THE HAUSDORFF DIMENSION OF THE BOUNDARY OF A SELF-SIMILAR TILE

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ABSTRACT

An effective method is given for computing the Hausdorff dimension of the boundary of a self-similar digit tile T in n-dimensional Euclidean space:

$$\dim_H(\partial T) = \frac{\log \lambda}{\log c},$$

where $1/c$ is the contraction factor and λ is the largest eigenvalue of a certain contact matrix first defined by Gröchenig and Haas.

1. *Introduction*

In the book *Classics on fractals* [**6**], Edgar asked what the Hausdorff dimension of the boundary of the Lévy dragon might be. In general Edgar asked what could be said about the dimension of the boundary of a self-similar tile [**6**, p. 236]. In [**5**] Duvall and Keesling determined precisely the Hausdorff dimension of the boundary of the Lévy dragon. In [**11**] Keesling showed that the Hausdorff dimension of the boundary of any self-similar tile in $\mathbb{R}^d$ is less than d, but that this dimension could be arbitrarily close to d. In this paper we give a general method for determining the Hausdorff dimension of the boundary of a self-similar digit tile. The only condition that is needed on the self-similar digit tile T is that one of the equivalent conditions given in Theorem 1 and below holds for T. The method given in this paper either determines precisely the Hausdorff dimension of the boundary of T or it determines that one of these conditions fails. One does not have to check beforehand whether or not the conditions hold. The outcome of the algorithm itself will tell.

Having given some motivation for the main result of this paper, we now proceed to give some basic definitions and results needed in the subsequent sections. A well-known method of constructing fractals is by using an iterated function system $\{f_i\}_{i=1}^N$ consisting of contractions on Euclidean space $\mathbb{R}^d$. On the space H of nonempty compact subsets of $\mathbb{R}^d$, with respect to the Hausdorff metric, define $F: H \longrightarrow H$ by

$$F(X) = \bigcup_{i=1}^{N} f_i(X),$$

for any nonempty compact set X. It is well known that F is a contraction on H and that H is a complete metric space with the Hausdorff metric. Hence, by the contraction mapping theorem, F has a unique fixed point or *attractor* T satisfying

$$T = \bigcup_{i=1}^{N} f_i(T) \quad \text{and given by} \quad T = \lim_{n\to\infty} F^{(n)}(T_0), \tag{1.1}$$

where $F^{(n)}$ denotes the nth iterate of F, T_0 is an arbitrary nonempty compact subset

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of $\mathbb{R}^d$ and the limit is with respect to the Hausdorff metric. The attractor T will be called a *self-similar tile* if

(1) each f_i is a similitude with the same contraction factor $1/c$, $c > 1$;
(2) the attractor T is the closure of its interior;
(3) $f_i(T)$ and $f_j(T)$ do not overlap for any $i \neq j$.

Non-overlapping means that the intersection of the interiors is empty. We use the term 'tile' because it is not hard to show that, under the above conditions, $\mathbb{R}^d$ can be tiled by copies of T.

In this paper attention is restricted to a certain class of self-similar tiles that are common in the literature on fractals. A *self-similar digit tile* is a self-similar tile whose iterated function system is of the form

$$f_i(x) = A^{-1}(x+d_i), \quad d_i \in D \tag{1.2}$$

where

(1) A is a similitude given by an expansive integer matrix;
(2) D is a set of coset representatives of $\mathbb{Z}^d/A(\mathbb{Z}^d)$ with $0 \in D$.

Expansive means that the eigenvalues of A are greater than 1 or, equivalently, that the ratio c of the similitude A is greater than 1. The set D is called a *digit set*.

Self-similar tiles, and the generalization to self-affine tiles, have been studied by, among others, Bandt [**1**], Dekking [**4**], Gröchenig and Haas [**9**], Kenyon [**12**], Lagarias and Wang [**14**–**16**], and Vince [**21**–**23**]. One of the motivations for studying this type of tile is its use in higher-dimensional wavelet multiresolution analysis. This is dealt with in the papers by Gröchenig and Madych [**10**] and Strichartz [**18**]. After the results of this study were obtained we came across unpublished preprints by Veerman [**20**] and by Strichartz and Wang [**19**] which also compute the Hausdorff dimension of the boundary of self-similar tiles of $\mathbb{R}^n$. Those papers obtain similar results by different methods. Some additional calculations based on [**19**] are done in [**13**].

