_2, \dots, i_d)$ and $\mathbf{j} := (j_1, j_2, \dots, j_d)$ be two arbitrary vertices of G_d. For $k = 1, 2, \dots, d$, let $e_{k,1}\, e_{k,2} \cdots e_{k,m}$ be a path of length m in G from vertex i_k to vertex j_k. For $q = 1, 2, \dots, m$, let $\mathbf{e_q} = (e_{1,q}, e_{2,q}, \dots, e_{d,q})$. Then $\mathbf{e_1}\, \mathbf{e_2} \cdots \mathbf{e_m}$ is a path of length m from vertex $\mathbf{i}$ to vertex $\mathbf{j}$ in G_d.

For each $\mathbf{i} := (i_1, i_2, \dots, i_d) \in V_d$, let $B_{\mathbf{i}} = I_{i_1} \times I_{i_2} \times \cdots \times I_{i_d}$, which is a box in $\mathbb{R}^d$. We leave it as an exercise to verify that

$$B_{\mathbf{i}} = \bigcup_{\mathbf{e} \in (E_d)_{\mathbf{i}}} f_{\mathbf{e}}(B_{\mathbf{e}^+})$$

for all $\mathbf{i} \in V_d$ and that the union is non-overlapping. This shows that $\{B_{\mathbf{i}} : \mathbf{i} \in V_d\}$ is the set of attractor components of $\mathcal{F}_d$. □

Theorem 9.1 suggests the following questions.

Question 9.1. For which strongly connected digraphs G and integers $d \geq 2$, does there exist a self-similar GIFS on $\mathbb{R}^d$ whose digraph is G?

⟨sec:rt⟩ 9.1. **The Dual of a Self-Similar GIFS on $\mathbb{R}$.**

Definition 9.1 (Dual GIFS)**.** A slightly more general definition of the dual appers in [31]. Let $G = (V, E)$ be a strongly connected digraph of order $d+1$, $d \geq 2$, not a cycle, and let ρ be the Perron-Frobenius eignevalue of the adjacency matrix of G. According to Theorem 9.1, there is a self-similar GIFS $\mathcal{F} = (G, F)$ on $\mathbb{R}$ (often many) such that

$$f_e(x) = \frac{1}{\rho}\big(x + d_e\big),$$

where $d_e \in \mathbb{Q}(\rho)$ for all $e \in E$. Such 1-dimensional self-similar GIFSs can be obtained explicitly and easily using the method provided in the proof of Theorem 9.1. Call any such self-similar GIFS a 1-*dimensional self-similar GIFS* or simply a 1-*dim self-similar GIFS*.

Let $\mathcal{F} = (G, F)$ be a 1-dim self-similar GIFS. If ρ is a Pisot unit, then the *dual GIFS* $\mathcal{F}^* = (\overleftarrow{G}, F^*)$ of $\mathcal{F}$ is defined as follows. The reverse digraph $\overleftarrow{G}$ is as defined in Section 4. Let $\rho = \rho_0, \rho_1, \dots, \rho_d$ be the Galois conjugates of ρ ordered as follows:

$$\rho > 1 > |\rho_1| \geq |\rho_2| \geq \cdots \geq |\rho_d|,$$

where complex conjugates appear consecutively. Let

$$B' = diag(\rho_1, \rho_2, \dots, \rho_d),$$

be the diagonal matrix and replace each pair $z, \overline{z}$ of complex conjugates in B' by the 2×2 real block $\begin{pmatrix} Re\, z & -Im\, z \\ Im\, z & Re\, z \end{pmatrix}$ to obtain the matrix B. Since $1, \rho, \rho^2, \dots, \rho^d$ is a basis for $\mathbb{Q}(\rho)$, we have $x = \sum_{i=0}^d x_i \rho^i$ for any $x \in \mathbb{Q}(\rho)$, where the $x_i \in \mathbb{Q}$. The *dual* of x is defined as

$$x^* = \sum_{i=0}^{d} x_i (\rho_1^i, \dots, \rho_d^i)^t \in \mathbb{C}^d,$$

where pair $z, \overline{z}$ of complex conjugates is replaced with $Re\, z, Im\, z$. Equivalently, the star operator is the unique linear map $* : \mathbb{Q}(\rho) \to \mathbb{R}^d$ such that $(\rho\, x)^* = Bx^*$ for all $x \in \mathbb{Q}(\rho)$. Now set $F^* = \{ f_{\overleftarrow{e}}(x) = Bx + d_e^* : e \in E\}$. The *dual* of $\mathcal{F}$ is $\mathcal{F}^* := (\overleftarrow{G}, F^*)$, which is a GIFS on $\mathbb{R}^d$.

