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# A framework for the greedy algorithm 

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#### Abstract

Perhaps the best known algorithm in combinatorial optimization is the greedy algorithm. A natural question is for which optimization problems does the greedy algorithm produce an optimal solution? In a sense this question is answered by a classical theorem in matroid theory due to Rado and Edmonds. In the matroid case, the greedy algorithm solves the optimization problem for every linear objective function. There are, however, optimization problems for which the greedy algorithm correctly solves the optimization problem for many-but not all-linear weight functions. Our intention is to put the greedy algorithm into a simple framework that includes such situations. For any pair $(S, \mathscr{P})$ consisting of a finite set $S$ together with a set $\mathscr{P}$ of partial orderings of $S$, we define the concepts of greedy set and admissible function. On a greedy set $L \subseteq S$, the greedy algorithm correctly solves the naturally associated optimization problem for all admissible functions $f: S \rightarrow \mathbb{R}$. Indeed, when $\mathscr{P}$ consists of linear orders, the greedy sets are characterized by this property. A geometric condition sufficient for a set to be greedy is given in terms of a polytope and roots that generalize Lie algebra root systems. © 2002 Elsevier Science B.V. All rights reserved.


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## 1. Introduction

This paper concerns a classical algorithm in combinatorial optimization, the greedy algorithm. The MINIMAL SPANNING TREE problem, for example, is solved by the greedy algorithm: Given a finite graph $G$ with weights on the edges, find a spanning tree of $G$ with minimum total weight. At each step in the greedy algorithm that solves this problem, there is chosen a set of edges $T$ comprising the partial tree; an edge $e$

[^0]of minimum weight among the edges not in $T$ (the greedy choice) is added to $T$ so long as $T+e$ contains no cycle.

A greedy algorithm makes a locally optimal choice in the hope that this will lead to a globally optimal solution. Clearly, greedy algorithms do not always yield the optimal solution. But for a wide range of important problems the greedy algorithm is quite powerful; a variety of such applications can be found in standard texts such as those by Lawler [8] or Papadimitriou and Steiglitz [9]. A natural question, precisely posed below, is the following. For which optimization problems does the greedy algorithm give the correct solution. In a sense this question is answered by a classical theorem in matroid theory due to Rado and Edmonds [3]. In the matroid case, the greedy algorithm always solves the optimization problem. That is, the greedy algorithm solves the optimization problem for every linear objective function. There are situations, however, for which the greedy algorithm works for many-but not all-linear objective functions. A simple framework for such problems is suggested below.

To make the question precise, consider a set system $(S, \mathscr{I})$ consisting of a finite set $S$ together with a nonempty collection $\mathscr{I}$ of subsets of $S$, called independent sets, closed under inclusion. Given a weight function $f: S \rightarrow \mathbb{R}$, extend this function linearly to the collection of subsets $A \subseteq S$ by defining

$$
f(A)=\sum_{a \in A} f(a) .
$$

There is a natural combinatorial optimization problem associated with the set system $(S, \mathscr{I})$.

## Optimization problem

Given a weight function $f: S \rightarrow \mathbb{R}$, find a maximal independent set with the greatest weight.

In the spanning tree problem, $S$ is the set of edges of the graph $G$ and the independent sets are the acyclic subsets of edges. Minimum and maximum in this problem are interchanged by negating the weights.

The greedy algorithm for this optimization problem is simply:

## Greedy algorithm

$I=\emptyset$
while $S \neq \emptyset$ do
remove from $S$ an element $a$ of largest weight.
if $I \cup\{a\}$ is independent then
$I=I \cup\{a\}$
end
end
By the theorem of Edmonds and Rado referred to earlier, the following statements are equivalent for the set system $(S, \mathscr{I})$. Here $\mathscr{B}$ denotes the set of bases, a basis being a maximum independent set.

1. $(S, \mathscr{I})$ is a matroid.
2. The greedy algorithm correctly solves the combinatorial optimization problem associated with $(S, \mathscr{I})$ for any weight function $f: S \rightarrow \mathbb{R}$.
3. Every basis has the same cardinality and, for every linear ordering $\preccurlyeq$ on $S$, there exists a $B \in \mathscr{B}$ such that for any $A \in \mathscr{B}$, if we write $B=\left(b_{1}, b_{2}, \ldots, b_{k}\right)$ and $A=$ $\left(a_{1}, a_{2}, \ldots, a_{k}\right)$ with the elements of $B$ and $A$ both in increasing order, then $a_{i} \preccurlyeq b_{i}$ for all $i$. This ordering on $k$-element subsets of $X$ is called the Gale order [4].

In the spanning tree problem, the acyclic subsets of edges comprise the independent sets of a matroid. Many well-known optimization problems, besides the spanning tree problem, can be put into the framework of matroids. Texts by Lawler [8] and by Papadimitriou and Steiglitz [9] contain numerous examples.
Matroids are characterized by the property that the greedy algorithm correctly solves the optimization problem for any weight function. There are, however, nonmatroids for which the greedy algorithm correctly solves the appropriate optimization problem for many-but not for every weight function. This is the case for the following, in order of increased generality: symmetric matroids [1], sympletic matroids [2] and the Coxeter matroids of Gelfand and Serganova [5,6,11]. It is our intention in this paper to put the greedy algorithm into a simple framework that includes such examples. In particular, Theorems 4.1 and 4.2 in this paper contain, as a special case, the classical matroid theorem of Rado and Edmonds stated earlier. In [1, Theorem 16], Borovik et al. prove that for symplectic matroids the greedy algorithm correctly solves the optimization problem for all "admissible" functions. This is also a special case of Theorem 4.1; symplectic matroids are used as an example in Sections 2-4 of this paper. Given a Coxeter system ( $W, P$ ) consisting of a finite irreducible Coxeter group $W$ and maximal parabolic subgroup $P$, in [11, Theorem 1] a concrete realization of the set $W / P$ of left cosets is given as a collection of subsets $\mathscr{B}$ of an appropriate partially ordered set $S$. Theorem 3 of [11], in part, states that the natural optimization problem for Coxeter matroids is solved, for all appropriate "admissible" objective functions, by the greedy algorithm applied to $(S, \mathscr{B})$. This result is again a special case of Theorem 4.1. The framework in this paper, however, is conceptually simpler than the usual approaches to Coxeter matroids. The theory of greedoids, developed by Korte and Lovász [7], concerns a framework for optimization problems for which the greedy algorithm finds the optimal for all generalized bottleneck functions. Since our results concern linear objective functions, they do not subsume results on greedoids. Our results on the greedy algorithm apply to situations not previously appearing in the literature, for example the cyclic and bipartite cases mentioned in Sections 3 and 4. Concerning open avenues of research, it would be interesting to formulate additional optimization problems, analogous to the minimum spanning tree problem for matroids, to which the theory is applicable.

