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The Lattice of Flats and its Underlying Flag Matroid Polytope

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Abstract. Let M be a matroid and $\mathcal{F}$ the collection of all linear orderings of bases of M, or *flags* of M. We define the flag matroid polytope $\Delta(\mathcal{F})$. We determine when two vertices of $\Delta(\mathcal{F})$ are adjacent, and provide a bijection between maximal chains in the lattice of flats of M and certain maximal faces of $\Delta(\mathcal{F})$.

Keywords: matroid, Coxeter matroid, flag matroid, flag matroid polytope

1. Introduction

An appealing aspect of matroid theory is the multiple approaches to the subject. A matroid can be defined in terms of independent sets, bases, circuits, closure, rank, lattice of flats, etc. In this paper, we concentrate on a less familiar characterization, but one deserving of more attention. This approach is by way of the matroid polytope, whose definition and properties are given in Section 2. The matroid polytope plays a special role in the subject of Coxeter matroids [BGW2]. The results in this paper are an example of how the machinery of Coxeter matroids can motivate new insights to ordinary matroids. For basic definitions and notations concerning matroids, see [O,W].

The main concepts in this paper are a matroid M, its underlying flag matroid $\mathcal{F}$ defined in Section 3, and the respective matroid and flag matroid polytopes $\Delta(M)$ and $\Delta(\mathcal{F})$. The elements of $\mathcal{F}$ (called its *bases*) are ordered bases of the matroid M. Thus, if the rank of M is k, then every basis in M gives rise to $k!$ bases in $\mathcal{F}$. By the way the polytopes are constructed, the vertices of $\Delta(M)$ are in one-to-one correspondence with the bases of M and the vertices of $\Delta(\mathcal{F})$ are in one-to-one correspondence with the bases in $\mathcal{F}$. In this setting,

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if two vertices δ_F and δ_G of $\Delta(\mathcal{F})$ are connected by an edge, then the two corresponding bases F and G of $\mathcal{F}$ are either reorderings of the same basis in M or span the same flag of flats in M.

Thus, two different aspects of the matroid M, its bases, and its flats are both encoded and interrelated in the adjacency graph of the convex polytope $\Delta(\mathcal{F})$.

The main results concerning this relationship between the lattice of flats of M and the underlying flag matroid polytope $\Delta(\mathcal{F})$ are contained in Section 4. In particular, let $\mathcal{L}$ be the lattice of flats of M ordered by inclusion. Theorem 4.1 gives a bijection between the maximal flags in $\mathcal{L}$ and certain maximal faces of $\Delta(\mathcal{F})$. From this bijection, $\mathcal{L}$ can be retrieved from $\Delta(\mathcal{F})$. This result implies (Corollary 4.3) the existence of a covering map from $\Delta(\mathcal{F})$ onto $\Delta(M)$.

2. The Matroid Polytope

Let M be a matroid of rank k on the set $[n] = \{1, 2, \dots, n\}$ and $\mathcal{B}$ the collection of bases of M. One of the many equivalent definitions of a matroid is that, for any two distinct bases A and B in $\mathcal{B}$ and any element $a \in A \setminus B$, there is an element $b \in B \setminus A$ such that the set $A \setminus \{a\} \cup \{b\}$ belongs to $\mathcal{B}$. Note that this definition implies that all bases of M have the same cardinality. The basis axiom is often referred to as the *exchange property*. We say that the basis $A \setminus \{a\} \cup \{b\}$ is obtained from A by an *elementary exchange* (a, b).

Construct the matroid polytope of M as follows. Let $\mathbf{R}^n$ be n-dimensional Euclidean space with the canonical orthonormal basis $\varepsilon_1, \dots, \varepsilon_n$. For any k-element subset B of $[n]$, set

$$\delta_B = \sum_{i \in B} \varepsilon_i. \tag{2.1}$$

Let $\Delta(\mathcal{B})$ be the convex hull of the points δ_B, $B \in \mathcal{B}$. If $\mathcal{B}$ is the set of bases of a matroid M, denote $\Delta(\mathcal{B})$ by $\Delta(M)$, called the *matroid polytope* of M. Notice that $\Delta(M)$ lies in the hyperplane $\sum x_i = k$, which will be denoted Π. In the matroid theory literature, the term matroid polytope usually refers to the convex hull of the points δ_I for all independent sets I of M. Our matroid polytope is the face of the classical matroid polytope supported by Π.