If T is a self-similar digit tile, then it is known that T is the closure of its interior, and that the Lebesgue measure of the boundary ∂T is 0. In fact the Hausdorff dimension of ∂T is less than d [**11**, Theorem 2.1]. Lagarias and Wang [**15**, **16**] have shown that there is a tiling of $\mathbb{R}^d$ by translates of T by some lattice, not necessarily $\mathbb{Z}^d$. The following special case of a 2-dimensional self-similar digit tile is illustrative.

1.1. *Block tiling*

Given an integer $k > 1$ consider the iterated function system whose functions are indexed by pairs (i,j) of integers with $0 \leqslant i, j \leqslant k-1$:

$$f_{ij}(\mathbf{x}) = \frac{1}{k}(\mathbf{x}+(i,j))+(a_{ij}, b_{ij}),$$

where a_{ij} and b_{ij} are integers. In this case

$$A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

and the digit set is $D = \{(i+ka_{ij}, j+kb_{ij}) \mid 0 \leqslant i, j \leqslant k-1\}$.

Let T_0 be the unit square in $\mathbb{R}^2$ with vertices at $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, and let $T_n = F^{(n)}(T_0)$. The set T_1 may be constructed by dividing T_0 into k^2 smaller squares by

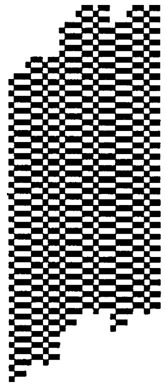

FIGURE 1. *The limit tile has measure* 2.

horizontal and vertical lines in the obvious way, and then translating each smaller square an integral distance a_{ij} in the horizontal direction and an integral distance b_{ij} in the vertical direction. One can rescale and repeat the same construction for each daughter square. By repeating for $n = 2, 3, \ldots$ one gets the sequence T_n. The Lebesgue measure $m(T_n) = 1$ for each n, and, moreover, $\mathscr{T}_n = \{a + T_n \mid a \in \mathbb{Z}^2\}$ is a tiling of the plane by translation by the integer lattice.

The self-similar digit tile T that is the attractor of this iterated function system is, according to equation (1.1), the limit of the T_n in the Hausdorff metric. However T may not have Lebesgue measure 1 and may not tile the plane by translation by the integer lattice. As an example, consider the case where $k = 3$ and

$$D = \{(0,0), (0,2), (0,4), (1,1), (1,3), (1,5), (2,1), (2,3), (2,5)\}.$$

The third tile in the sequence T_n is shown in Figure 1. The limit tile T has area 2 and, consequently, does not tile the plane by translation by the integer lattice. Also note that, in this example, the boundary ∂T_n does not approach ∂T in the Hausdorff metric as $n \to \infty$. In fact, $\lim_{n\to\infty} \partial T_n = T$ and is space filling.

The main result of this paper is an effective method for computing the Hausdorff dimension of the boundary of a self-similar digit tile in Euclidean d-dimensional space. It states that, under a certain natural condition, the Hausdorff dimension of the boundary ∂T of a self-similar digit tile $T = T(A, D)$ is given by

$$\dim_H(\partial T) = \frac{\log \lambda}{\log c},$$

where c is the expansion factor of A and λ is the largest eigenvalue of certain matrices first defined by Gröchenig and Haas [**9**]. The exact statements are Theorems 2 and 3. The hypotheses of these theorems insure that situations like those in the last example do not occur. Theorem 1 contains several equivalent conditions for what we call *well-behaved boundary*. In particular, four of these equivalent conditions are that, for the approximating tiles $T_n = F^{(n)}(T_0)$, where T_0 is the unit square centered at the origin with edges parallel to the axes,

(1) $\lim_{n\to\infty} \partial T_n = \partial T$;
(2) $\lim_{n\to\infty} \partial T_n$ is not space filling;
(3) $m(T) = 1$;
(4) $\{T + x \mid x \in \mathbb{Z}^d\}$ is a tiling of $\mathbb{R}^d$.