The basic notion in this paper is that of a pair $(S, \mathscr{P})$ consisting of a finite set $S$ together with a set $\mathscr{P}$ of partial orderings of $S$. The notions of greedy set and admissible function are defined in Section 2 and examples are given in Sections 2 and 3. It is shown in Section 4 that, for greedy sets, the greedy algorithm correctly solves the naturally associated optimization problem for all admissible functions.

Indeed, when $\mathscr{P}$ contains only linear orders, the greedy sets are characterized by this property. It is also proved in Section 4 that there is essentially no loss of generality in assuming that $\mathscr{P}$ contains only linear orders. Our results naturally lead to the problem of effectively characterizing greedy sets. A geometric approach via polytopes and a generalization of Lie algebra root systems is taken in Section 5. It is proved that if every edge of the polytope of a collection $L$ is parallel to a root, then $L$ is a greedy set.

## 2. Greedy sets and admissible functions

A partial ordering $\preccurlyeq$ of a set $S$ is a binary relation on $S$ that is reflexive, transitive and antisymmetric. If, for any $a$ and $b$ in $S$, either $a \preccurlyeq b$ or $b \preccurlyeq a$ then the partial ordering is called a linear ordering of $S$. The notation $a \prec b$ will mean that $a \preccurlyeq b$ but $a \neq b$.

Consider a pair $(S, \mathscr{P})$ consisting of a finite set $S$ and a collection $\mathscr{P}$ of partial orderings of $S$. A subset $L \subseteq S$ will be called a greedy set for the pair $(S, \mathscr{P})$ if $L$ has a maximum for every ordering in $\mathscr{P}$. That is, for every ordering $\preccurlyeq$ in $\mathscr{P}$, there is an $a \in L$ such that $b \preccurlyeq a$ for all $b \in L$.

Let $S_{k}$ denote the collection of all $k$-element subsets of $S$. Each partial order $\preccurlyeq$ on $S$ induces a partial order on $S_{k}$, namely the Gale order. If $A, B \in S_{k}$, then $A \preccurlyeq B$ in the Gale order if there are arrangements

$$
\begin{aligned}
& A=\left(a_{1}, a_{2}, \ldots, a_{k}\right), \\
& B=\left(b_{1}, b_{2}, \ldots, b_{k}\right)
\end{aligned}
$$

of the elements of the two sets such that $a_{i} \leqslant b_{i}$ for all $i$. The set of Gale orderings induced on $S_{k}$ by the orderings in $\mathscr{P}$ will be denoted $\mathscr{P}_{k}$. A greedy set for the pair $\left(S_{k}, \mathscr{P}_{k}\right)$ will be referred to as a rank $k$ greedy set for ( $S, \mathscr{P}$ ).

A weight function $f: S \rightarrow \mathbb{R}$ is called compatible with a partial order $\preccurlyeq$ on $S$ if $f(a)<f(b)$ whenever $a \prec b$. A weight function $f$ is said to be admissible for $(S, \mathscr{P})$ if $f$ is compatible with some partial order in $\mathscr{P}$. An admissible weight function $f$ for $(S, \mathscr{P})$ can be extended to an admissible weight function $f: S_{k} \rightarrow \mathbb{R}$ for $\left(S_{k}, \mathscr{P}_{k}\right)$ by defining

$$
f(A)=\sum_{a \in A} f(a) .
$$

That this function is indeed admissible is the statement of the corollary below. The first proposition is obvious from the definitions of greedy set and admissible weight function.

Proposition 2.1. If $L \subseteq S$ is a greedy set for $(S, \mathscr{P})$ and $f$ is an admissible weight function, then $f$ attains a unique maximum on $L$.

Proposition 2.2. Let $S$ be a partially ordered set and $S_{k}$ the set of all k-element subsets of $S$ with the Gale order. For any $A, B \in S_{k}$ we have $B \preccurlyeq A$ if and only if $f(B) \leqslant f(A)$ for every weight function $f$ compatible with the order on $S$.

Proof. Assume that $B \preccurlyeq A$. Then there are orderings $A=\left(a_{1}, a_{2}, \ldots, a_{k}\right)$ and $B=$ $\left(b_{1}, b_{2}, \ldots, b_{k}\right)$ such that $b_{i} \leqslant a_{i}$, and hence $f\left(b_{i}\right) \leqslant f\left(a_{i}\right)$, for $1 \leqslant i \leqslant k$. Therefore $f(B)=\sum_{b \in B} f(b) \leqslant \sum_{a \in A} f(a)=f(A)$.

