Linear Algebra and its Applications 435 (2011) 3124–3138

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

# The eigenvalue problem for linear and affine iterated function systems ☆

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ARTICLE INFO

*Article history:*
Received 23 April 2010
Accepted 21 May 2011
Available online 21 June 2011

Submitted by R.A. Brualdi

*AMS classification:*
15A18
28A80

*Keywords:*
Eigenvalue problem
Iterated function system
Joint spectral radius

ABSTRACT

The eigenvalue problem for a linear function $L$ centers on solving the eigen-equation $Lx = \lambda x$. This paper generalizes the eigenvalue problem from a single linear function to an iterated function system $F$ consisting of possibly an infinite number of linear or affine functions. The eigen-equation becomes $F(X) = \lambda X$, where $\lambda > 0$ is real, $X$ is a compact set, and $F(X) = \bigcup_{f \in F} f(X)$. The main result is that an irreducible, linear iterated function system $F$ has a unique eigenvalue $\lambda$ equal to the joint spectral radius of the functions in $F$ and a corresponding eigenset $S$ that is centrally symmetric, star-shaped, and full dimensional. Results of Barabanov and of Dranisnikov–Konyagin–Protasov on the joint spectral radius follow as corollaries.



## 1. Introduction

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with no nontrivial invariant subspace, equivalently no real eigenvalue. We use the notation $L(X) := \{Lx : x \in X\}$. Although $L$ has no real eigenvalue, $L$ does have an eigen-ellipse. By *eigen-ellipse* we mean an ellipse $E$, centered at the origin, such that $L(E) = \lambda E$, for some real $\lambda > 0$. An example of an eigen-ellipse appears in Example 1 of Section 2 and in Fig. 1. Although easy to prove, the existence of an eigen-ellipse is not universally known.

**Theorem 1.** *If $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map with no real eigenvalue, then there is an ellipse $E$ and a real number $\lambda > 0$ such that $L(E) = \lambda E$.*

☆ Supported in part by ARC Discovery Grant DP0558974 and ARC Discovery Grant DP0984353.
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doi:10.1016/j.laa.2011.05.011

**Proof.** Using the real Jordan canonical form for $L$, there exists an invertible $2 \times 2$ matrix $S$ such that

$$M := S^{-1}LS = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for some angle $\theta$ and $\lambda > 0$. If $D$ is the unit disk centered at the origin and if $E = S(D)$, then

$$L(E) = SMS^{-1}(E) = SM(D) = \lambda S(D) = \lambda E.$$

Note that the eigenvalues of $L$ are $a \pm b\,i$. Therefore if $z = a + b\,i$, then the relationship between the eigenvalues of $L$ and the rotation angle $\theta$ and stretching factor $\lambda$ is: $\theta = arg(z)$ and $\lambda = |z|$. $\square$

The intent of this paper is to investigate the existence of eigenvalues and corresponding eigensets in the context of fractal geometry.

**Definition 1** (*iterated function system*)**.** If $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in I$, are continuous mappings, then $F = (\mathbb{R}^n; f_i,\ i \in I)$ is called an **iterated function system** (IFS). The set $I$ is the index set. Call IFS $F$ **linear** if each $f \in F$ is a linear map and **affine** if each $f \in F$ is an affine map.

In the literature the index set $I$ is usually finite. This is because, in constructing deterministic fractals, it is not practical to use an infinite set of functions. We will, however, allow an infinite set of functions in order to obtain certain results on the joint spectral radius. In the case of an infinite linear IFS $F$ we will always assume that the set of functions in $F$ is compact in the compact open topology. For linear maps, this just means, regarding each linear map as an $n \times n$ matrix, that the set $F$ of linear maps is a compact subset of $\mathbb{R}^{n\times n}$.

Let $\mathbb{H} = \mathbb{H}(\mathbb{R}^n)$ denote the collection of all nonempty compact subsets of $\mathbb{R}^n$, and, by slightly abusing the notation, let $F : \mathbb{H} \to \mathbb{H}$ also denote the function defined by

$$F(B) = \bigcup_{f\in F} f(B).$$

Note that, if $B$ is compact and $F$ is compact, then $F(B)$ is also compact. Let $F^k$ denote $F$ iterated $k$ times with $F^0(B) = B$ for all $B$. Our intention is to investigate solutions to the eigen-equation

$$F(X) = \lambda X, \tag{1}$$

where $\lambda \in \mathbb{R},\ \lambda > 0$, and $X \neq \{0\}$ is a compact set in Euclidean space.

**Definition 2** (*eigenvalue-eigenset*)**.** The value $\lambda$ in Eq. (1) above will be called an **eigenvalue** of $F$, and $X$ a corresponding **eigenset**.

When $F$ consists of a single linear map on $\mathbb{R}^2$, the eigen-ellipse is an example of an eigenset. Section 2 contains other examples of eigenvalues and eigensets of linear IFSs. Section 3 contains background results on the joint spectral radius of a set of linear maps and on contractive IFSs. Both of these topics are germane to the investigation of the IFS eigenvalue problem. Section 4 contains the main result on the eigenvalue problem for a linear IFS.

**Theorem 2.** *A compact, irreducible, linear IFS $F$ has exactly one eigenvalue which is equal to the joint spectral radius $\rho(F)$ of $F$. There is a corresponding eigenset that is centrally symmetric, star-shaped, and full dimensional.*

If $F = (\mathbb{R}^n;\ f_i,\ i \in I)$ is an IFS, let $F_\lambda := \frac{1}{\lambda} F = (\mathbb{R}^n;\ \frac{1}{\lambda} f_i,\ i \in I)$. Another way to view the above theorem is to consider the family $\{F_\lambda : \lambda > 0\}$ of IFSs. If $\lambda > \rho(F)$, then the attractor of $F_\lambda$, defined