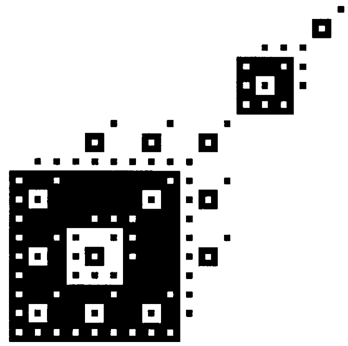

FIGURE 2. *A Sierpinski tile.*

The limits are with respect to the Hausdorff metric. Our methods are not, in general, equipped to compute the dimension of the boundary when $\lim \partial T_n \neq \partial T$, as is the case for the tile in Figure 1. However, certain 'nonprimitive' cases for which the conditions above fail can be reduced, as explained in §4, to cases for which the conditions hold.

The whole calculation of the Hausdorff dimension of the boundary ∂T of the digit tile T depends only on the expansion matrix A and the set of digits D. A computer program was written with input A and D and output λ so that, under any of the above four conditions, $\dim_H(\partial T) = \log \lambda / \log c$. In this case $\dim_H(\partial T) < d$; in other words $\lambda < c^d = |\det A|$. The algorithm yields a value λ even for the situation where conditions (1)–(4) fail. In this case Theorem 3 shows that the output must be $\lambda = |\det A|$, so, in fact, the algorithm detects well-behaved boundary. If $\lambda < |\det A|$, then conditions (1)–(4) hold. If $\lambda = |\det A|$, then they do not and $\dim_H \partial T$ is not determined by the method. The equivalence of conditions (3) and (4) above was proved by Gröchenig and Haas [**9**, Proposition 4.1]. A different algorithmic condition equivalent to conditions (1)–(4) is given by Vince [**21**, Theorem 4].

As a simple example of a tile where our methods apply, consider the following modification of the Sierpinski carpet. Let

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

and let the digit set be $D = \{(0,0), (1,0), (2,0), (0,1), (4,4), (2,1), (0,2), (1,2), (2,2)\}$. Figure 2 plots T_4 to approximate the limit tile. Our methods compute the dimension of the boundary of this tile:

$$\dim_H \partial T = \frac{\log(3 + 2\sqrt{2})}{\log 3} = 1.604\,522\ldots.$$

Several additional examples will be computed in detail in §5.

2. *Well-behaved boundary*

THEOREM 1. *Let $T = T(A, D)$ be a self-similar digit tile. If $T_n = F^{(n)}(T_0)$, where T_0 is the unit square centered at the origin with edges parallel to the axes, are approximating tiles, then the following statements are equivalent.*

(1) $\lim_{n\to\infty} \partial T_n = \partial T$.
(2) $\lim_{n\to\infty} \partial T_n$ *is not space filling.*
(3) $\{T+x \mid x\in\mathbb{Z}^d\}$ *is a tiling of* $\mathbb{R}^d$.
(4) $m(T) = 1$.

Moreover, if these conditions hold and c is the expansion factor of A, then there is a positive constant a such that

$$d(\partial T, \partial T_n) < a/c^n,$$

where d denotes the Hausdorff metric.

Proof. Various parts of the theorem appear in the literature. In particular, a proof of the equivalence of the four conditions for the two-dimensional case appears in Vince [**22**, Theorem 3]. The arguments in that paper generalize without change to the d-dimensional case. We will prove the last statement in Theorem 1. First note that there is a constant a such that $d(T, T_n) \leqslant a/c^n$. This can be proved by induction; just note that

$$d(T, T_n) = d(F(T), F(T_n)) \leqslant \frac{1}{c} d(T, T_{n-1}). \tag{2.1}$$

To show that $d(\partial T, \partial T_n) \leqslant a/c^n$, we first show that each point $x\in\partial T_n$ is at distance at most a/c^n from some point of ∂T. Consider three cases: x is in the interior of T, $x\in\partial T$, and $x\notin T$. If $x\in\partial T$, then the assertion is trivial. If $x\notin T$, then by (2.1) there is a point $y\in T$ such that $d(x,y)\leqslant a/c^n$. Hence there must be a point $z\in\partial T$ on the segment xy such that $d(x,z)\leqslant d(x,y)\leqslant a/c^n$. If x is in the interior of T, then consider the two tilings of $\mathbb{R}^d$:

$$\mathcal{T}_n = \{p+T_n \mid p\in\mathbb{Z}^d\}$$
$$\mathcal{T} = \{p+T \mid p\in\mathbb{Z}^d\}.$$