Conversely assume that it is not the case that $B \preccurlyeq A$. We will construct a function $f$ that is compatible with the order on $S$ but for which $f(B)>f(A)$. For each element $a \in A$, let $B_{a}=\{x \in B \mid x \preccurlyeq a\}$. Now $B \preccurlyeq A$ in the Gale order if and only if there is a set of distinct representatives of the sets $B_{a}, a \in A$. Hence, there is no such set of representatives, and, by Philip Hall's theorem on distinct representatives, there must be a set $A^{\prime} \subseteq A$ such that $\left|g\left(A^{\prime}\right)\right|<\left|A^{\prime}\right|$, where $g\left(A^{\prime}\right)=\bigcup\left\{B_{a} \mid a \in A^{\prime}\right\}$. Now define $f(x)=0$ if $x \preccurlyeq a^{\prime}$ for some $a^{\prime} \in A^{\prime}$ and $f(x)=1$ otherwise. Note that $f(x)=0$ if $x \in g\left(A^{\prime}\right)$. For this function we have $f(B)=\sum_{b \in B} f(b)=\left|B \backslash g\left(A^{\prime}\right)\right|>\left|A \backslash A^{\prime}\right| \geqslant \sum_{a \in A} f(a)=f(A)$. It is easy to see that this function $f$ can be perturbed slightly to be compatible with the order on $S$ and to retain the property that $f(B)>f(A)$.

Remark. By the same reasoning as in the proof above, it is also true that, for any $A, B \in S_{k}$, we have $B \prec A$ if and only if $f(B)<f(A)$ for every weight function $f$ compatible with the order on $S$.

Corollary 2.3. If $f: S \rightarrow \mathbb{R}$ is admissible for $(S, \mathscr{P})$, then $f: S_{k} \rightarrow \mathbb{R}$ is admissible for $\left(S_{k}, \mathscr{P}_{k}\right)$.

This paper uses notation that is common in the literature. In particular, we use [ $n$ ] for the set $\{1,2, \ldots, n\}$ of the first $n$ positive integers. For readability, brackets may be deleted in denoting a set; for example $\{2,4,6\}$ may be denoted simply 246 .

Example 2.4 (Matroids are a special case). Let $S=[n]$ and let $\mathscr{P}$ be the set of all linear orderings of $S$. Then clearly every injective weight function $f: S \rightarrow \mathbb{R}$ is admissible. By definition, a rank $k$ greedy set $L$ is a collection of $k$-element subsets of $S$ such that, for every linear ordering of [ $n$ ], there is a unique maximum in $L$ in the induced Gale ordering on $S_{k}$. But this is exactly the third characterization of matroid given in the introduction. In other words, $L$ is a rank $k$ greedy set for $(S, \mathscr{P})$ if and only if $L$ is the set of bases of a rank $k$ matroid.

Example 2.5 (Symplectic matroids). Let $[n]^{*}=\left\{1^{*}, \ldots, n^{*}\right\}$ and let $S=[n] \cup[n]^{*}$. By convention $i^{* *}=i$. Let $\mathscr{P}$ be the set of all linear orderings $\leqslant$ of $S$ with the property that $i \preccurlyeq j$ if and only if $j^{*} \preccurlyeq i^{*}$ for any $i, j \in S$. Equivalently, the pair $i$ and $i^{*}$ appear symmetrically in the order. For example, with $n=3$ one such order is $2 \prec$ $1^{*} \prec 3 \prec 3^{*} \prec 1 \prec 2^{*}$. Consequently, the admissible weight functions include all injective weights $f: S \rightarrow \mathbb{R}$ such that $f\left(i^{*}\right)=-f(i)$ for each $i \in[n]$. A symmetric matroid of Bouchet [2] is exactly a rank $n$ greedy set $L$ for the pair ( $S, \mathscr{P}$ ) with the additional property that $A \cap A^{*}=\emptyset$ for each $A \in L$. More generally, the rank $k$ greedy
sets, $0 \leqslant k \leqslant n$, satisfying this same property are exactly the symplectic matroids of Borovik et al. [2]. The significance of the added assumption will be discussed in the next section.

## 3. The group case

Let $S$ be a partially ordered set and $G$ a transitive group of permutations of $S$. If $\preccurlyeq$ denotes the order on $S$ and $\pi \in G$, let $\leqslant_{\pi}$ denote the order defined by

$$
a \preccurlyeq \pi b \quad \text { if } \quad \pi^{-1} a \preccurlyeq \pi^{-1} b .
$$

For rank $k$ subsets, the corresponding Gale order will likewise be denoted by $A \preccurlyeq{ }_{\pi} B$. This shifted order will be called $\pi$-order. Let $\mathscr{P}=\mathscr{P}(G)$ be the set of all $\pi$-orders on $S$ for $\pi \in G$. For example, if $S=[n]$ with the order $1 \prec 2 \prec \cdots \prec n$, then $\mathscr{P}$ is the set of all orders $\pi(1) \prec \pi(2) \prec \cdots \prec \pi(n)$ where $\pi \in G$. The pair $(S, \mathscr{P}(G))$ is referred to as the group case.

Although $G$ acts transitively on $S$, the induced action of $G$ on $S_{k}$ may not be transitive. There will therefore be situations where attention will be restricted to a single orbit $O_{k}$ of $G$ acting on $S_{k}$. The rank $k$ greedy sets of ( $S, \mathscr{P}(G)$ ) will then be restricted to being contained in a set $O_{k}$ on which $G$ acts transitively.

Example 3.1 (Matroids). If $S=[n]$ with the linear order

$$
1 \prec 2 \prec \cdots \prec n,
$$

and $\Sigma_{n}$ is the symmetric group consisting of all permutations of [ $n$ ], then the greedy sets for $\left(S, \mathscr{P}\left(\Sigma_{n}\right)\right)$ are exactly the ordinary matroids of Example 2.4.