The following central theorem concerning the matroid polytope is originally due to Gelfand, Goresky, MacPherson, and Serganova [GGMS] and was subsequently generalized by Gelfand and Serganova [GS] (see also Serganova, Vince and Zelevinsky [SVZ]).

Theorem 2.1. *A set $\mathcal{B}$ of subsets of $[n]$ of the same cardinality are the bases of a matroid if and only if each edge of $\Delta(\mathcal{B})$ is parallel to $\varepsilon_i - \varepsilon_j$ for some i and j. Moreover, in the matroid case, vertices of $\Delta(M)$ are adjacent if and only if the corresponding bases of M can be obtained from each other by an elementary exchange.*

Because of this result, a matroid polytope can be taken as a fundamental matroid concept, like basis, independent set, circuit, etc. Consider the faces of dimension $k-1$ in the $(n-1)$-simplex Δ and let Δ' be the convex hull of the set of barycenters of some set of such faces. Call Δ' a *matroid polytope* (of rank k) if each edge of Δ' is parallel to $\varepsilon_i - \varepsilon_j$ for some i and j. This gives another equivalent definition of matroid.

3. Underlying Flag Matroid

A *flag* F in $[n]$ is a strictly increasing sequence $A_1 \subset A_2 \subset \cdots \subset A_m$ of finite subsets of $[n]$. Denote by k_i the cardinality of the set A_i; the m-tuple $(k_1,\ldots,k_m)$ will be called the *rank* of F. A flag will be denoted by $F = (A_1,\ldots,A_m)$, and the set A_i will be called the ith *component* of the flag F.

Let M and M' matroids be on the same set $[n]$. We say that M' is a *quotient* of M if every circuit of M is a union of circuits of M'. This is synonymous with the phrase that the identity map on $[n]$ is a strong map from M to M'. There are many equivalent characterizations of quotients or strong maps (see [W, Prop. 7.4.7 and Chpt. 8]). Every truncation of a matroid M is a quotient of M, where, if $k \leq \text{rank}(M)$, the truncation of M to rank k is the matroid whose bases are all independent sets of M of cardinality k.

Let $\mathcal{F}$ be a set of flags of the same rank $(k_1,k_2,\ldots,k_m)$. We say that $\mathcal{F}$ is a *flag matroid* if, for every $i = 1,\ldots,m$, the ith components of flags in $\mathcal{F}$ form a matroid M_i of rank k_i called the ith *component matroid* of $\mathcal{F}$, and furthermore, M_i is a quotient of M_{i+1} for all i, such that $1 \leq i \leq m-1$. Note that a matroid is a special case of a flag matroid.

A polytope can be assigned to a set $\mathcal{F}$ of flags of the same rank by generalizing the construction in Section 2. If $F = (A_1,\ldots,A_m) \in \mathcal{F}$, let c_i denote the number of components of F to which element i belongs and set

$$\delta_F = \sum_{i=1}^{n} c_i \varepsilon_i. \tag{3.1}$$

Note that formula (2.1) is the special case of formula (3.1) when the flag has just one component. Let $\Delta(\mathcal{F})$ be the convex hull of the points $\delta_F, F \in \mathcal{F}$. If $\mathcal{F}$ is a flag matroid, then $\Delta(\mathcal{F})$ is called the *matroid polytope* of $\mathcal{F}$.

There is an equivalent way to understand the flag matroid polytope $\Delta(\mathcal{F})$. Consider the set of all flags of $[n]$. This is an abstract simplicial complex Δ whose faces are the flags in $[n]$. Combinatorially, this complex is isomorphic to the barycentric subdivision of the regular $(n-1)$-simplex. Using this geometric realization of Δ, construct $\Delta(\mathcal{F})$ as follows. Identify each flag $F \in \mathcal{F}$ with the barycenter δ_F of the corresponding face of Δ. Then $\Delta(\mathcal{F})$ is the convex hull of $\{\delta_F : F \in \mathcal{F}\}$.

Theorem 2.1 has the following generalization which is a special case of the Gelfand-Serganova Theorem [GS].