That $\mathcal{T}_n$ is a tiling follows easily from the fact that D is a set of coset representatives of $\mathbb{Z}^d/A(\mathbb{Z}^d)$. Since $x\in\partial T_n$, there is another copy $y+T_n$ in the tiling $\mathcal{T}_n$ such that $x\in\partial(y+T_n)$ but $x\notin y+T$. Now (2.1) implies that there is a point $z\in y+T$ such that $d(z,x)\leqslant a/c^n$, and hence a point $w\in\partial T$ on segment zx such that $w\in\partial(y+T)$ and $d(w,x)\leqslant d(z,x)\leqslant a/c^n$.

That each point of ∂T is at distance at most a/c^n from a point of ∂T_n is similarly proved. □

3. *Main result*

Throughout this section A denotes an expansive similitude in $\mathbb{R}^d$ and D a digit set as defined in §1. Let $T := T(A, D)$ denote the self-similar digit tile as constructed from A and D. By the sum of two sets A and B of points in $\mathbb{R}^d$ we always mean the Minkowski sum $A+B=\{a+b \mid a\in A,\ b\in B\}$ and by AX we likewise mean $AX=\{Ax \mid x\in X\}$. Let $\{e_1,\ldots,e_d\}$ denote the canonical basis of $\mathbb{R}^d$. The following lemma appears in [**9**, Lemma 4.5], but our proof is included because it is short.

LEMMA 1. *Let* $N_0=\{0\}\cup\{\pm e_1,\ldots,\pm e_d\}$. *There is a unique smallest finite set* $N\subset\mathbb{Z}^d$ *such that* $N_0\subseteq N$ *and* $D+N\subseteq AN+D$.

The set N of lattice points in Lemma 1 will be called the (A, D)-*neighborhood*, or simply *neighborhood* when A and D are understood. It is clear that the neighborhood can be easily computed using the following algorithm. Because D is a set of coset representatives of $\mathbb{Z}^d/A(\mathbb{Z}^d)$, for any lattice point y the equation $Ax+d=y$ has a unique solution pair (x, d), where $x \in \mathbb{Z}^d$ and $d \in D$. It only remains to show that the algorithm terminates after a finite number of steps.

ALGORITHM 1.

$$N = N_0$$

Repeat until the two sets are equal:

$$N \longleftarrow N \cup \{x \in \mathbb{Z}^d \mid Ax+d = y \text{ for some } d \in D \text{ and } y \in D+N\}.$$

Proof of algorithm termination. Let δ denote the maximum distance from the origin to any point of D. Let N_j denote the set N in the algorithm at the jth iteration and r_j the maximum distance from the origin to any point of N_j. If $c > 1$ is the expansion constant of A, then according to the algorithm $r_{j+1} \leqslant (1/c)(r_j + 2\delta)$. Therefore the sequence $\{r_j\}$ is bounded and so N is finite. □

The contact matrix C will now be introduced. For each $x \in N$ and $d \in D$ let x_d denote the unique solution to $d+x \in Ax_d + D$. Let $k = |N|$ and let C' be the $k \times k$ matrix whose rows and columns are indexed by the elements in N and whose entries are as follows. For $x, y \in N$

$$c_{xy} = |\{d \in D \mid x_d = y\}|.$$

By convention let the first index of C' correspond to the element $0 \in N$. Note that $c_{00} = |D|$ and $c_{0y} = 0$ for $y \neq 0$. Thus the first row of C' consists of all zeros except for one entry. Let C denote the $(k-1) \times (k-1)$ matrix obtained from C' by removing the first row and column. Call C the *contact matrix* for the pair (A, D). (In **[9]** it is actually C' that is referred to as the contact matrix.) In Lemma 2 the following notation is used:

$$D_n := D + AD + \ldots + A^{n-1}D. \tag{3.1}$$

LEMMA 2. *Let x and y be points in the (A, D)-neighborhood. Then the entry c^n_{xy} in the contact matrix C^n counts the number of elements $d \in D_n$ such that $d + x \in A^n y + D_n$.*

Proof. The statement will be proved by induction on n. For $n = 1$ the quantity in question is the number of $d \in D$ such that $d+x \in Ay+D$. In previous notation this is the number of $d \in D$ such that $x_d = y$, which is precisely the definition of the entry of C in position (x, y).