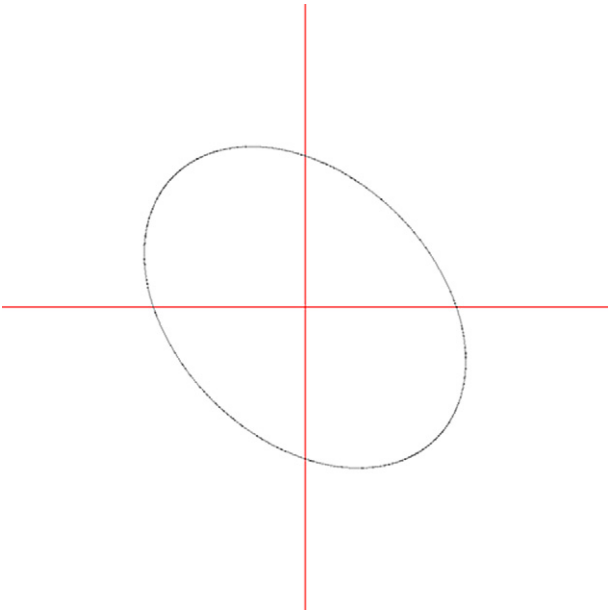

**Fig. 1**. The eigen-ellipse in Example 1.

formally in the next section, is the trivial set $\{0\}$. If $\lambda < \rho(F)$, then $F_\lambda$ has no attractor. So $\lambda = \rho(F)$ can be considered as a "phase transition", at which point a somewhat surprising phenomenon occurs – the emergence of the centrally symmetric, star-shaped eigenset.

Theorems of Dranisnikov–Konyagin–Protasov and of Barabanov follow as corollaries of Theorem 2. These results are discussed in Section 5. Because some of the material on joint spectral radius may be unfamiliar to those whose background is mainly IFS theory, and because it does not take much extra effort, we prove Theorem 2 from scratch. It would be shorter, but perhaps not necessarily more illuminating, to give a proof assuming the Barabanov result.

No such transition phenomenon occurs in the case of an affine, but not linear, IFS. A result for the affine case is the following, whose proof appears in Section 6.

**Theorem 3.** *For a compact, irreducible, affine, but not linear, IFS $F$, a real number $\lambda > 0$ is an eigenvalue if $\lambda > \rho(F)$ and is not an eigenvalue if $\lambda < \rho(F)$. There are examples where $\rho(F)$ is an eigenvalue and examples where it is not.*

The transition phenomenon resurfaces in the context of projective IFSs, which will be the subject of a subsequent paper.

## 2. Examples

**Example 1.** Fig. 1 shows the eigen-ellipse for the IFS $F = (\mathbb{R}^2; L)$, where

$$L = \begin{pmatrix} 0 & -3 \\ 3 & 2 \end{pmatrix}.$$

The eigenvalue is 3.

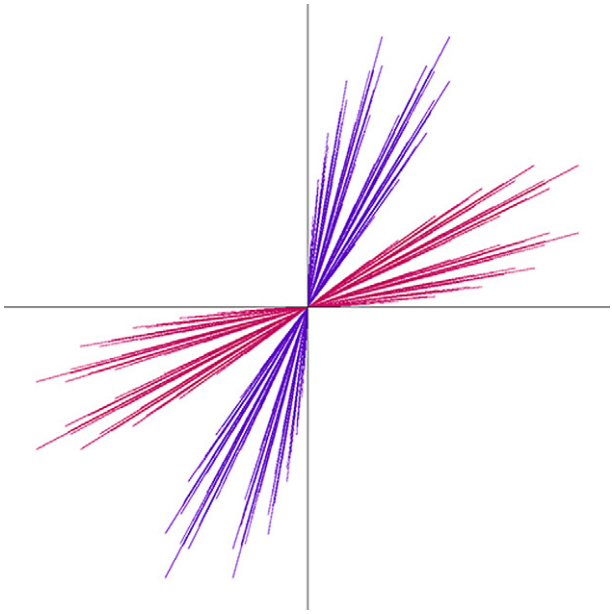

**Fig. 2**. An eigenset of Example 2.

**Example 2.** Fig. 2 shows an eigenset for the IFS $F = (\mathbb{R}^2; L_1, L_2)$, where

$$L_1 = \begin{pmatrix} 10 & 10 \\ 8 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 8 & 0 \\ 10 & 10 \end{pmatrix}.$$

The eigenvalue appears to be $5 + \sqrt{105}$, the value of the largest eigenvalue of $L_1$. The part of the set shown in red is, to viewing accuracy, the image of the whole set under $L_1$. The part of the set shown in blue is, similarly, the image of the whole set under $L_2$. The coordinate axes are indicated in black. (Colors appear in the online version.)

**Example 3.** Fig. 3 shows an eigenset for the IFS $F = (\mathbb{R}^2; L_1, L_2)$, where

$$L_1 = \begin{pmatrix} 0.02 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0.0594 & -1.98 \\ 0.495 & 0.01547 \end{pmatrix}.$$

The eigenvalue is 1, which will be proved in the next section after the joint spectral radius is introduced. The coordinate axes are indicated in red. (Colors appear in the online version.)

## 3. Background

This section concerns the following three basic notions: (1) the joint spectral radius of an IFS, (2) contractive properties of an IFS, and (3) the attractor of an IFS. Theorems 1 and 4 provides the relationship between these three notions for a linear and an affine IFS, respectively.

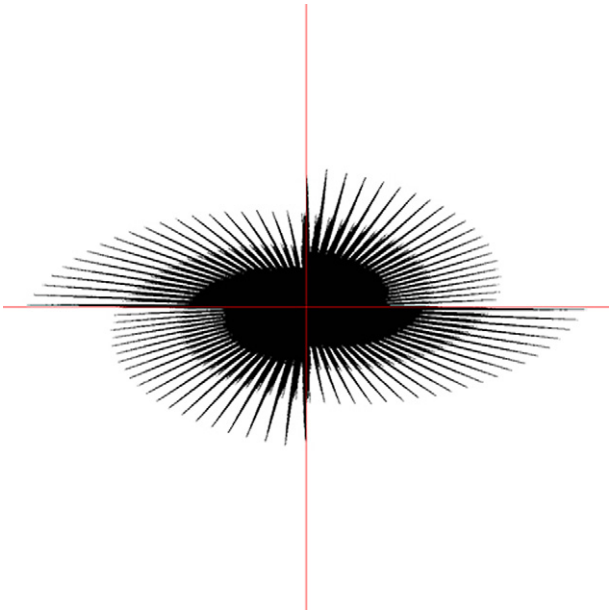

**Fig. 3**. An eigenset of Example 3.

### 3.1. Norms and metrics

Any vector norm $\|\cdot\|$ on $\mathbb{R}^n$ induces a matrix norm on the space of linear maps taking $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$\|L\| = \max\left\{\frac{\|Lx\|}{\|x\|} \ : \ x \in \mathbb{R}^n\right\}.$$

Since it is usually clear from the context, we use the same notation for the vector norm as for the matrix norm. This induced norm is *sub-multiplicative*, i.e., $\|L \circ L'\| \leqslant \|L\| \cdot \|L'\|$ for any linear maps $L, L'$.