Theorem 3.1. *Let $\mathcal{F}$ be a set of flags of the same rank. Then $\mathcal{F}$ is a flag matroid if and only if each edge of $\Delta(\mathcal{F})$ is parallel to $\varepsilon_i - \varepsilon_j$ for some i and j.*

Given a matroid M of rank k on $[n]$, there is a particular flag matroid associated with M that is important in the context of this paper. Let $\mathcal{B}$ be the set of bases of M. If $B = (b_1,\ldots,b_k)$ is a basis in $\mathcal{B}$ written in some order, then consider the flag of independent sets

$$\{b_1\} \subset \{b_1,b_2\} \subset \cdots \subset \{b_1,\ldots,b_k\},$$

which will be denoted simply by $[b_1,\ldots,b_k]$. Let $\mathcal{F}(M)$ be the set of all flags obtained from all orderings of bases of $\mathcal{B}$. Then $\mathcal{F}(M)$ is a flag matroid, since the lower component matroids are just truncations of M, and we will call $\mathcal{F}(M)$ the *underlying flag matroid* of M.

A loop in M is an element of $[n]$ that is dependent; in other words, a nonloop is an element that belongs to at least one basis.

Theorem 3.2. *If M is a matroid with p nonloops and $\mathcal{F}$ is its underlying flag matroid, then* $\dim\Delta(\mathcal{F}) = p-1$.

Proof. Let $P \subseteq [n]$ be the set of nonloops of $\mathcal{B}$, and let $U = \text{affine span}(\{e_i \mid i \in P\})$. With the δ_F's appropriately scaled, we can assume without loss of generality that $\Delta(\mathcal{F}) \subseteq \mathcal{U}$. Suppose H is a hyperplane of U which contains $\Delta(\mathcal{F})$. If $j \in P$, then there exists $B \in \mathcal{B}$, such that $j \in B$. Let $\mathcal{F}_B$ denote the set of flags of independent sets obtained from all orderings of B. By definition, $\mathcal{F}_B \subseteq \mathcal{F}$. Let $h, i \in B$. Then we may construct flags $F = [h, i, \ldots]$ and $G = [i, h, \ldots]$, where both are obtained from orderings of B, and these two flags are equal after the first two entries. Then $\delta_F - \delta_G$ is a scalar multiple of $e_h - e_i$, which is a vector parallel to H. By adding appropriate scalar multiples of $e_j - e_i$ for various i to, say, δ_G, we find that $e_j \in H$. Since this is true for all $j \in P$, no such H exists, and it follows that $\dim\Delta(\mathcal{F}) = \dim U = p-1$. ∎

4. The Lattice of Flats of a Matroid

Let M be a rank k matroid on $[n]$ and $\mathcal{L}$ its lattice of flats, ordered by inclusion. The lattice $\mathcal{L}$ is a geometric lattice (semimodular, point lattice) and is ranked; the rank of a flat being the cardinality of any basis of that flat. Let $\mathcal{M}$ denote the set of all *maximal flags of flats* or maximal chains in $\mathcal{L}$. If $F = F_1 \subset F_1 \subset \ldots \subset F_k$ is such a maximal flag of flats, then the flat F_i has a basis $\{b_1, b_2, \ldots, b_i\}$, each such basis obtained from the previous by extension:

$$\{b_1\} \subset \{b_1, b_2\} \subset \cdots \subset \{b_1, \ldots, b_k\}.$$

This flag of independent sets has already been denoted $[b_1, \ldots, b_k]$. Conversely, for any such flag of independent sets $[b_1, \ldots, b_k]$, the closures

$$\overline{\{b_1\}} \subset \overline{\{b_1, b_2\}} \subset \cdots \subset \overline{\{b_1, \ldots, b_k\}}$$

form a maximal flag of flats of M. This flag of flats is referred to as the flag of flats *spanned* by the flag of independent sets $[b_1, \ldots, b_k]$. Of course, a given flag of flats can be spanned by many different flags of independent sets.