Assume that the statement is true for $n-1$. Now $c^n_{xy} = \sum_{z \in N} c_{xz} c^{n-1}_{zy}$. (Recall that $c_{x0} = c_{0x}$ for all $x \neq 0$.) We first show that (1) for each pair $d_1 \in D$, $d_2 \in D_{n-1}$ with $d_1 + x \in Az + D$ and $d_2 + z \in A^{n-1}y + D_{n-1}$, the element $d = Ad_2 + d_1 \in D_n$ satisfies $x + d \in A^n y + D_n$. Second we show that (2) each $d \in D_n$ such that $d + x \in A^n y + D_n$ is of the form $d = Ad_2 + d_1$, where $d_1 \in D$, $d_2 \in D_{n-1}$, $d_1 + x \in Az + D$ and $d_2 + z \in A^{n-1}y + D_{n-1}$. This will complete the proof.

Concerning (1), we have $d_1 + Ad_2 + x = Az + d' + Ad_2 = A^n y + Ad'' + d'$, where $d' \in D$ and $d'' \in D_{n-1}$. Since $Ad'' + d' \in D_n$ we have $d + x = d_1 + Ad_2 + x \in A^n y + D_n$.

Concerning (2), assume that $d+x\in A^n y+D_n$, where $d=d_1+Ad_2$ and $d_1\in D$ and $d_2\in D_{n-1}$ are uniquely determined. By Lemma 1 we have $d_1+x=Az+d'$, for some uniquely determined $z\in N$ and $d'\in D$. Then $A(d_2+z)+d'=x+d_1+Ad_2=x+d\in A^n y+D_n=A(A^{n-1}y+D_{n-1})+D$. By uniqueness of the representation $d_2+z\in A^{n-1}y+D_{n-1}$. □

According to the Perron–Frobenius theorem for non-negative matrices, C has a real eigenvalue λ such that for any other eigenvalue μ we have $\lambda \geqslant |\mu|$. In other words, the spectral radius of C is an eigenvalue.

THEOREM 2. *Let $T=T(A,D)$ be a self-similar digit tile where A has expansion factor c and the contact matrix C has largest eigenvalue λ. Under any of the conditions in Theorem 1 we have*

$$\dim_H(\partial T)=\frac{\log\lambda}{\log c}.$$

Proof. Let T_0 be the unit cube centered at the origin with edges parallel to the axes and let $T_n=F^n(T_0)$ be the nth approximation to the self-similar digit tile T as in equation (1.1). In the case of a self-similar digit tile the iterated function system is given by equation (1.2), which implies that

$$T_n=\bigcup\{A^{-n}(T_0+d_0+Ad_1+\ldots+A^{n-1}d_{n-1})\mid d_i\in D\}$$

and $T=\lim_{n\to\infty}T_n$. Note that T_n is the non-overlapping union of copies of $A^{-n}(T_0)$, each copy being a cube of edge length $1/c^n$. Under the mapping A^n there is a bijection between this set of cubes of T_n and the set of lattice points D_n as defined in (3.1).

For the given pair (A,D), let $N'=N(A,D)\backslash\{0\}$. For any matrix M, let $|M|$ denote the sum of the entries of M. By Lemma 2, $|C^n|$ counts the number of triples (x,y,d) that are solutions to the equation $d+x\in A^n y+D_n$, where $x,y\in N'$ and $d\in D_n$. Let B_n denote the set of $d\in D_n$ such that $d+x\in A^n y+D_n$ for some $x,y\in N'$, and let β_n denote the cardinality of B_n. Thus

$$\beta_n\leqslant|C^n|\leqslant(k-1)^2\beta_n, \tag{3.2}$$

where k is the cardinality of N. Under the bijection in the first paragraph of this proof, B_n also corresponds to a certain set of cubes in T_n. By abuse of language we also refer to this set of cubes as B_n.