Example 3.2 (Symplectic matroids). Let $S=[n] \cup[n]^{*}$ with the linear order

$$
1 \prec 2 \prec \cdots \prec n \prec n^{*} \prec \cdots \prec 2^{*} \prec 1^{*},
$$

and let $G$ be the hyperoctahedral group of permutations of $S$ generated by all transpositions of the form $\left(i i^{*}\right)$ and all involutions of the form $(i j)\left(i^{*} j^{*}\right)$. Note that the set of all $k$-element sets $A$ with the property that $A \cap A^{*}=\emptyset$ comprises a single orbit of $G$ acting on $S_{k}$; call this orbit $O_{k}$. Then the greedy sets $L \subseteq O_{k}$ for $(S, \mathscr{P}(G))$ are exactly the symplectic matroids of Example 2.5 above.

Example 3.3 (Cyclic case). Let $S=[n]$ be a poset with the order

$$
1 \prec 2 \prec \cdots \prec n,
$$

and let $G$ be the cyclic group acting on [ $n$ ] and generated by the cycle $(12 \cdots n)$. For example, $3 \prec 4 \prec 1 \prec 2$ is a cyclic ordering for $n=4$. It is interesting that, even for this elementary example, characterization of the collection of greedy sets is evasive. For example, it is easy to check that the orbit of 652 under the action of the cyclic group $C_{7}$ acting on $S=[7]$ is a greedy set, but the orbit of 652 under the action of $C_{6}$ acting on $S=[6]$ is not a greedy set. (As noted earlier, the set $\{6,5,2\}$ is denoted simply by 652 .)

Example 3.4 (Bipartite case). Let $S=[n] \cup[n]^{*}$ and let $\mathscr{P}$ consist of any linear order such that either all the unstarred elements precede the starred elements or all the starred elements precede the unstarred elements. This is the group case where, if $[n]$ and $[n]^{*}$ are considered as vertex sets of the two parts of a complete bipartite graph $K_{n, n}$, then the group is the automorphism group of $K_{n, n}$.

## 4. The greedy algorithm

As previously, $S$ is a finite set and $\mathscr{P}$ a collection of partial orderings of $S$. The optimization problem applied to the pair $(S, \mathscr{P})$ is the following.

## Optimization problem

Given an admissible weight function $f: S \rightarrow \mathbb{R}$ and a set $L \subseteq S_{k}$, find an element of $L$ that maximizes the induced weight function $f: S_{k} \rightarrow \mathbb{R}$.

Given $L \subseteq S_{k}$, call a subset $I \subseteq S$ independent with respect to $L$ if $I$ is a subset of some element of $L$. The greedy algorithm, precisely as stated in the introduction, applied to this optimization problem, merely chooses the largest weight element at each stage subject to the condition that the resulting set is independent with respect to $L$. The following result is basic.

Theorem 4.1. Let $(S, \mathscr{P})$ be a pair consisting of a collection $\mathscr{P}$ of orderings of the finite set $S$. If $L \subseteq S_{k}$ is a rank $k$ greedy set, then the greedy algorithm correctly solves the optimization problem for every admissible weight function $f: S \rightarrow \mathbb{R}$.

Proof. Assume that $L$ is a greedy set and that $f$ is compatible with some order, say $\preccurlyeq$, in $\mathscr{P}$. Since $L$ is a greedy set, it contains a unique set $A$ that is maximum with respect to the Gale order. We claim that the greedy algorithm selects this set $A$. Suppose, instead that $B$ is chosen where $B \preccurlyeq A$. Order the sets $A=\left(a_{1}, a_{2}, \ldots, a_{k}\right)$ and $B=\left(b_{1}, b_{2}, \ldots, b_{k}\right)$ so that the elements of $B$ appear in the order selected by the greedy algorithm and $b_{i} \leqslant a_{i}$ for $1 \leqslant i \leqslant k$. Assume that $a_{i}=b_{i}$ for $1 \leqslant i<j$, but $a_{j} \neq b_{j}$. Then $b_{j} \prec a_{j}$ implies, by the compatibility of $f$, that $f\left(b_{j}\right)<f\left(a_{j}\right)$. But this contradicts the fact that, at this stage, the greedy algorithm chooses $b_{j}$.

In the case that $\mathscr{P}$ contains only linear orderings of $S$, the greedy sets are actually characterized by the property that the greedy algorithm correctly solves the optimization problem for every admissible weight function. The assumption that $\mathscr{P}$ contains only linear orderings of $S$ is a reasonable one in light of Theorem 4.3 below.

Theorem 4.2. Let $\mathscr{P}$ be a set of linear orderings of $S$ and $L \subseteq S_{k}$. Then the greedy algorithm correctly solves the optimization problem for every admissible weight function if and only if $L$ is a greedy set.

Proof. In one direction, this result is a corollary of Theorem 4.1. To prove it in the other direction, assume that $L$ is not a greedy set. Then there exists an order on $S$, say $\preccurlyeq$, for which $L$ does not attain a unique maximum. Since $\preccurlyeq$ is a linear ordering of $S$, order the elements in each set in $L$ in decreasing order. Let $A$ denote the lexicographically maximum element of $L$, and let $B \neq A$ be a set in $L$ that is a maximum with respect to Gale order. According to Proposition 2.2 there exists a weight function compatible with $\preccurlyeq$ such that $f(B)>f(A)$. On the other hand, the greedy algorithm chooses $A$.

It is desirable to choose $\mathscr{P}$ so that there are many admissible weight functions and many rank $k$ greedy sets. Then the greedy algorithm will correctly solve a large collection of optimization problems. Unfortunately, these two objectives - many admissible functions and many greedy sets-are often conflicting. If $(S, \mathscr{P})$ and $(S, \mathscr{Q})$ have the same collection of admissible functions, but, for each $k$, each rank $k$ greedy set for $(S, \mathscr{P})$ is a rank $k$ greedy set for $(S, \mathscr{Q})$, then clearly it is preferable, for algorithmic purposes, to use $(S, \mathscr{Q})$ rather than $(S, \mathscr{P})$.