Let M be a matroid, $\mathcal{F}$ the underlying flag matroid of M, and $\Delta(\mathcal{F})$ the matroid polytope of $\mathcal{F}$. The *graph of a polytope* means the 1-skeleton, consisting of the vertices and edges. Recall that each vertex of $\Delta(\mathcal{F})$ corresponds to a flag of independent sets $[b_1, \ldots, b_k]$. Define an equivalence relation $\sim$ on the set of vertices of $\Delta(\mathcal{F})$ by declaring $u \sim v$ if u and v are adjacent in the graph of $\Delta(\mathcal{F})$ and the *unordered* bases associated with u and v are distinct; then take the transitive closure of this relation. Let $\Sigma(M)$ denote the set of convex hulls in $\Delta(\mathcal{F})$ of the equivalence classes.

Theorem 4.1. *Let M be a matroid and $\mathcal{M}$ the set of all maximal flags in its lattice of flats. With the above notation,*

(1) *the elements of* $\Sigma(M)$ *are faces of* $\Delta(\mathcal{F})$ *of dimension* $p-k$, *where* p *is the number of nonloops in* M;

(2) *there is a bijection* Φ *between* $\mathcal{M}$ *and* $\Sigma(M)$ *defined as follows: If* $F \in \mathcal{M}$ *and* $\overline{F}$ *denotes the set of all flags of independent sets that span* F, *then* $\Phi(F) = \Delta(\overline{F})$.

Before giving the proof, we describe how the lattice of flats $\mathcal{L}$ can be reconstructed from the underlying flag matroid polytope. The two examples that follow are illustrative. For a matroid M of rank k on the set $[n]$ with underlying flag matroid $\mathcal{F}$, consider the underlying flag matroid polytope $\Delta(\mathcal{F})$ with each vertex labeled by the appropriate flag of independent sets. Then the set of faces $\Sigma(M)$ is determined. Define an equivalence relation on the set of jth components of the set of all flags of independent sets of M, for $j = 0, 1, 2, \ldots, k$. Declare two such j-subsets *equivalent* if these jth components appear at vertices in the same element of $\Sigma(M)$; then take the transitive closure. Let $\mathcal{L}(M)$ denote the set of equivalence classes. Put an order on $\mathcal{L}(M)$ by declaring that $\alpha \leq \beta$ if there are representatives $A \in \alpha$ and $B \in \beta$, such that $A \subseteq B$.

Corollary 4.2. *With the above notation,* $\mathcal{L}(M)$ *is a lattice isomorphic to the lattice of flats of* M.

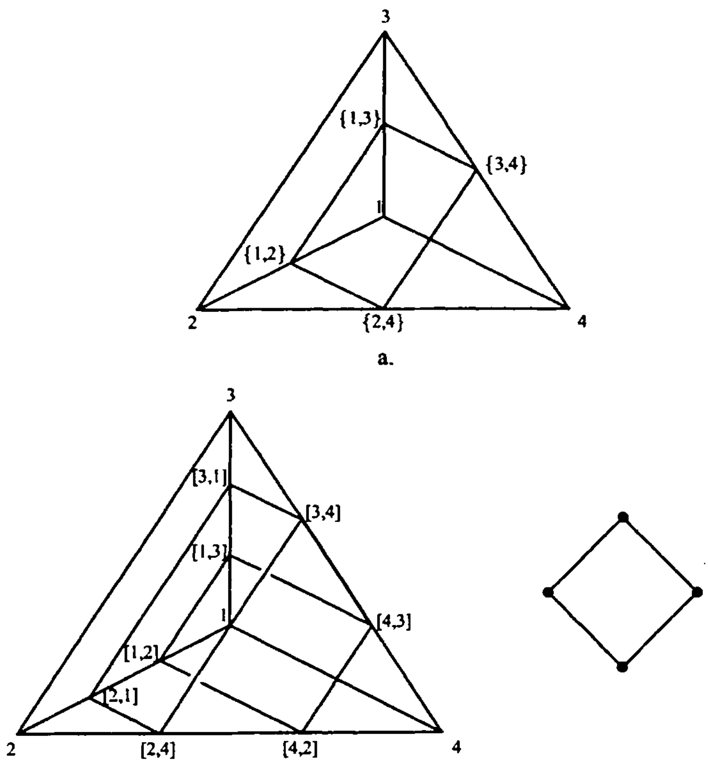


Figure 1: Matroid Polytope, Underlying Flag Matroid Polytope, and Lattice of Flats