By Lemma 1 we have $D+N\subseteq AN+D$. By a straightforward induction it is also true that $D_n+N\subseteq A^nN+D_n$ for $n=1,2,\ldots$. Also recall that $\{\pm e_1,\ldots,\pm e_d\}\in N$. Let $b=b(A,D)$ denote the largest Euclidean distance from the origin to any point in the neighborhood $N(A,D)$. By the comments above and the last statement in Theorem 1, the center of each cube in B_n has distance at most $(a+b)/c^n$ from some point on ∂T, and each point of ∂T has distance at most $(a+b)/c^n$ from a center of such a cube.

Consider the following tiling of $\mathbb{R}^d$ by cubes of edge length $1/c^n$:

$$\{x+A^{-n}(T_0)\mid x\in A^{-n}(\mathbb{Z}^d)\}.$$

The number of such tiles of edge length $1/c^n$ within distance $(a+b)/c^n$ of, say, the origin is bounded by a constant h that depends only on the dimension d, not on n.

Let α_n be the smallest number of tiles of edge length $1/c^n$ whose union covers ∂T. Thus $\beta_n \leqslant h\alpha_n$ and $\alpha_n \leqslant h\beta_n$. Moreover, by (3.2) and the paragraph above, there are positive constants a' and b' such that

$$a'|C^n| \leqslant \alpha_n \leqslant b'|C^n|. \tag{3.3}$$

By a standard result for non-negative matrices [**3**, Lemma 12, p. 198] we have $\lim_{n\to\infty}(|C^n|)^{1/n} = \lambda$ which implies that

$$\lim_{n\to\infty} \frac{1}{n}\log|C^n| = \log\lambda. \tag{3.4}$$

Now ∂T is a *sub-self-similar set* in the sense of Falconer [**5**, §2f]. By a result in [**8**, Theorem 3.5], the box counting dimension and the Hausdorff dimension coincide for ∂T. Let $\underline{\dim}_B\,\partial T$ and $\overline{\dim}_B\,\partial T$ denote the upper and lower box counting dimensions of ∂T, respectively. Then by [**8**, Theorem 3.5], $\underline{\dim}_B\,\partial T = \overline{\dim}_B\,\partial T = \dim_H \partial T$. Thus

$$\dim_H \partial T = \dim_B \partial T = \lim_{n\to\infty} \frac{\log(\alpha_n)}{\log c^n}; \tag{3.5}$$

in particular the above limit exists. Together (3.3), (3.4) and (3.5) yield

$$\dim_H(\partial T) = \lim_{n\to\infty} \frac{\log \alpha_n}{n \log c} = \lim_{n\to\infty} \frac{\log|C^n|}{n\log c} = \frac{\log\lambda}{\log c}. \qquad \square$$

4. *Testing for well-behaved boundary*

THEOREM 3. *Let $T = T(A, D)$ be a self-similar digit tile in $\mathbb{R}^d$, and let λ be the largest eigenvalue of the contact matrix C. Then the four conditions in Theorem 1 are equivalent to*

$$\lambda < |\det A|.$$

Proof. Assume the conditions in Theorem 1 are satisfied. Then by Theorem 2 we have $\dim_H(\partial T) = \log\lambda/\log c$ and by [**11**, Theorem 2.1] we have $d > \dim_H(\partial T)$. Therefore $|\det A| = c^d > \lambda$, the equality implied by standard results in linear algebra.