Theorem 4.3. For any pair $(S, \mathscr{P})$ there is a pair $(S, \mathscr{Q})$ such that

1. 2 contains only linear orders;
2. $(S, \mathscr{P})$ and $(S, \mathscr{Q})$ have the same admissible injective functions; and
3. for each $k$, each rank $k$ greedy set for $(S, \mathscr{P})$ is also a rank $k$ greedy set for $(S, \mathscr{Q})$.

Proof. Let $\mathscr{2}$ be the collection of all linear extensions of the orders in $\mathscr{P}$. Clearly condition (1) is satisfied. Concerning condition (2) assume that weight function $f$ is admissible for $(S, \mathscr{2})$. Then $f$ is compatible with some linear ordering $\leqslant$ in $\mathscr{2}$ and hence is also compatible with any ordering in $\mathscr{P}$ having $\leqslant$ as a linear extension. Therefore, $f$ is admissible for $(S, \mathscr{P})$. Conversely, assume that $f$ is admissible for $(S, \mathscr{P})$ and is injective. Then $f$ is compatible with some ordering $\preccurlyeq$ in $\mathscr{P}$. Assume that the elements of $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ of $S$ are indexed such that $f\left(x_{1}\right) \leqslant f\left(x_{2}\right) \leqslant \cdots \leqslant f\left(x_{n}\right)$. Then, by definition, the linear order $x_{1} \leqslant x_{2} \leqslant \cdots \leqslant x_{n}$ is a linear extension of $\preccurlyeq$ and $f$ is compatible with this linear order. Therefore, $f$ is admissible for ( $S, \mathscr{Q}$ ).

Concerning condition (3) assume that $L$ is a rank $k$ greedy set for $(S, \mathscr{P})$. To show that $L$ must also be a greedy set for $(S, \mathscr{2})$, let $\leqslant$ be any linear order in $\mathscr{2}$. Then $\leqslant$ is a linear extension of some order $\preccurlyeq$ in $\mathscr{P}$. Since $L$ is greedy for $(S, \mathscr{P})$, there is a unique maximum set $A=\left(a_{1}, \ldots, a_{k}\right)$ in $L$ such that, for any $B=\left(b_{1}, \ldots, b_{k}\right)$, we have $b_{i} \preccurlyeq a_{i}$ for all $i$ for some ordering of the elements of $A$ and $B$. Because $\leqslant$ is a linear extension of $\preccurlyeq$, it is also true that $b_{i} \leqslant a_{i}$ for all $i$. Therefore, $A$ is also the unique maximum with respect to the Gale order relative to $\leqslant$. So $L$ is a greedy set for $(S, \mathscr{2})$.

Remark. The assumption in condition (2), that the admissible functions be injective, is not a serious restriction because, for any admissible function $f$, there is an injective admissible function that is small perturbation of $f$.

The following examples are intentionally kept simple in order to illustrate the main ideas.

Example 4.4 (Symplectic matroids). This is a continuation of Example 2.5 of Section 2. Consider the case $n=3$. It is not hard to check that the set

$$
L=\left\{12^{*}, 13,2^{*} 3^{*}, 1^{*} 3^{*}\right\}
$$

is a rank 2 greedy set. Consider, as an example, the particular function $f:[3] \cup[3]^{*} \rightarrow$ $\mathscr{R}$ defined by

$$
\begin{array}{ll}
f(1)=1, & f\left(3^{*}\right)=4, \\
f(2)=2, & f\left(2^{*}\right)=5, \\
f(3)=3, & f\left(1^{*}\right)=6 .
\end{array}
$$

This is an admissible function because it is compatible with the ordering $1 \prec 2 \prec 3 \prec$ $3^{*} \prec 2^{*} \prec 1^{*}$. The greedy algorithm applied to $L$ chooses the set $1^{*} 3^{*}$ whose total weight is 10 , greater than that of any other set in $L$.

On the other hand, for the function

$$
\begin{array}{ll}
f(1)=4, & f\left(3^{*}\right)=1, \\
f(2)=2, & f\left(2^{*}\right)=5, \\
f(3)=3, & f\left(1^{*}\right)=6,
\end{array}
$$

which is not admissible, the greedy algorithm again chooses the set $1^{*} 3^{*}$ whose total weight is 7 , but the total weight of $12^{*}$ is 9 . The greedy algorithm fails in this case. (Note that the collection $L$ is not an ordinary matroid on the set $[3] \cup[3]^{*}$.)

Example 4.5 (Cyclic case). This is a continuation of Example 3.3 of Section 3. Consider the case $n=4$, and take the admissible weight function

$$
f(1)=3 \quad f(2)=4 \quad f(3)=1 \quad f(4)=2 .
$$

The set

$$
L=\{13,24\}
$$

is a rank 2 greedy set for $(S, \mathscr{P})$. The greedy algorithm chooses 24 whose weight is 6 , greater than the other element of $L$. On the other hand, for the weight function

$$
f(1)=1, f(2)=3, f(3)=5, f(4)=4,
$$

which is not admissible, the greedy algorithm fails for $L$.
Example 4.6 (Bipartite case). This is a continuation of Example 3.4 of Section 3. Consider the case $n=2$; clearly

$$
L=\left\{12,1^{*} 2^{*}\right\}
$$

is a rank 2 greedy set. An admissible function is one for which either $f(i)<f(j)$ for every unstarred $i$ and starred $j$ or $f(i)<f(j)$ for every starred $i$ and unstarred $j$. As a
simple example, let

$$
\begin{array}{ll}
f(1)=1, & f\left(1^{*}\right)=3, \\
f(2)=2, & f\left(2^{*}\right)=4 .
\end{array}
$$

Then $f$ is admissible, and the greedy algorithm chooses $1^{*} 2^{*}$ which has total weight 7 compared to the total weight 3 of 12 . On the other hand the function

$$
\begin{array}{ll}
f(1)=3, & f\left(1^{*}\right)=1, \\
f(2)=4, & f\left(2^{*}\right)=5
\end{array}
$$

is not admissible. The greedy algorithm chooses $1^{*} 2^{*}$ which has total weight 6 , although 12 has greater total weight 7 .