Example 1. Consider the rank 2 matroid M on the set [4] with bases $\mathcal{B} = \{12, 13, 24, 34\}$. Then the matroid polytope $\Delta(M)$ of M is a square inscribed in the tetrahedron whose vertices are labeled $1, 2, 3, 4$ (see Figure 1(a)). The matroid polytope $\Delta(\mathcal{F})$ of the underlying flag matroid $\mathcal{F}$ has eight vertices. The convex hull of these vertices is combinatorially equivalent to a cube, but has two rectangular faces and four trapezoidal faces (see Figure 1(b)). The set of faces in $\Sigma(M)$ are the rectangular faces, which correspond to two maximal flags of flats spanned by the flags of independent sets $[12]$ and $[21]$. The equivalence classes that comprise the lattice $\mathcal{L}(M)$ are

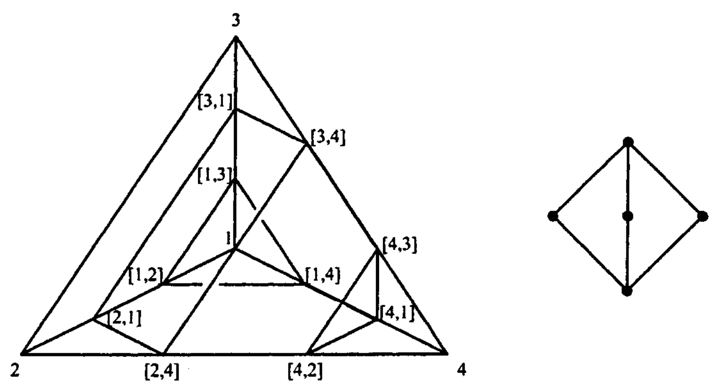
Rank 0. $\emptyset$.
Rank 1. $\{1,4\}$ and $\{2,3\}$.
Rank 2. $\{12, 13, 24, 34\}$.

The Hasse diagram of the lattice $\mathcal{L}(M)$ is shown in Figure 1(c).

Example 2. Again with M a rank 2 matroid on [4], take $\mathcal{B}$ to consist of all 2-element sets in [4] with the exception of [23]. Then $\Delta(\mathcal{F})$, the underlying flag matroid polytope, has two hexagonal, two trapezoidal, one rectangular, and two triangular faces (see Figure 2(a)). The faces of $\Sigma(M)$ are the two triangles, corresponding to the flags of flats spanned by [12] and [41], and the rectangular face, corresponding to the flag of flats spanned by [21]. The equivalence classes that comprise the lattice $\mathcal{L}(M)$ are

Rank 0. $\emptyset$.
Rank 1. $\{1\}, \{4\}$ and $\{2,3\}$.
Rank 2. $\{12, 13, 14, 24, 34\}$.

The Hasse diagram of the lattice $\mathcal{L}(M)$ is shown in Figure 2(b).



a. b.

Figure 2: Underlying Flag Matroid Polytope and Lattice of Flats

Proof of Theorem 4.1. An outline of the proof is as follows. We will show that

(I) if A and B are two different bases in $\mathcal{B}$, the collection of bases of the matroid M, and $[A] = [a_1,\dots,a_k]$ and $[B] = [b_1,\dots,b_k]$ are two corresponding flags of independent sets in the underlying flag matroid $\mathcal{F}$, and if the vertices $\delta_{[A]}$ and $\delta_{[B]}$ of the matroid polytope $\Delta(\mathcal{F})$ of the flag matroid $\mathcal{F}$ are adjacent in $\Delta(\mathcal{F})$, then the flags of independent sets F and G span the same flag of flats of the matroid M; and

(II) if $F \in \mathcal{M}$, then $\Delta(\overline{F})$ is a face of $\Delta(\mathcal{F})$.

By statement (II), the mapping $F \mapsto \Delta(\overline{F})$ defines a function from $\mathcal{M}$ to certain faces of $\Delta(\mathcal{F})$.

Since two flags of flats spanned by different orderings of the same basis must be distinct, and each face is a polytope whose graph is connected, statement (I) implies that these certain faces are exactly the elements of $\Sigma(M)$ and Φ is injective. Since every flag of flats is spanned by some flag of independent sets, Φ is surjective. Next, we prove statements (I) and (II); the statement about dimension is considered at the end.