Conversely, assume that the conditions in Theorem 1 fail. Then there exists a point $x \in \mathbb{Z}^d$ such that distinct tiles T and $x + T$ overlap. As in the proof of the last statement in Theorem 1, there is a constant a such that $d(T, T_n) \leqslant a/c^n$. Likewise $d(x+T, x+T_n) \leqslant a/c^n$. (Note that the hypotheses of Theorem 1 were not used to prove this result.) We will first show that any point in $T \cap (x+T)$ is within distance a/c^n of some point in ∂T_n. If $y \in T \cap (x+T)$ there are points $z \in T_n$ and $w \in x + T_n$ such that $d(y, z) \leqslant a/c^n$ and $d(y, z) \leqslant a/c^n$. Since $T_n \cap (x + T_n) = \varnothing$, there is a point $u \in \partial T_n$ lying on the segment zw such that $d(y, u) \leqslant a/c^n$.

The number of small cubes in T_n equals $|\det A|^n$. Let β_n denote the number of cubes of T_n contained in $T \cap (x+T)$ and γ_n the number of cubes of T_n in $T \cap (x+T)$ that intersect ∂T_n. Then there are constants a_1 and a_2 such that

$$|\det A|^n \leqslant a_1\beta_n \leqslant a_2\gamma_n \leqslant a_2|C^n|,$$

where C is the contact matrix. The last inequality is a consequence of Lemma 2, the bijection between the points of D_n and the cubes in T_n described in the proof of

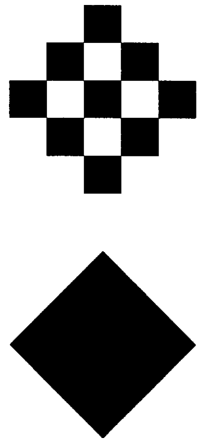

FIGURE 3. *Non-primitive pair: digits and limit tile.*

Theorem 2, and the fact that the points $\{\pm e_1, \ldots, \pm e_d\}$ are in the (A, D)-neighborhood. By equation (3.4)

$$|\det A| \leqslant \lim_{n\to\infty} |C^n|^{1/n} = \lambda. \qquad \square$$

REMARK 1. There are certain digit tiles which do not satisfy the conditions in Theorem 1, but for which our method can, nevertheless, be made to work. A pair (A, D) is called *primitive* if D is contained in no proper A-invariant sublattice of $\mathbb{Z}^d$. By A-invariant we mean that $A(L) \subset L$. An example of a non-primitive pair is

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

with digit set

$$D = \{(0,0), (1,-1), (2,0), (-1,1), (1,1), (3,1), (0,2), (1,3), (2,2)\}$$

because D is contained in the A-invariant sublattice generated by $(1,1)$ and $(-1,1)$.

The limit tile, shown in Figure 3, is a square with side of length $\sqrt{2}$. Thus $m(T) = 2$ and, according to Theorems 1 and 2, our method of computing the dimension of the boundary fails. However, by [**16**, Lemma 2.1], there is an easily computable matrix $\tilde{A}$ and digit set $\tilde{D}$ such that $(\tilde{A}, \tilde{D})$ is primitive and

$$T(\tilde{A}, \tilde{D}) = g(T(A, D))$$

where g is an invertible affine map. Since Hausdorff dimension is preserved by bi-Lipschitz maps,

$$\dim_H \partial T(\tilde{A}, \tilde{D}) = \dim_H \partial T(A, D).$$

Our method may well apply to $\tilde{T} = T(\tilde{A}, \tilde{D})$ although it fails for $T = T(A, D)$. In the above example, for instance,

$$\tilde{A} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

and

$$\tilde{D} = \{(0,0), (1,0), (2,0), (0,1), (1,1), (2,1), (0,2), (1,2), (2,2)\}.$$

Theorem 3 applies and gives $\dim_H \partial \tilde{T} = 1$, the measure of the unit square in Figure 3.

5. *Some examples*

5.1. *Twin dragon*

The twin dragon is a well-known and well-studied example. The dimension of the boundary is also known and has been computed by various means. By our method we start with the expansion similitude

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and the digit set $D = \{(0,0),(1,0)\}$. The expansion factor of A is $\sqrt{2}$. An approximation of the twin dragon is given in Figure 4.