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent* if there are positive constants $a, b$ such that $a\|x\|_1 \leqslant \|x\|_2 \leqslant b\|x\|_1$ for all $x \in \mathbb{R}^n$. Two metrics $d_1(\cdot,\cdot)$ and $d_2(\cdot,\cdot)$ are *equivalent* if there exist positive constants $a, b$ such that $a\, d_1(x, y) \leqslant d_2(x, y) \leqslant b\, d_1(x, y)$ for all $x, y \in \mathbb{R}^n$. It is well known that any two norms on $\mathbb{R}^n$ are equivalent [9]. This implies that any two $n \times n$ matrix norms are equivalent. Any norm $\|\cdot\|$ on $\mathbb{R}^n$ induces a metric $d(x, y) = \|x - y\|$. Therefore any two metrics induced from two norms are equivalent.

A set $B \subset \mathbb{R}^n$ is called *centrally symmetric* if $-x \in B$ whenever $x \in B$. A *convex body* in $\mathbb{R}^n$ is a convex set with nonempty interior. If $C$ is a centrally symmetric convex body, define the *Minkowski functional* with respect to $C$ by

$$\|x\|_C = \inf\{\mu \geqslant 0 \ : \ x \in \mu C\}.$$

The following result is well known.

**Lemma 1.** *The Minkowski functional is a norm on $\mathbb{R}^n$. Conversely, any norm $\|\cdot\|$ on $\mathbb{R}^n$ is the Minkowski functional with respect to the closed unit ball $\{x \ : \ \|x\| \leqslant 1\}$.*

Given a metric $d(\cdot,\cdot)$, there is a corresponding metric $d_{\mathbb{H}}$, called the *Hausdorff metric*, on the collection $\mathbb{H}(\mathbb{R}^n)$ of all non-empty compact subsets of $\mathbb{R}^n$:

$$d_{\mathbb{H}}(B,C) = \max\left\{\sup_{b\in B}\inf_{c\in C} d(b,c),\ \sup_{c\in C}\inf_{b\in B} d(b,c)\right\}.$$

### 3.2. Joint spectral radius

The joint spectral radius of a set $\mathbb{L} = \{L_i,\ i \in I\}$ of linear maps was introduced by Rota and Strang [12] and the generalized spectral radius by Daubechies and Lagarias [6,7]. Berger and Wang [3] proved that the two concepts coincide for bounded sets of linear maps. The concept has received much attention in the recent research literature; see for example the bibliographies of [13,14]; we note in particular [4,10,5,15]. What follows is the definition of the joint spectral radius of $\mathbb{L}$. Let $\Omega_k$ be the set of all words $i_1\, i_2\, \cdots\, i_k$, of length $k$, where $i_j \in I,\ 1 \leqslant j \leqslant k$. For $\sigma = i_1\, i_2\, \cdots\, i_k \in \Omega_k$, define

$$L_\sigma := L_{i_1} \circ L_{i_2} \circ \cdots \circ L_{i_k}.$$

A set of linear maps is *bounded* if there is an upper bound on their norms. Note that if $\mathbb{L}$ is compact, then $\mathbb{L}$ is bounded. For a linear map $L$, let $\rho(L)$ denote the ordinary spectral radius, i.e., the maximum of the moduli of the eigenvalues of $L$.

**Definition 3.** For any set $\mathbb{L}$ of linear maps and any norm, the **joint spectral radius** of $\mathbb{L}$ is

$$\hat{\rho} = \hat{\rho}(\mathbb{L}) := \limsup_{k\to\infty} \hat{\rho}_k^{1/k} \quad \text{where } \hat{\rho}_k := \sup_{\sigma\in\Omega_k} \|L_\sigma\|.$$

The **generalized spectral radius** of $\mathbb{L}$ is

$$\rho = \rho(\mathbb{L}) := \limsup_{k\to\infty} \rho_k^{1/k} \quad \text{where } \rho_k := \sup_{\sigma\in\Omega_k} \rho(L_\sigma).$$

The following are well known properties of the joint and generalized spectral radius:

1. The joint spectral radius is independent of the particular norm.
2. For an IFS consisting of a single linear map $L$, the generalized spectral radius is the ordinary spectral radius of $L$.
3. For any real $\alpha > 0$ we have $\rho(\alpha\,\mathbb{L}) = \alpha\,\rho(\mathbb{L})$ and $\hat{\rho}\,(\alpha\,\mathbb{L}) = \alpha\,\hat{\rho}(\mathbb{L})$.
4. For any sub-multiplicative norm used to define $\hat{\rho}$ and for all $k \geqslant 1$ we have

$$\rho_k^{1/k} \leqslant \rho \leqslant \hat{\rho} \leqslant \hat{\rho}_k^{1/k}.$$

5. If $\mathbb{L}$ is bounded, then the joint and generalized spectral radius are equal.

From here on we always assume that $\mathbb{L}$ is bounded. So, in view of Property 5, we denote by $\rho(\mathbb{L})$ the common value of the joint and generalized spectral radius.

**Example 3** (*continued*)**.** Assuming Theorem 2, the eigenvalue of $F$ equals the joint spectral radius $\rho(F)$. We will prove that $\rho(F) = 1$. First $\rho(F) \geqslant 1$ because $L_1$ already has eigenvalue 1. To show that $\rho(F) \leqslant 1$, we will find a norm with respect to which $\hat{\rho}_k \leqslant 1$ for all $k \geqslant 1$. Consider the convex hull $C$ in $\mathbb{R}^2$ of the points $\pm e_2, \pm L_2 e_2, \pm L_2^2 e_2, \ldots$. Since it is easy to check that the sequence $\{L_2^k\}$ converges to 0, there is a $K$ such that $C$ is the convex hull (a polygon) of the points $\pm e_2, \pm L_2 e_2, \pm L_2^2 e_2, \ldots, \pm L_2^K$. Clearly both $L_1$ and $L_2$ take $C$ into $C$. Therefore $\hat{\rho}_k \leqslant 1$ with respect to the Minkowski norm $\|\cdot\|_C$.