Proof of Statement (I). Let $[A] = [a_1,a_2,\dots,a_k]$ and $[B] = [b_1,b_2,\dots,b_k]$, with $A \neq B$. By the definition in Section 3, $\delta_{[A]} = \sum_i (k+1-i)\varepsilon_{a_i}$ and $\delta_{[B]} = \sum_i (k+1-i)\varepsilon_{b_i}$. Let L be a linear functional which attains its maximum on $\Delta(\mathcal{F})$ on the edge $\delta_{[A]}\delta_{[B]}$. Since L is linear, it is determined by its values on the basis elements $\varepsilon_i, i \in [n]$. For simplicity of notation, we will abbreviate $L(\varepsilon_i)$ by $L(i)$. Since $A \neq B$ implies that $[A]$ and $[B]$ are not reorderings of each other, all reorderings of $[A]$ or $[B]$ must give a strictly smaller L-values. It follows that $L(a_1) > L(a_2) > \dots > L(a_k)$ and $L(b_1) > L(b_2) > \dots > L(b_k)$.

Since $A \neq B$, there exists $a_j \in A \setminus B$. Without loss of generality, we may assume $a_i = b_i$ for all $i < j$. By symmetric matroid basis exchange, there exists $b_l \in B \setminus A$, $l \geq j$, such that $A' := (A \setminus \{a_j\}) \cup \{b_l\}, B' := (B \setminus \{b_l\}) \cup \{a_j\} \in \mathcal{B}$. Let $[A']$ and $[B']$ be the flags obtained from $[A]$ and $[B]$ by interchanging a_j and b_l, but otherwise retaining the ordering of $[A]$ and $[B]$, and let $[A'']$ and $[B'']$ be obtained from $[A']$ and $[B']$ by reordering the elements according to descending L-values. Then

$$L(\delta_{[A'']}) \geq L(\delta_{[A']}) = L(\delta_{[A]}) + (k+1-j)(L(b_l) - L(a_j)),$$

$$L(\delta_{[B'']}) \geq L(\delta_{[B']}) = L(\delta_{[B]}) + (k+1-l)(L(a_j) - L(b_l));$$

both cannot be strictly less than $L(\delta_{[A]}) = L(\delta_{[B]})$. It follows that $A' = B$ and $B' = A$, and that $L(a_j) = L(b_l)$.

Suppose $l > j$. Then $L(a_j) = L(b_l) < L(b_j)$ and, hence, $b_j \notin A$. This contradicts $B' = A$, and since we already had $l \geq j$, we now have $l = j$.

Now, suppose $\{a_1,a_2,\dots,a_j,b_j\}$ is independent in the matroid M. Then, for some m, $\{a_1,a_2,\dots,a_j,b_j,a_{j+1},\dots,a_{m-1},a_{m+1},\dots,a_k\} \in \mathcal{B}$. Since $L(b_j) > L(a_m)$, we have a contradiction to the maximality of $L(\delta_{[A]})$. Therefore, $\{a_1,a_2,\dots,a_j,b_j\}$ is dependent in the matroid. On the other hand, $\{a_1,a_2,\dots,a_{j-1},b_j\} = \{b_1,b_2,\dots,b_j\} \subseteq B$ is independent. From elementary matroid theory, it follows that $\{a_1,a_2,\dots,a_j\}$ and $\{b_1,b_2,\dots,b_j\}$ have the same closure, and consequently, so do $\{a_1,a_2,\dots,a_m\}$ and $\{b_1,b_2,\dots,b_m\}$ for all m. Thus, $[A]$ and $[B]$ span the same flag of flats of the matroid.