## 5. Roots and polytopes

In light of Theorems 4.1 and 4.2, it is important to have an efficient method to determine whether a collection $L \subseteq S_{k}$ is a greedy set. If ( $S, \mathscr{P}$ ) is such that $|S|=n$ and $\mathscr{P}$ consists of $N$ linear orderings of $S$, then $N$ may well be exponential as a function of $n$. If $L$ is a collection of $k$-element subsets of $S$, then it will take no less than exponential time to check, using the definition, whether $L$ is a greedy set. An alternative procedure is sought that is polynomial in $n$ and the cardinality of $L$. In the matroid case, any one of many cryptomorphic definitions of matroid can be used to do this; that is one of the nice properties of matroids. For ordinary and symplectic matroids, the associated matroid polytope [10] furnishes an efficient procedure to determine whether a set is greedy. For the general case, we give a geometric approach (Theorem 5.7 and the remarks that follow it) using polytopes and roots.

Because of Theorem 4.3, it will be assumed throughout this section that $S=[n]$ and $\mathscr{P}$ contains only linear orders. Each linear order $\preccurlyeq$ in $\mathscr{P}$ can be denoted by the permutation $\pi$ for which $\pi 1 \prec \pi 2 \prec \cdots \prec \pi n$. Given the pair ( $S, \mathscr{P}$ ), we will associate a polytope $\Delta(L)$ to each subset $L \subseteq S_{k}$ as follows. Let $\varepsilon_{1}, \ldots, \varepsilon_{n}$ be the canonical orthonormal basis for $n$-dimensional Euclidean space $\mathbb{R}^{n}$. For any $k$-element subset $A$ of $S$, set

$$
\begin{equation*}
\delta_{A}=\sum_{i \in A} \varepsilon_{i} . \tag{1}
\end{equation*}
$$

Let $\Delta(L)$ be the convex hull of the points $\left\{\delta_{A} \mid A \in L\right\}$. Note that $\Delta(L)$ lies in the ( $n-1$ )-dimensional hyperplane in $\mathbb{R}^{n}$ with equation $\sum_{i=1}^{n} x_{i}=k$.

Define a root of $(S, \mathscr{P})$ as a non-zero vector $\mathbf{r}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ such that

1. $x_{i}=0,1$ or -1 for all $i$,
2. $\sum_{i=1}^{n} x_{i}=0$ and,
3. for every $\pi \in \mathscr{P}$ there exists an $\eta_{\pi} \in\{-1,1\}$ such that $\eta_{\pi} \cdot \sum_{i=1}^{m} x_{\pi i} \geqslant 0$ for each $m=1, \ldots, n$.
In particular, note that $\varepsilon_{i}-\varepsilon_{j}$ is a root of any pair $(S, \mathscr{P})$ for all $i \neq j$. Our definition of root of $(S, \mathscr{P})$ is meant as a generalization of a Lie algebra root system. The two
notions coincide for root systems of Coxeter groups. In the group case, we refer to a root of $(S, \mathscr{P}(G))$ as a root of $G$. The group acts on the set of roots. The roots in the following examples are easy to compute.

Example 5.1. For either the symmetric group $\Sigma_{n}$ or the alternating group $A_{n}$ acting on [ $n$ ], the roots are

$$
R_{0}=\left\{\varepsilon_{i}-\varepsilon_{j} \mid i \neq j\right\}
$$

Example 5.2. For the cyclic group $\mathbb{Z}_{n}$ of Example 3.3 acting on [ $n$ ], the roots are

$$
\left\{ \pm \sum_{j=1}^{2 m}(-1)^{j} \varepsilon_{i_{j}}\right\}
$$

where $i_{1}<i_{1}<\cdots<i_{2 m}$. In other words, +1 and -1 alternate in the vector $\mathbf{r}$. For example, with $n=5$, the vector $(1,0,-1,1,-1)$ is a root while $(1,-1,0,-1,1)$ is not.

Example 5.3. For the hyperoctahedral group of Example 3.2 acting on $[n] \cup[n]^{*}$ the roots are

$$
R_{0} \cup\left\{\varepsilon_{i}+\varepsilon_{j}-\varepsilon_{i^{*}}-\varepsilon_{j^{*}} \mid i \neq j, j^{*}\right\}
$$

For example, for $n=3$, the vector $(1,-1,0,0,1,-1)$ is a root.
Example 5.4. For the bipartite group of Example 3.4 acting on $[n] \cup[n]^{*}$, let $\varepsilon=\varepsilon_{1}+$ $\cdots+\varepsilon_{n}$ and $\varepsilon^{*}=\varepsilon_{1^{*}}+\cdots+\varepsilon_{n^{*}}$. The roots are

$$
R_{0} \cup\left\{\delta_{A}-\delta_{A^{*}}\left|A \subseteq[n], A^{*} \subseteq[n]^{*},|A|=\left|A^{*}\right|\right\}\right.
$$

The idea now is, given a set $L$, to find a computationally efficient algorithm, in terms of its polytope $\Delta(L)$, for deciding whether $L$ is a greedy set. A vector $\mathbf{v}=$ $\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in \mathbb{R}^{n}$ will be called $\pi$-admissible for $(S, \mathscr{P})$ if $0<x_{\pi 1}<x_{\pi 2}<\cdots<x_{\pi n}$. A vector that is $\pi$-admissible for some $\pi \in \mathscr{P}$ will simply be called admissible.