The neighborhood N, computed using the algorithm in §3, is the following set of lattice points:

$$N = \{(0,0),(0,1),(1,0),(1,-1),(0,-1),(-1,0),(-1,1)\}.$$

Ordering the elements of $N \backslash \{0\}$ as above (clockwise around a hexagon) the contact matrix C, computed using the definition, is the following integer matrix with cyclical structure:

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial is easy to compute because of the near diagonal structure of the matrix:

$$\det(C - \lambda I) = \lambda^4(1-\lambda)^2 - 4 = (\lambda+1)(\lambda^2 - 2\lambda + 2)(\lambda^3 - \lambda^2 - 2).$$

Thus the largest eigenvalue of C is the real root of $\lambda^3 - \lambda^2 - 2$. One can easily verify that $\lambda < 2 = \det A$. Theorem 3 implies that the formula in Theorem 2 successfully computes the Hausdorff dimension of the boundary of the twin dragon:

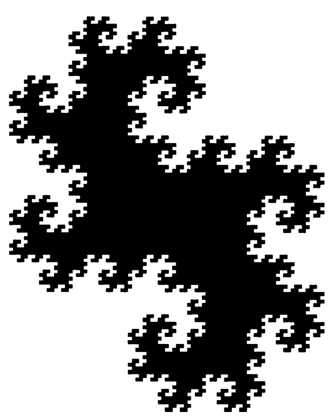

FIGURE 4. *The twin dragon.*

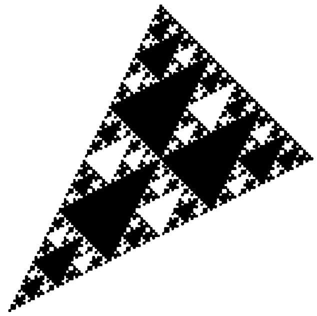

FIGURE 5. *The gasket.*

$$\dim_H \partial K = \frac{\log\left(\frac{\sqrt[3]{3\cdot\sqrt{87}+28}}{3} + \frac{1}{3\cdot\sqrt[3]{3\cdot\sqrt{87}+28}} + \frac{1}{3}\right)}{\log\sqrt{2}} = 1.523\,627\ldots$$

5.2. *Gasket*

For the example we call the *gasket* the matrix is

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and the digit set is $D = \{(0,0),(1,0),(0,1),(-1,-1)\}$. An approximation to the gasket is given in Figure 5.

The neighborhood N is again in a hexagonal pattern:

$$N = \{(0,0),(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1)\}.$$

The contact matrix is a cyclic matrix with three 1s in each row:

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Hence the Perron–Frobenius eigenvector, the unique eigenvector with positive entries, is the all-1s vector. The corresponding eigenvalue is $\lambda = 3$.

$$\dim_H \partial K = \frac{\log 3}{\log 2} = 1.584\,962\,5\ldots$$

5.3. *Rocket and lander*

We conclude with two examples, shown in Figures 6 and 7, having successively greater Hausdorff dimension.

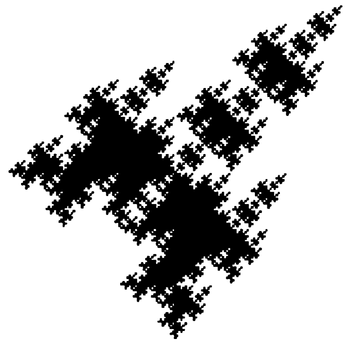

FIGURE 6. *The rocket.*

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$D = \{(0,0),(1,1),(2,2),(-1,0),(-2,0),(-1,1),(0,-1),(0,-2),(1,-1)\}$$

$$\dim_H \partial K = \frac{\log(3+2\cdot\sqrt{2})}{\log 3} = 1.604\,522\ldots$$

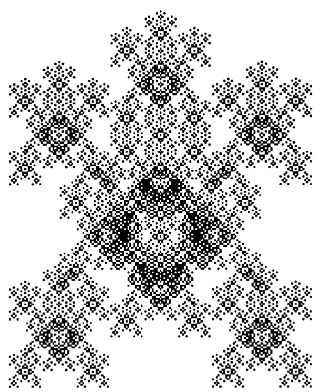

FIGURE 7. *The lander.*

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$D = \{(0,0),(1,1),(1,-1),(2,0),(-1,-2),(3,-2),(-1,2),(3,2),(1,3)\}$$

$$\dim_H \partial K = 1.913\,624\ldots$$

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