Proof of Statement (II). Let $F = F_1 < F_2 < \dots < F_k$ be a maximal flag of flats of M and $\overline{F}$ the set of flags of bases that span F. Denote by P_1 the set of all nonloop elements of $[n]$ in F_1, and by P_j, $j = 2, \ldots, k$, the sets of all elements in $F_j \setminus F_{j-1}$. Denote by L_i the linear functional on $\mathbf{R}^n$, which takes value 1 on ε_j for $j \in P_i$ and 0 on other basis vectors ε_j. If $[B] = [i_1, \ldots, i_k]$ is an ordered basis in $\mathcal{F}$, then the corresponding vertex of $\Delta(\mathcal{F})$ has the form

$$\delta_{[B]} = k\varepsilon_{i_1} + (k-1)\varepsilon_{i_2} + \cdots + 2\varepsilon_{i_{k-1}} + 1\varepsilon_{i_k}.$$

Notice that at most one of the elements $i_1, \ldots, i_k$ belongs to P_1. For this reason, the maximal value of the functional L_1 is reached only at those vertices $\delta_{[B]}$ in $\Delta(\mathcal{F})$ which satisfy $i_1 \in P_1$; this maximum value is k. If Δ_1 is the convex hull of all such vertices, then Δ_1 is a face of $\Delta(\mathcal{F})$. Consider a vertex $\delta_{[B]}$ in Δ_1. Notice again that at most one element in B belongs to P_2, which means that the maximum of the functional L_2 restricted to Δ_1 is reached at only those vertices $\delta_{[B]}$ of Δ_1 which satisfy $i_2 \in P_2$. Denote by Δ_2 the convex hull of all these vertices; then Δ_2 is a face of Δ_1 and, hence, of $\Delta(\mathcal{F})$. Continuing this process, we find that the face of $\Delta(\mathcal{F})$ obtained by maximizing the sequence of functionals $L_1, \ldots, L_k$, is exactly the convex hull of all vertices $\delta_{[B]}$ with $[B] = [i_1, \ldots, i_k]$ satisfying $i_j \in P_j$ for all $j = 1, \ldots, k$, i.e., the polytope $\Delta(\overline{F})$.

Concerning the statement about dimension, let p be the number of nonloops of the matroid M. Notice that, given a flag of flats $F_1 < \cdots < F_k$, the set of all flags of independent sets in $\overline{F}$ spanning this flag of flats has the natural structure of the direct sum of the uniform rank 1 matroids on the sets $P_1, \ldots, P_k$, and for this reason, the statement about dimension $p - k$ follows from Corollary 15 in [BGW2]. ∎

Let M be a matroid on the set $[n]$, $\mathcal{F}$ its underlying flag matroid, and $\Delta(M)$ and $\Delta(\mathcal{F})$ the respective matroid and flag matroid polytopes. If $[a_1, \ldots, a_k]$ is a flag of independent sets, then $[a_1, \ldots, a_k] \mapsto \{a_1, \ldots, a_k\}$ defines a map from the set of vertices of $\Delta(\mathcal{F})$ to the set of vertices of $\Delta(M)$. This map can be extended to a map ϕ from the lattice of faces of $\Delta(\mathcal{F})$ to the lattice of faces of $\Delta(M)$. The last result gives several properties of this covering map ϕ. Two polytopes are *combinatorially isomorphic* if the respective lattices of faces are isomorphic.

Corollary 4.3. *With M a rank k matroid and with the above notation, the map ϕ from the lattice of faces of $\Delta(\mathcal{F})$ to the lattice of faces of $\Delta(M)$ has the following properties.*

(1) *Each vertex of $\Delta(M)$ is the image of $k!$ vertices of $\Delta(\mathcal{F})$.*
(2) *The map ϕ takes each face in $\Sigma(M)$ to a parallel face of the same dimension of $\Delta(M)$.*
(3) *Corresponding edges of Γ and $\phi(\Gamma)$ are parallel for $\Gamma \in \Sigma(M)$.*
(4) *The faces Γ and $\phi(\Gamma)$ are combinatorially isomorphic for $\Gamma \in \Sigma(M)$.*