Theorem 5.5. A subset $L \subseteq S_{k}$ is a rank $k$ greedy set for $(S, \mathscr{P})$ if and only if, for each admissible vector $\mathbf{v}$, the linear function $f_{\mathbf{v}}(\mathbf{x})=\langle\mathbf{x}, \mathbf{v}\rangle$ attains a maximum on $\Delta(L)$ at a unique vertex.

Proof. If $\mathbf{v}=\left(x_{1}, \ldots, x_{n}\right)$, then

$$
f_{\mathbf{v}}\left(\delta_{A}\right)=\sum_{i \in A} x_{i}
$$

Note that, for $A, B \in S_{k}$ and $\pi \in \mathscr{P}$, we have, by Proposition 2.2 and the remark following it, that $A \prec B$ in the $\pi$-Gale order if and only if $f(A)<f(B)$ for all positive weight functions $f$ compatible with $\prec_{\pi}$. This is equivalent to $\sum_{i \in A} f(i)<\sum_{i \in B} f(i)$ for all positive weight functions such that $0<f(\pi 1)<f(\pi 2)<\cdots<f(\pi n)$. But this, in turn, is the same as $f_{\mathbf{v}}\left(\delta_{A}\right)=\sum_{i \in A} x_{i}<\sum_{i \in B} x_{i}=f_{\mathbf{v}}\left(\delta_{B}\right)$ for all $\pi$-admissible vectors
v. Thus the linear function $f_{\mathbf{v}}(\mathbf{x})=\langle\mathbf{x}, \mathbf{v}\rangle$ attains a maximum on $\Delta(L)$ at a unique vertex for each admissible vector $\mathbf{v}$ if and only if there is a unique $\pi$-Gale maximum in $L$ for each $\pi \in \mathscr{P}$. But the latter condition is the definition of greedy set.

Lemma 5.6. Let $\mathbf{r}$ be a vector satisfying conditions (1) and (2) in the definition of root. Then $\mathbf{r}$ is a root if and only if $\mathbf{r}^{\perp}$ contains no admissible vector.

Proof. Assume that $\mathbf{r}=\left(y_{1}, \ldots, y_{n}\right)$ is orthogonal to some admissible vector $\mathbf{v}=$ $\left(x_{1}, \ldots, x_{n}\right)$. Since $\mathbf{v}$ is admissible, there is a $\pi \in \mathscr{P}$ such that $0<x_{\pi 1}<x_{\pi 2}<\cdots<x_{\pi n}$. We have $\sum_{i=1}^{n} x_{\pi i} y_{\pi i}=\sum_{i=1}^{n} x_{i} y_{i}=0$. Let $A=\left\{i \mid y_{\pi i}=+1\right\}$ and $B=\left\{i \mid y_{\pi i}=-1\right\}$. Then $\sum_{i \in A} x_{\pi i}=\sum_{i \in B} x_{\pi i}$. Assume, by way of contradiction, that $\mathbf{r}$ is a root. Condition (3) in the definition of root implies (without loss of generality) that there is a bijection $\phi$ from $A$ onto $B$ so that $\phi(i)>i$ for all $i \in A$. But this implies that $\sum_{i \in A} x_{\pi i}<\sum_{i \in B} x_{\pi i}$, a contradiction.

Conversely, assume that $\mathbf{r}=\left(y_{1}, \ldots, y_{n}\right)$ is not a root. We use the same notation $A$ and $B$ as above for some fixed $\pi \in \mathscr{P}$ violating condition (3). Without loss of generality we can ignore the 0 entries in $\mathbf{r}$ and assume that $A \cup B=[n]$ and that $1 \in A$. Define a bijection $\phi$ from $A$ to $B$ recursively as follows. Let $\phi(1)$ be the least element of $B$. Having defined $\phi(i)$ for $i<j$, define $\phi(j)$ as the least element of $B$ not already in the image of $\phi$. Let $C=\{i \in A \mid \phi(i)>i\} \cup\{\phi(i) \in B \mid \phi(i)>i\}$ and $D=\{i \in A \mid \phi(i)<i\} \cup$ $\{\phi(i) \in B \mid \phi(i)<i\}$. Since $\mathbf{r}$ is not a root, $C$ and $D$ are nonempty. A $\pi$-admissible vector $\mathbf{v}=\left(x_{1}, \ldots, x_{n}\right)$ can now be chosen so that $\sum_{i \in D} x_{\pi i} y_{\pi i}$ and $\sum_{i \in C} x_{\pi i} y_{\pi i}$ take arbitrary positive and negative values, respectively. In particular, a $\pi$-admissible vector $\mathbf{v}$ can be chosen so that $\langle\mathbf{r}, \mathbf{v}\rangle=\sum_{i \in C \cup D} x_{i} y_{i}=0$.

Theorem 5.7. Let $\mathscr{P}$ be a collection of linear orderings of a finite set $S$ and let $L \subseteq S_{k}$. If every edge of $\Delta(L)$ is parallel to a root, then $L$ is a greedy set for $(S, \mathscr{P})$.

Proof. Assume that $L$ is not a greedy set. By Theorem 5.5 there exists an admissible vector $\mathbf{v}=\left(x_{1}, \ldots, x_{n}\right)$ such that the linear function $f_{\mathbf{v}}$ achieves its maximum on at least two vertices of $\Delta(L)$. Since $\Delta(L)$ is convex, $f_{\mathrm{v}}$ achieves its maximum on some edge $\mathbf{e}$ of $\Delta(L)$. Therefore, $\langle\mathbf{e}, \mathbf{v}\rangle=0$. By Lemma 5.6, $\mathbf{e}$ is not parallel to a root.