If $F$ is an affine IFS, then each $f \in F$ is of the form $f(x) = Lx + a$, where $L$ is the *linear part* and $a$ is the *translational part*. Let $\mathbb{L}_F$ denote the set of linear parts of $F$.

**Definition 4.** The **joint spectral radius of an affine IFS** $F$ is the joint spectral radius of the set $\mathbb{L}_F$ of linear parts of $F$ and is denoted $\rho(F)$.

**Definition 5.** A set $\{L_i,\ i \in I\}$ of linear maps is called **reducible** if these linear maps have a common nontrivial invariant subspace. The set is **irreducible** if it is not reducible. An IFS is **reducible (irreducible)** if the set of linear parts is reducible (irreducible).

A set of linear maps is reducible if and only if there exists an invertible matrix $T$ such that each $L_i$ can be put simultaneously in a block upper-triangular form:

$$T^{-1}L_iT = \begin{pmatrix} A_i & * \\ 0 & B_i \end{pmatrix},$$

with $A_i$ and $B_i$ square, and $*$ is any matrix with suitable dimensions. The joint spectral radius $\rho(F)$ is equal to $\max\left(\rho(\{A_i\}),\ \rho(\{B_i\})\right)$.

### 3.3. A contractive IFS

Basic to the IFS concept is the relationship between the existence of an attractor and the contractive properties of the functions of the IFS. The proofs of Theorems 2 and 3 depend on this relationship, which is given by Theorem 4 and Corollary 1.

**Definition 6** (*contractive IFS*)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a **contraction** with respect to a metric $d$ if there is an $s,\ 0 \leqslant s < 1$, such that $d(f(x), f(y)) \leqslant s\, d(x, y)$ for all $x, y \in \mathbb{R}^n$. An IFS $F = (\mathbb{R}^n;\ f_i,\ i \in I)$ is said to be **contractive** if there is a metric $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$, equivalent to the standard metric on $\mathbb{R}^n$, such that each $f \in F$ is a contraction with respect to $d$.

**Definition 7** (*attractor*)**.** A nonempty compact set $A \subset \mathbb{R}^n$ is said to be an **attractor** of the affine IFS $F$ if

1. $F(A) = A$ and
2. $\lim_{k\to\infty} F^k(B) = A$, for all compact sets $B \subset \mathbb{R}^n$, where the limit is with respect to the Hausdorff metric.

A proof of the equivalence of the first four conditions in the following theorem appears in [1] for a finite IFS. Only the equivalence of condition (5) will be proved – and one modification needed to extend the result from the finite case to the case of a compact IFS. The notation *int(X)* will be used to denote the interior of a subset $X$ of $\mathbb{R}^n$ and the notation $conv(X)$ for the convex hull of $X$.

**Theorem 4.** *If $F = (\mathbb{R}^n; f_i,\ i \in I)$ is a compact, affine IFS, then the following statements are equivalent.*

1. ***[contractive]*** *The IFS $F$ is contractive on $\mathbb{R}^n$.*
2. ***[F-contraction]*** *The map $F : \mathbb{H}(\mathbb{R}^n) \to \mathbb{H}(\mathbb{R}^n)$ defined by $F(B) = \bigcup_{L\in F} L(B)$ is a contraction with respect to a Hausdorff metric.*
3. ***[topological contraction]*** *There exists a compact set $C$ such that $F(C) \subset int(C)$.*
4. ***[attractor]*** *$F$ has a unique attractor, the basin of attraction being $\mathbb{R}^n$.*
5. ***[JSR]*** *$\rho(F) < 1$.*

**Proof.** Concerning the equivalence of statements 1–4, the only modification required in going from the finite to compact case is in the implication $(1 \Rightarrow 2)$. In the case of an IFS $F = (\mathbb{R}^n; f_i,\ i \in I)$, where $I$ is finite (and the $f_i$ are assumed only to be continuous), this is a basic result whose proof can

be found is most texts on fractal geometry, for example [8]. Since $F$ is assumed contractive,

$$\sup\left\{\frac{d(f_i(x), f_i(y))}{d(x,y)} \ : \ x \neq y\right\} = s_i < 1,$$

for each $i \in I$. The only sticking point in extending the proof for the finite IFS case to the infinite IFS case is to show that $\sup\{s_i \ : \ i \in I\} < 1$. But if there is a sequence $\{s_k\}$ such that $\lim_{k\to\infty} s_k = 1$, then, by the compactness of $F$, the limit $f := \lim_{k\to\infty} f_k \in F$. Moreover,

$$\frac{d(f(x), f(y))}{d(x,y)} = \lim_{k\to\infty} \frac{d(f_k(x), f_k(y))}{d(x,y)} = \lim_{k\to\infty} s_k = 1,$$

contradicting the assumption that each function in $F$ is a contraction.

Concerning the equivalence of statement (5) to the other statements, first assume that $F$ is linear. In one direction assume that $F$ is contractive. Hence there is an $0 \leqslant s < 1$ such that $\|Lx\| \leqslant s\,\|x\|$ for all $x \in \mathbb{R}^n$ and all $L \in F$. By Property (4) of the joint spectral radius

$$\rho(F) \leqslant \hat{\rho}_1 = \sup_{L\in F} \frac{\|Lx\|}{\|x\|} \leqslant s < 1.$$

The last inequality is a consequence of the compactness of $F$, the argument identical to the one used above in showing that $(1 \Rightarrow 2)$.