Proof. Statement (1) is clear since there are $k!$ permutations of $\{a_1, \ldots, a_k\}$. Concerning statements (2) and (3), let $F_1, \ldots, F_m$ be the vertices of a face Γ in $\Sigma(M)$, that is, the flags of independent sets spanning a given flag of flats $F = F_1 < F_1 < \ldots < F_k$ of M. Let $A_1, \ldots, A_m$ be the corresponding (unordered) bases of M. All the bases $A_1, \ldots, A_l$ are distinct since two different orderings of the same basis produce distinct flags of flats. Hence, Γ and $\phi(\Gamma)$ have the same number of vertices. Denote by P_1 the set of all nonloop elements of $[n]$ in F_1, and by P_j, $j = 2,\ldots,k$, the sets of all elements in $F_j \setminus F_{j-1}$. Denote by L_i the linear functional on $\mathbf{R}^n$ which takes value 1 on ε_j for $j \in P_i$ and 0 on other basis vectors ε_j. This is the same sequence of linear functionals used in the proof of the second part of Theorem 4.1. On $\Delta(M)$, using the same arguments as in that proof, the sequence of functionals $L_1,\ldots,L_k$ reach their maximum at the vertices $\delta_{A_1},\ldots,\delta_{A_l}$, and therefore, the convex hull of these vertices is a face Γ' of $\Delta(M)$ parallel to Γ. Now use formulas (2.1) and (3.1) in Sections 2 and 3 and the fact that the edges of $\Delta(\mathcal{F})$ and $\Delta(M)$ are parallel to $\varepsilon_i - \varepsilon_j$ for some i and j (Gelfand-Serganova Theorem [GS]) to conclude that the edges of Γ are parallel to the corresponding edges of Γ'. Moreover, the dimension of Γ is equal to the dimension of Γ'. This is because the dimension of either polytope equals the dimension of the vector space spanned by the vectors $\overline{uv}$ corresponding to all pairs of adjacent vertices u, v in the polytope.

Concerning statement (4), let Δ be the n-simplex with vertices at the origin and at the points ε_i, $i = 1,\ldots,n$. The construction using the linear functionals $L_1,\ldots,L_m$ implies that Γ is the intersection of Δ with an affine subspace U parallel to some face of Δ. The same is true of Γ', for an affine subspace U' parallel to the same face of Δ. This implies that Γ and Γ' are combinatorially isomorphic, since the fact that the faces Γ and Γ' have the same number of vertices precludes any degeneracy. ∎

There are open questions concerning the covering map in Corollary 4.3. Assume, for simplicity, that the matroid M has no loops or coloops (loops in the dual matroid). Then, in view of Corollary 4.3, the maximal flags of flats in M are represented by certain faces of $\Delta(M)$ of dimension $n-k$, those in the image $\phi(\Sigma(M))$. Let us call these faces *flag* faces. It is clear from formula (2.1) of Section 2 that the matroid polytope $\Delta(M^*)$ of the dual matroid M^* is similar to $\Delta(M)$. Therefore, the maximal flags of flats in the dual matroid M^* can be represented by certain faces of $\Delta(M)$ of dimension k (called *coflag* faces). Every vertex of $\Delta(M)$ belongs to $k!$ flag faces and $(n-k)!$ coflag faces (with repetition, since several distinct flags can be represented by the same face), and every edge of $\Delta(M)$ belongs to at least one flag and one coflag face. It would be interesting to find a combinatorial criterion for recognizing the flag and coflag faces of $\Delta(M)$, and to study other properties of the mosaic of flag and coflag faces on $\Delta(M)$. For example, we have the following result.

Corollary 4.4. *Each flag face of a matroid polytope is combinatorially isomorphic to the product of k simplices (of various dimensions).*

Proof. Let F be the flag of flats in M corresponding to a flag face Γ' of the matroid polytope and let $P_1,\ldots,P_k$ be as in the proof of Corollary 4.3. Then the set of all ordered bases spanning F (corresponding to the set of vertices of Γ') has the natural matroid structure of the direct product of the uniform rank 1 matroids on the sets $P_1,\ldots,P_k$. Hence, the vertices of Γ' are represented by the set of all k-sets $a_1,\ldots,a_k$, where $a_i \in P_i$. By Theorem 2.1, two vertices of Γ' are adjacent if and only if they can be obtained from each other by an elementary exchange. This translates to the fact that two vertices $a_1,\ldots,a_k$ and $b_1,\ldots,b_k$ of Γ' are adjacent if and only if $a_i = b_i$ for all i except one. This means that the graph of Γ' is the direct product of the graphs of simplices of dimensions $|P_1|-1,\ldots,|P_k|-1$. However, such a direct product of simplices is a simple polytope, and hence, its combinatorial structure is completely determined by its graph [BM]. ∎

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