Unfortunately the converse of Theorem 5.7 is, in general, false. There exist greedy sets $L$ for which the polytope $\Delta(L)$ has non-root edges. As an example, consider the cyclic group $\mathbb{Z}_{5}$ acting on the poset $[5]=\{1,2,3,4,5\}$. This is the group case $n=5$ in Example 3.3. If

$$
L=\{12,23,34,15,35\}
$$

then $L$ is a greedy set: The (Gale) maximum is 35 for the order 12345; 15 for the order 23451; 12 for the order $34512 ; 23$ for the order 45123 ; and 34 for the order 51234. It is easy to check that the segment joining $\delta_{12}$ and $\delta_{34}$ is an edge $\mathbf{e}$ of $\Delta(L)$ but that $\mathbf{e}=(1,1,-1,-1,0)$ is not a root because condition (3) in the definition fails.

Recall from Section 3 that in the group case it is natural to require that $L$ be contained in a single orbit $O_{k} \subseteq S_{k}$ under the action of $G$. This is, however, not the
case in the above example; $L$ is not contained in a single orbit of $S_{2}$ under the action of $\mathbb{Z}_{5}$. This motivates the following question.

Question 5.8. Let $G$ denote a permutation group acting on $S$, and let $O_{k}$ be an orbit of $G$ acting on $S_{k}$. Is it true that $L \subseteq O_{k}$ is a rank $k$ greedy set for $(S, \mathscr{P}(G))$ if and only if every edge of $\Delta(L)$ is parallel to a root of $G$ ?

In certain cases the question can be answered in the affirmative. An edge joining vertices $\mathbf{x}$ and $\mathbf{y}$ of a polytope $\Delta$ in $\mathbb{R}^{n}$ will be called supporting if it is contained in a supporting hyperplane of $\Delta$ that is orthogonal to an admissible vector $\mathbf{v}$ with $\langle\mathbf{x}, \mathbf{v}\rangle=\langle\mathbf{y}, \mathbf{v}\rangle$. According to Lemma 5.6, an edge of a polytope $\Delta(L)$ not parallel to a root must be orthogonal to some admissible vector; so it is possible for such an edge to be supporting. Also according to Lemma 5.6, an edge that is parallel to a root cannot be supporting. A set $L \subseteq S_{k}$ is called supporting if each edge of $\Delta(L)$ is either supporting or parallel to a root.

Theorem 5.9. Let $\mathscr{P}$ be a collection of linear orderings of a finite set $S$ such that $\mathscr{P}$ is closed under the operation of taking the inverse (reversing order). Assume that $L \subseteq S_{k}$ is supporting. Then $L$ is a greedy set for $(S, \mathscr{P})$ if and only if every edge of $\Delta(L)$ is parallel to a root.

Proof. In one direction the statement follows directly from Theorem 5.7. Conversely, assume that some edge $\mathbf{e}$ of $\Delta(L)$ is not parallel to a root. Because $L$ is supporting, there is an admissible vector $\mathbf{v}$ such that $\mathbf{e}$ is contained in a supporting hyperplane of $\Delta(L)$ that is orthogonal to $\mathbf{v}$, and the linear function $f_{\mathbf{v}}(\mathbf{x})=\langle\mathbf{x}, \mathbf{v}\rangle$ takes equal values at the two endpoints of $\mathbf{e}$. This implies that $f_{\mathbf{v}}(\mathbf{x})$ attains a maximum on $\Delta(L)$ at both endpoints of the edge $\mathbf{e}$ or a minimum at both endpoints of the edge e. By Theorem 5.5, if $f_{\mathbf{v}}(\mathbf{x})$ attains a maximum at both endpoints, then $L$ is not a greedy set. If $f_{\mathbf{v}}(\mathbf{x})$ attains a minimum at both endpoints, say $\delta_{A}$ and $\delta_{B}$, then, as in the proof of Theorem 5.5, there is a $\pi \in \mathscr{P}$ such that $A$ and $B$ are both minimum in the $\pi$-Gale order. But a Gale minimum for $\pi$ is a Gale maximum for the inverse of $\pi$, which is also in $\mathscr{P}$. Hence $L$ is not a greedy set.

Let $G$ be a permutation group acting on $S$ such that $\mathscr{P}(G)$ is closed under the operation of taking the inverse. Let $O_{k}$ be an orbit under the induced action of $G$ on $S_{k}$. Sometimes it is the case that any $L \subseteq O_{k}$ is supporting. This is so, for example, if each vector determined by a point on the boundary of $\Delta\left(O_{k}\right)$ is either admissible or orthogonal to a root. It can be shown that this is the case for ordinary and symplectic matroids. In general, it is not the case that any $L \subseteq O_{k}$ is supporting. Again consider the cyclic group $\mathbb{Z}_{5}$ acting on the poset $\{1,2,3,4,5\}$. If

$$
L=\{12,23,34,51\}
$$

then $L$ lies on one orbit under the action of the cyclic group acting on the set of pairs $S_{2}$, but the edge $\mathbf{e}=(1,1,-1,-1,0)$ joining $\delta_{12}$ and $\delta_{34}$ in $\Delta(L)$ is not a root and is not supporting. For $\mathbf{e}$ to be supporting, there would have to exist an admissible vector $\mathbf{v}=\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\right)$ such that $x_{1}+x_{2}=x_{3}+x_{4}$. This would imply that either
$0<x_{2}<x_{3}<x_{4}<x_{5}<x_{1}$ or $0<x_{4}<x_{5}<x_{1}<x_{2}<x_{3}$. In both cases, the vertices $\delta_{51}$ and $\delta_{23}$ lie on different sides of the hyperplane containing $\mathbf{e}$ and orthogonal to $\mathbf{v}$. So there does not exist a supporting hyperplane of $\Delta(L)$ containing $\mathbf{e}$ which is orthogonal to some admissible vector.

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