Conversely, assuming

$$\limsup_{k\to\infty} \hat{\rho}_k^{1/k} = \rho(F) < 1,$$

we will show that $F$ has attractor $A = \{0\}$. The inequality above implies that there is an $s$ such that $\hat{\rho}_k{}^{1/k} \leqslant s < 1$ for all but finitely many $k$. In other words

$$\sup_{\sigma\in\Omega_k} \|L_\sigma\| = \hat{\rho}_k \leqslant s^k$$

for all but finitely many $k$. For $k$ sufficiently large, this in turn implies, for any $x \in \mathbb{R}^n$ and any $\sigma \in \Omega_k$, that $\|L_\sigma x\| \leqslant s^k\|x\|$. Therefore, for any compact set $B \subset \mathbb{R}^n$, with respect to the Hausdorff metric, $\lim_{k\to\infty} F^k(B) = \{0\}$. So $\{0\}$ is the attractor of $F$.

For the more general affine case, assuming $\rho(F) < 1$ we show that $F$ is contractive. Let $F'$ be the linear IFS obtained from $F$ by removing the translational component from each function in $F$. By the proof above for the linear case, the IFS $F'$ is contractive. Hence there is a norm $\|\cdot\|$ with respect to which each $L \in F'$ is a contraction. Define a metric by $d(x,y) = \|x-y\|$ for all $x, y \in \mathbb{R}^n$. For any $f(x) = Lx + a \in F$ we have $d(f(x), f(y)) = \|f(x) - f(y)\| = \|(Lx+a) - (Ly+a)\| = \|L(x-y)\|$. Therefore each function $f \in F$ is a contraction with respect to metric $d$.

Conversely, assume that the affine IFS $F$ is contractive. With linear IFS $F'$ as defined above, it is shown in [1, Theorem 6.7] that there is a norm with respect to which each $L \in F'$ is a contraction. It follows from the linear case proved above that $\rho(F) < 1$. $\square$

Note that this last equivalence implies that, if a linear IFS $F$ has an attractor and $F'$ is obtained from $F$ by adding any translational component to each function in $F$, then $F'$ also has an attractor.

**Corollary 1.** *For a compact, linear IFS $F = (\mathbb{R}^n; L_i, \ i \in I)$ the following statements are equivalent:*

1. ***[contractive]*** *There exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ and an $0 \leqslant s < 1$ such that $\|Lx\| \leqslant s\,\|x\|$ for all $L \in F$ and all $x \in \mathbb{R}^n$.*

2. ***[F-contraction]*** *The map* $F : \mathbb{H}(\mathbb{R}^n) \to \mathbb{H}(\mathbb{R}^n)$ *defined by* $F(B) = \bigcup_{L\in F} L(B)$ *is a contraction with respect to a Hausdorff metric.*
3. ***[topological contraction]*** *There is a compact, centrally symmetric, convex body* $C$ *such that* $F(C) \subset int(C)$.
4. ***[attractor]*** *The origin is the unique attractor of* $F$.
5. ***[JSR]*** $\rho(F) < 1$.

**Proof.** According to Theorem 4, any of the statements besides (3) implies that there is a compact set such that $F(C) \subset int(C)$. We must show that $C$ can be chosen to be a symmetric, convex set with nonempty interior.

Let $B$ denote a closed unit ball centered at the origin. Since $\{0\}$ is the attractor (statement 4), $\lim_{k\to\infty} d_{\mathbb{H}}(F^k(B), \{0\}) = 0$, which implies that there is an integer $m$ such that

$$F^m(B) \subset int(B)$$

and hence also

$$conv\, F^m(B) \subset int(B).$$

Consider the Minkowski sum

$$C := \sum_{k=0}^{m-1} conv\, F^k(B).$$

For any $L \in F$

$$\begin{aligned} L(C) &= \sum_{k=0}^{m-1} L\Big(conv\, F^k(B)\Big) = \sum_{k=0}^{m-1} conv\Big(L\left(F^k(B)\right)\Big) \\ &\subseteq \sum_{k=0}^{m-1} conv\, F^{k+1}(B) = conv\, F^m(B) + \sum_{k=1}^{m-1} conv\, F^k(B) \\ &\subseteq int\,(B) + \sum_{k=1}^{m-1} conv\, F^k(B) \\ &= int(C). \end{aligned}$$

The last equality follows from the fact that if $K$ and $K'$ are convex bodies in $\mathbb{R}^n$, then $int(K) + K' = int\,(K + K')$. □

**Corollary 2.** *If a compact, linear IFS* $F$ *is contractive and* $F(A) = A$ *for* $A$ *compact, then* $A = \{0\}$.

**Proof.** According to Corollary 1 the IFS has the $F$-contractive property. According to the Banach fixed point theorem, $F$ has a unique invariant set, i.e., a unique compact $A$ such that $F(A) = A$. Since $F$ is linear, clearly $F(\{0\}) = \{0\}$. □

## 4. The eigenvalue problem for a linear IFS

Just as for eigenvectors of a single linear map, an eigenset of an IFS is defined only up to scalar multiple, i.e., if $X$ is an eigenset, then so is $\alpha X$ for any $\alpha > 0$. Moreover, if $X$ and $X'$ are eigensets corresponding to the same eigenvalue, then $X \cup X'$ is also a corresponding eigenset. For an eigenvalue of a linear IFS, call a corresponding eigenset $X$ *decomposable* if $X = X_1 \cup X_2$, where $X_1$ and $X_2$ are also corresponding eigensets and $X_1 \not\subseteq X_2$ and $X_2 \not\subseteq X_1$. Call eigenset $X$ *indecomposable* if $X$ is not decomposable.

**Example.** It is possible for a linear IFS to have infinitely many indecomposable eigensets corresponding to the same eigenvalue. Consider $F = \{\mathbb{R}^2; L_1, L_2\}$ where

$$L_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

Let

$$\begin{aligned} S(r) = \{ & (\pm 1, \pm r/2^k),\ (\pm 1, \mp r/2^k),\ (\pm r/2^k, 1),\ (\pm r/2^k, \mp 1), \\ & (\pm r, \pm 1/2^k),\ (\pm r, \mp 1/2^k),\ (\pm 1/2^k, r),\ (\pm 1/2^k, \mp r)\ :\ k \geqslant 0\}. \end{aligned}$$

It is easily verified that, for any $1 \geqslant r > 0$, the set $\{\alpha S(r) : 0 \leqslant \alpha \leqslant 1\}$ is an eigenset corresponding to eigenvalue 1. In addition, the unit square with vertices $(1, 1), (1, -1), (-1, 1), (-1, -1)$ is also an eigenset corresponding to eigenvalue 1.