The following theorem is essentially [31, Theorem 1.2].

⟨thm:P1⟩ **Theorem 9.3.** *The dual of a* 1*-dim self-similar GIFS* $\mathcal{F}$ *for which* $1/\lambda(\mathcal{F})$ *is a Pisot unit is also a self-similar GIFS.*

Proof sketch. A GIFS satisfies the OSC if there exists open sets $U_1, \dots, U_n$ such that

$$\bigcup_{e \in E_i} f_e(U_{e+}) \subset U_i$$

for $i = 1, 2, \dots, n$, where n is the order of the digraph of the GIFS. Three results lead to the proof of Theorem 9.3.

First, necessary and sufficient conditions for a GIFS to satisfy the OSC, in terms of certain "digits" is given in [24].

Second, a result in [31], using the above necessary and sufficient conditions, immediately implies that if 1-dim self-similar GIFS $\mathcal{F}$ satisfies (1) the open set condition (OSC) and (2) that $1/\lambda(\mathcal{F})$ is a Pisot number, then $\mathcal{F}^*$ also satisfies the OSC. Concerning condition (1), the attractor components of a 1-dim self-similar GIFS are closed intervals; hence the OSC is satisfied by taking the U_i as the set of corresponding open intervals. Condition (2) is, by assumption, satisfied. Therefore $\mathcal{F}^*$ satisfies the OSC.

Third, a result in [22] immediately implies that if $\mathcal{F}^* = (\overleftarrow{G}, F^*)$ is such that (1) the functions in F^* are affine of the form $f_e(x) = B(x) + d_e$ for all $e \in E$, (2) $\mathcal{F}^*$ satisfies the OSC, and (3) $1/\rho^* = |\det(B)|$, where ρ^* is the Perron-Frobenius eigenvalue of $M(\overleftarrow{G})$, then the attactor components of $\mathcal{F}^*$ have nonempty interior. Condition (1) is clearly true, and condition (2) follows immediatly from the previous result from [24]. Concerning condition (3), we have

$$|\det(B)| = |\rho_1 \cdot \rho_2 \cdots \rho_d| = \frac{1}{\rho} = \frac{1}{\rho^*}.$$

The second equality follows from the fact that ρ is a unit and the last equality from the fact that $M(\overleftarrow{G})$ is the transpose of $M(G)$.

Therefore the attractor components of $\mathcal{F}^*$ have nonempty interior. Moreover, because $\lambda(\mathcal{F}^*) = \sqrt{|\det(B)|} = 1/\sqrt{\rho^*}$, Lemma 5.1. implies that $\mathcal{F}^*$ is a self-similar GIFS. □

⟨ex:R⟩ **Example 9.1** (Rauzy Tilings from a Dual Self-Similar GIFS)**.** Duality is used here to obtain fractal, non-periodic self-similar GIFS-tilings of $\mathbb{R}^2$. We start with an initial 1-dim self-similar GIFS $\mathcal{F} = (G, F)$ on $\mathbb{R}$, where G is an order 3 strongly connected digraph and the Perron-Frobenius eigenvalue ρ of the adjacency matrix $M(G)$ is a Pisot unit. It is not hard to find many such 1-dim self-similar GIFSs as follows. Regarding the 9 integer entries of $M(G)$ as variables, it is easy to solve for values of these variables so that (1) $\det(M(G)) = \pm 1$, (2) the degree 3 characteristic polynomial of $M(G)$ has a pair of complex roots, and (3) G is strongly connected. That $\rho > 1$ together with condition (2) immediately implies that ρ is a Pisot number. Condition (1) further implies that ρ is a unit. Because $M(G)$ is a 3×3 matrix, the dual $\mathcal{F}^*$ is a self-similar GIFS on $\mathbb{R}^2$. With the digraph G in hand, we use Theorem 9.1 (and its proof) to obtain possibilities for F and hence possibilities for the 1-dim self-similar GIFS $\mathcal{F}$. We then use Theorem 9.3 to obtain F^* and hence $\mathcal{F}^* = (\overleftarrow{G}, F^*)$. The tiling method of this paper is then used to produce the $\mathcal{F}^*$-tiling.