The proof of the following lemma is straightforward. A set $B \subset \mathbb{R}^n$ is called *star shaped* if $\lambda x \in B$ for all $x \in B$ and all $0 \leqslant \lambda \leqslant 1$.

**Lemma 2**

1. *If $\{A_k\}$ is a sequence of centrally symmetric, convex, compact sets and $A$ is a compact set such that $\lim_{k\to\infty} A_k = A$, then $A$ is also centrally symmetric and convex.*
2. *If $F$ is a compact, linear IFS, $B$ a centrally symmetric, convex, compact set and $A = \lim_{k\to\infty} F^k(B)$, then $A$ is a centrally symmetric, star-shaped, compact set.*

**Lemma 3.** *If $F$ is an compact, irreducible, linear IFS with $\rho(F) = 1$, then there exists a compact, centrally symmetric, convex body $A$ such that $F(A) \subseteq A$.*

**Proof.** Since, for each $k \geqslant 2$, we have $\rho((1 - \frac{1}{k})F) = 1 - \frac{1}{k} < 1$, Corollary 1 implies that there is a compact, centrally symmetric, convex body $A_k$ such that

$$\left(1 - \frac{1}{k}\right) F(A_k) \subseteq int(A_k).$$

Since $F$ is linear and the above inclusion is satisfied for $A_k$, it is also satisfied for $\alpha A_k$ for any $\alpha > 0$. So, without loss of generality, it can be assumed that $\max\{\|x\| : x \in A_k\} = 1$ for all $k \geqslant 2$. Since the sequence of sets $\{A_k\}$ is bounded in $\mathbb{H}(\mathbb{R}^n)$, this sequence has an accumulation point, a compact set $A$. Therefore, there is a subsequence $\{A_{k_i}\}$ such that $\lim_{i\to\infty} A_{k_i} = A$ with respect to the Hausdorff metric. Since

$$\left(1 - \frac{1}{k_i}\right) F(A_{k_i}) \subseteq int(A_{k_i}),$$

it is the case that $\left(1 - \frac{1}{k_i}\right) f(A_{k_i}) \subseteq int(A_{k_i})$ for all $f \in F$. From this it is straightforward to show that $f(A) \subseteq A$ for all $f \in F$ and hence that $F(A) \subseteq A$. Moreover, by Lemma 2, since the $A_{k_i}$ are centrally symmetric and convex, so is $A$. Notice also that $A$ is a convex body, i.e., has nonempty interior; otherwise $A$ spans a subspace $E \subset \mathbb{R}^n$ with $\dim E < n$ and $F(A) \subseteq A$ implies $F(E) \subseteq E$, contradicting that $F$ is irreducible. □

The *affine span* $\mathrm{aff}(B)$ of a set $B$ is the smallest affine subspace of $\mathbb{R}^n$ containing $B$. Call a set $B \subset \mathbb{R}^n$ *full dimensional* if $\dim(\mathrm{aff}(B)) = n$. Given an affine IFS $F = (\mathbb{R}^n; f_i,\ i \in I)$ let

$$F_\lambda = \left(\mathbb{R}^n;\ \frac{1}{\lambda} f_i,\ i \in I\right).$$

**Lemma 4.** *If an irreducible, affine IFS F has an eigenset X, then X must be full dimensional.*

**Proof.** Suppose that $F(X) = \lambda X$, i.e. $F_\lambda(X) = X$. For $x \in X$, let $g$ be a translation by $-x$. For the IFS $F$, let $F_g = (\mathbb{R}^n;\ gfg^{-1}, f \in F_\lambda)$. If $Y = g(X)$, then $0 \in Y$ and $F_g(Y) = Y$. In particular, $Y$ is full dimensional if and only if $X$ is full dimensional, and the affine span of $Y$ equals the ordinary (linear) span $E = span(Y)$ of $Y$. Moreover, the linear parts of the affine maps in $F_g$ are just scalar multiples of the linear parts of the affine maps in $F$. Therefore $F_g$ is irreducible if and only if $F$ is irreducible.

Let $f(x) = Lx + a$ be an arbitrary affine map in $F_g$. From $F_g(Y) \subset Y \subset E$ it follows that $L(Y) + a = f(Y) \subset E$. Since $0 \in Y$, also $a = L(0) + a = f(0) \in Y \subset E$. Therefore $L(Y) \subset -a + E = E$. Since $E = span(Y)$, also $L(E) \subset E$. Because this is so for all $f \in F_g$, the subspace $E$ is invariant under all linear parts of maps in $F_g$. Because $F_g$ is irreducible, $dim(E) = n$. Therefore $Y$, and hence $X$, must be full dimensional. □

**Proof of Theorem 2.** Given $F = (\mathbb{R}^n; L_i,\ i \in I)$, consider the family $\{F_\lambda\}$ of IFS's for $\lambda > 0$. Recall that $F_\lambda = \left(\mathbb{R}^n;\ \frac{1}{\lambda} f_i,\ i \in I\right)$.

It is first proved that $F$ has no eigenvalue $\lambda > \rho(F)$. By way of contradiction assume that $\lambda > \rho(F)$, which implies that $\rho(F_\lambda) < 1$. According to Corollary 1 the IFS $F_\lambda$ is contractive. By Corollary 2 the only invariant set of $F_\lambda$ is $\{0\}$, which means that the only solution to the eigen-equation $F(X) = \lambda X$ is $X = \{0\}$. But by definition, $\{0\}$ is not an eigenset.

The proof that $F$ has no eigenvalue $\lambda < \rho(F)$ is postponed because the more general affine version is provided in the proof of Theorem 3 in Section 6.

We now show that $\rho(F)$ is an eigenvalue of $F$. Again let $F_\lambda = \frac{1}{\lambda} F$, so that $\rho(F_\lambda) = 1$. With $A$ as in the statement of Lemma 3, consider the nested intersection

$$S = \bigcap_{k \geqslant 0} F_\lambda^k(A) = \lim_{k \to \infty} F_\lambda^k(A).$$

That $S$ is compact, centrally symmetric, and star-shaped follows from Lemma 2. Also

$$F_\lambda(S) = F_\lambda\left(\bigcap_{k \geqslant 0} F_\lambda^k(A)\right) = \bigcap_{k \geqslant 1} F_\lambda^k(A) = S,$$

the last equality because $A \supseteq F_\lambda(A) \supseteq F_\lambda^{(2)}(A) \supseteq \cdots$. From $F_\lambda(S) = S$ it follows that $F(S) = \lambda S$.

It remains to show that $S$ contains a non-zero vector. Since $A$ is a convex body and determined only up to scalar multiple, there is no loss of generality in assuming that $A$ contains a ball $B$ of radius 1 centered at the origin. Then

$$\sup\{\ \|L_\sigma(x)\|\ :\ \sigma \in \Omega_k,\ x \in B\} = \hat{\rho}_k(F_\lambda) \geqslant (\rho(F_\lambda))^k = 1.$$

So there is a point $a_k \in F_\lambda^k(A)$ such that $\|a_k\| \geqslant 1$. If $a$ is an accumulation point of $\{a_k\}$, then $\|a\| \geqslant 1$, and there is a subsequence $\{a_{k_i}\}$ of $\{a_k\}$ such that

$$\lim_{i \to \infty} a_{k_i} = a.$$

Since the sets $F_\lambda^{(k_i)}(A)$ are closed and nested, it must be the case that $a \in F_\lambda^{(k_i)}(A)$ for all $i$. Therefore $a \in S$.

That $S$ is full dimensional follows from Lemma 4. □

## 5. Theorems of Dranisnikov–Konyagin–Protasov and of Barabanov

Important results of Dranisnikov–Konyagin–Protasov and of Barabanov on the joint spectral radius turn out to be almost immediate corollaries of Theorem 2. The first result is attributed to Dranisnikov

and Konyagin by Protasov, who provided a proof in [11]. Barabanov's theorem appeared originally in [2].

**Corollary 3** (Dranisnikov–Konyagin–Protasov). *If $F = (\mathbb{R}^n;\ L_i,\ i \in I)$ is a compact, irreducible, linear IFS with joint spectral radius $\rho := \rho(F)$, then there exists a centrally symmetric convex body $K$ such that*

$$conv\, F(K) = \rho K.$$

**Proof.** According to Theorem 2 there is a centrally symmetric, full dimensional eigenset $S$ such that $F(S) = \rho\, S$. If $K = conv(S)$, then $K$ is also centrally symmetric and

$$conv\, F(K) = conv\, F(conv\, S) = conv\, F(S) = conv\, (\rho\, S) = \rho\, conv\, S = \rho\, K.$$

The second equality is routine to check. Since $S$ is full dimensional, $K$ is a convex body, i.e., has nonempty interior. □

The original form of the Barabanov theorem is as follows:

**Theorem 5** (Barabanov). *If a set $F$ of linear maps on $\mathbb{R}^n$ is compact and irreducible, then there exists a vector norm $\|\cdot\|_B$ such that*

$$\begin{aligned} &\text{for all } x \text{ and all } L \in F && \|Lx\|_B \leqslant \rho(F)\, \|x\|_B, \\ &\text{for any } x \in \mathbb{R}^n \text{ there exists an } L \in F \text{ such that} && \|Lx\|_B = \rho(F)\, \|x\|_B. \end{aligned}$$

Such a norm is called a *Barabanov norm*. The first property says that $F$ is *extremal*, meaning that

$$\|L\|_B \leqslant \rho(F) \tag{2}$$

for all $L \in F$. It is extremal in the sense that, by Property (4) of the joint spectral radius in Section 3,

$$\sup_{L \in F} \|L\| \geqslant \rho(F)$$

for any matrix norm. Since $F$ is assumed compact, the inequality (2) cannot be strict for all $L \in F$. Hence there exists an $L \in F$ whose Barabanov norm achieves the upper bound $\rho(F)$. Furthermore, the second property in the statement of Barabanov's Theorem says that, for any $x \in \mathbb{R}^n$, there is such an $L$ achieving a value equal to the joint spectral radius at the point $x$. See [16] for more on extremal norms.

In view of Lemma 1, Barabanov's theorem can be restated in the following equivalent geometric form. Here $\partial$ denotes the boundary.

**Corollary 4.** *If $F$ is a compact, irreducible, linear IFS with joint spectral radius $\rho := \rho(F)$, then there exists a centrally symmetric convex body $K$ such that*

$$F(K) \subseteq \rho K,$$

*and, for any $x \in \partial K$, there is an $L \in F$ such that $Lx \in \partial(\rho\, K)$.*

**Proof.** Let $F^t = (\mathbb{R}^n;\ L_i^t,\ i \in I)$, where $L^t$ denotes the adjoint (transpose matrix) of $L$. For a compact set $Y$, the *dual* of $Y$ (sometimes called the polar) is the set

$$Y^* = \{z \in \mathbb{R}^n\ :\ \langle y, z\rangle \leqslant 1 \text{ for all } y \in Y\}.$$

The first two of the following properties are easily proved for any compact set $B$:

1. $B^*$ is convex.

2. If $B$ is centrally symmetric, then so is $B^*$.
3. If $L$ is linear and $L^t(S) \subseteq S$, then $L(S^*) \subseteq S^*$.

To prove the third property above, assume that $L^t(S) \subseteq S$. and let $x \in S^*$. Then

$$\begin{aligned} x \in S^* &\Rightarrow \langle x, y\rangle \leqslant 1 \ \text{ for all } y \in S \\ &\Rightarrow \langle x, L^t y\rangle \leqslant 1 \ \text{ for all } y \in S \\ &\Rightarrow \langle Lx, y\rangle \leqslant 1 \ \text{ for all } y \in S \\ &\Rightarrow Lx \in S^* \end{aligned}$$