For each of the following examples we provide an adjacency matrix $M = M(G)$, which determines the digraph G, the characteristic polynomial $p(x)$ of M, whose Perron-Frobenius eigenvalue ρ is a Pisot unit and equals $1/\lambda(\mathcal{F})$, and the functions in F.

(1) The original Rauzy fractal is the non-overlapping union of three smaller similar copies of itself. A tiling based on the Rauzy fractal appears on the right in Figure 3. This self-similar GIFS-tiling can be constructed from the dual of this 1-dim self-similar GIFS:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad p(x) = x^3 - x^2 - x - 1 \qquad \begin{array}{l} f_{2,1}(x) = f_{31}(x) = f_{1,1}(x) = (1/\rho)x \\ f_{1,2}(x) = f_{2,3}(x) = (x-1)/\rho, \end{array}$$

where $f_{i,j}$ is the function in F that is the label of edge (i, j).

(2) A tiling from the the dual of the data below appears on the left in Figure 7.

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad p(x) = x^3 - x^2 - 1 \qquad \begin{array}{l} f_{2,1}(x) = f_{3,2}(x) = f_{1,1}(x) = (1/\rho)x \\ f_{1,3}(x) = (x-1)/\rho \end{array}$$

(3) The tilings on the left and right, respectively, in Figure 4 are from the duals of these 1-dim self-similar GIFSs:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad p(x) = x^3 - x^2 - 2x - 1 \qquad \begin{array}{l} f_{3,1}(x) = f_{2,1}(x) = f_{1,1}(x) = (1/\rho)x \\ f_{1,3}(x) = f_{2,3}(x) = (x+\rho)/\rho \\ f_{1,2}(x) = (x+\rho+1)/\rho \end{array}$$

(4)

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad p(x) = x^3 - 2x^2 - x - 1 \qquad \begin{array}{l} f_{2,3}(x) = f_{3,1}(x) = f_{1,1}(x) = (1/\rho)x \\ f_{2,1}(x) = (x-1)/\rho \\ f_{1,2}(x) = (x+\rho)/\rho \\ f_{2,2}(x) = (x+\rho+1)/\rho \end{array}$$

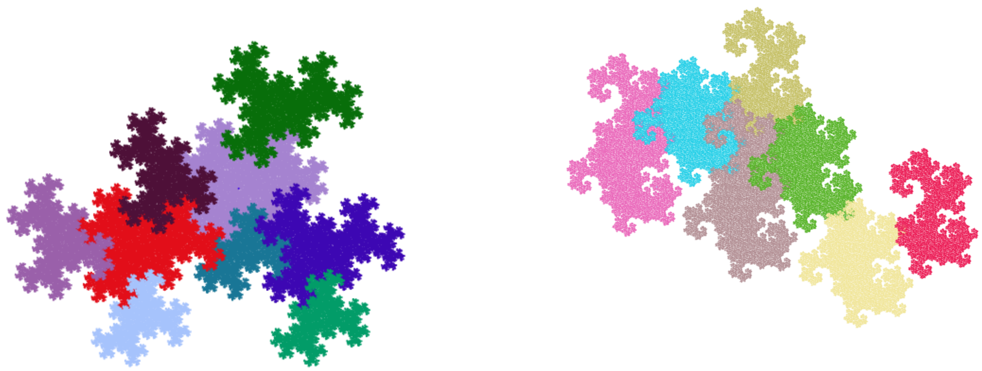

FIGURE 7. Tilings from duals given in Example 9.1.

⟨fig:RR⟩

(5) The tiling on the right in Figure 7 is from the dual of the 1-dim self-similar GIFS below. The three prototiles also appear in Figure 2.

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad p(x) = x^3 - 3x^2 + 2x - 1 \qquad \begin{aligned} f_{2,3}(x) &= f_{3,3}(x) = f_{1,1}(x) = (1/\rho)x \\ f_{3,1}(x) &= f_{2,1}(x) = (x-1)\rho \\ f_{2,2}(x) &= (x+\rho)/\rho \\ f_{1,2}(x) &= (x+\rho-1)/\rho \end{aligned}$$

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