Since $F$ is a compact, irreducible, linear IFS, so is $F^t$. Let $S$ be a centrally symmetric eigenset for $F^t$ as guaranteed by Theorem 2. By properties 1 and 2 above, $S^*$ is a centrally symmetric convex body. From the eigen-equation $F^t(S) = \rho S$, it follows that $\frac{1}{\rho} L^t(S) \subseteq S$ for all $L \in F$. From Property 3 above it follows that $\frac{1}{\rho} F(S^*) \subseteq S^*$ or $F(S^*) \subseteq \rho S^*$. Setting $K = S^*$ yields

$$F(K) \subseteq \rho K.$$

Concerning the second statement of the corollary, assume that $x \in \partial K = \partial S^*$. Then $\langle x, y\rangle \leqslant 1$ for all $y \in S$ and $\langle x, y\rangle = 1$ for some $y \in S$. Since $F(S) = \rho S$, the last equality implies that there is an $L \in F$ such that $\langle \frac{1}{\rho} Lx, z\rangle = \langle x, \frac{1}{\rho} L^t z\rangle = 1$ for some $z \in S$. Now we have $\langle \frac{1}{\rho} Lx, y\rangle \leqslant 1$ for all $y \in S$ and $\langle \frac{1}{\rho} Lx, z\rangle = 1$ for some $z \in S$. Therefore, $\frac{1}{\rho} Lx \in \partial S^* = \partial K$ or $Lx \in \rho(\partial K) = \partial(\rho K)$. $\square$

## 6. The eigenvalue problem for an affine IFS

For an affine IFS $F$, there is no theorem analogous to Theorem 2. More specifically, there are examples where $\rho(F)$ is an eigenvalue of $F$ and examples where $\rho(F)$ is not an eigenvalue of $F$. For an example where $\rho(F)$ is an eigenvalue, let

$$F_1 = (\mathbb{R}^2; f), \quad f(x) = Lx + (1, 0), \quad L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that $L$, a 90° rotation about the origin, is irreducible and $\rho(F_1) = 1$. If $S$ is the unit square with vertices $(0, 0), (1, 0), (0, 1), (1, 1)$, then $F_1(S) = S$. Therefore $\rho(F_1) = 1$ is an eigenvalue of $F_1$. On the other hand let

$$F_2 = (\mathbb{R}; f), \quad f(x) = x + 1.$$

In this case $\rho(F_2) = 1$, but it is clear that there exists no compact set $X$ such that $F(X) = X$. For the affine case, Theorem 3, as stated in Section 1, does holds. The proof is as follows.

**Proof of Theorem 3.** If $\lambda > \rho(F)$, then $\rho(F_\lambda) < 1$. According to Theorem 4, the IFS $F_\lambda$ has an attractor $A$ so that $F_\lambda(A) = A$. Since at least one function in $F_\lambda$ is not linear, $A \neq \{0\}$. Since $F_\lambda(A) = A$, also $F(A) = \lambda A$. Therefore $\lambda$ is an eigenvalue of $F$.

Concerning the second statement in the theorem assume, by way of contradiction, that such an eigenvalue $\lambda < \rho(F)$ exists, with corresponding eigenset $S$. Then $F_\lambda(S) = S$ and $\rho(F_\lambda) > 1$. According to Lemma 4, since $F$ is assumed irreducible, the eigenset $S$ is full dimensional. Exactly as in the proof of Lemma 4, using conjugation by a translation, there is an affine IFS $F'$ and a nonempty compact set $S'$ such that

1. $F'(S') = S'$.
2. $0 \in int(conv(S'))$.

3. The set $\mathbb{L}_{F'}$ of linear parts of the functions in $F'$ is equal to the set $\mathbb{L}_{F_\lambda}$ of linear parts of the functions in $F_\lambda$.
4. $\rho(F') = \rho(F_\lambda) > 1$.
5. $F'$ is irreducible.

In item 2 above, $int(conv(S'))$ denotes the interior of the convex hull of $S'$. If $K = conv(S')$ and $f(x) = Lx + a$ is an arbitrary affine function such that $f(S') \subseteq S'$, then

$$f(K) \subseteq K.$$

This follows from the fact that $f(S') \subseteq S'$ as follows. If $z \in K$, then $z = \alpha x + (1-\alpha) y$ where $0 \leqslant \alpha \leqslant 1$ and $x, y \in S'$. Therefore

$$\begin{aligned} f(z) &= \alpha Lx + (1-\alpha) Ly + a = \alpha(Lx + a) + (1-\alpha)(Ly + a) \\ &= \alpha f(x) + (1-\alpha) f(y) \in conv(f(S')) \subset conv(S') = K. \end{aligned}$$

Let $r > 0$ be the largest radius of a ball centered at the origin and contained in $K$ and $R$ the smallest radius of a ball centered at the origin and containing $K$. Let $x \in K$ such that $0 < \|x\| \leqslant r$. If $f(x) = Lx+a$ is any affine function such that $f(S') \subseteq S'$, then we claim that $\|Lx\| \leqslant R + r$. To prove this, first note that $-x \in K$. From $f(K) \subseteq K$ it follows that

$$\begin{aligned} &\|Lx + a\| = \|f(x)\| \leqslant R \\ &\|-Lx + a\| = \|L(-x) + a\| = \|f(-x)\| \leqslant R \\ &\|2a\| = \|(Lx + a) + (-Lx + a)\| \leqslant \|Lx + a\| + \|L(-x) + a\| \leqslant 2R \\ &\|Lx\| = \|f(x) - a\| \leqslant \|f(x)\| + \|a\| \leqslant R + r. \end{aligned}$$

From the definition of the joint spectral radius, $\rho(F') > 1$ implies that there is an $\epsilon > 0$ such that $(\hat{\rho}_k(F_\lambda))^{1/k} > 1 + \epsilon$ for infinitely many values of $k$. This, in turn, implies that, for each such $k$, there is an affine map $f_k \in \{f_\sigma : \sigma \in \Omega_k\}$ and its linear part $L_k \in \{L_\sigma : \sigma \in \Omega_k\}$ such that $\|L_k\| \geqslant (1+\epsilon)^k$. Choose $k = k_0$ sufficiently large that $\|L_k\| \geqslant (1+\epsilon)^{k_0} > \frac{R+r}{r}$. Then there is a $y \in K'$ with $\|y\| = r$ such that $\|L_{k_0} y\| > r\,\frac{R+r}{r} = R + r$. Since $L_{k_0}$ is the linear part of an affine function $f$ with the property $f(S') \subseteq S'$ (Property 1 above), this is a contradiction to what was proved in the previous paragraph. $\square$

## Acknowledgement

The authors thank the referee for valuable suggestions.

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