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DISCRETE LINES AND WANDERING PATHS*

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Abstract. The problem of finding an approximation to a geometric line by a discrete line using pixels is ubiquitous in computer graphics applications. We show that this discrete line problem in $\mathbb{R}^{n+1}$, for grids of any shape, is equivalent to a geometry problem in $\mathbb{R}^n$ concerning the minimization of the distance that a certain type of closed polygonal path wanders from the origin. This geometry problem is solved completely in dimension 1 (corresponding to 2-dimensional grids), and two simple and efficient algorithms provide near optimum solutions in higher dimensions.

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1. Introduction.

This paper concerns a geometry problem in n-dimensional Euclidean space motivated by the drawing of a discrete line with pixels. The generation of such line segment raster images is ubiquitous in computer graphics applications, the first such algorithm due to Bresenham [1, 5]. In 2001, Bresenham wrote:

> I was working in the computation lab at IBM's San Jose development lab. A Calcomp plotter had been attached to an IBM 1401 via the 1407 typewriter console. [The algorithm] was in production use by summer 1962, possibly a month or so earlier. Programs in those days were freely exchanged among corporations so Calcomp (Jim Newland and Calvin Hefte) had copies. When I returned to Stanford in Fall 1962, I put a copy in the Stanford comp center library. A description of the line drawing routine was accepted for presentation at the 1963 ACM national convention in Denver, Colorado. It was a year in which no proceedings were published, only the agenda of speakers and topics in an issue of *Communications of the ACM*. A person from the IBM Systems Journal asked me after I made my presentation if they could publish the paper. I happily agreed, and they printed it in 1965.

The Bresenham algorithm was designed for rectangular grids in the plane. More recent applications in visualization of 3-dimensional medical image data and in global image processing have led to an interest in nonrectangular grids, for example the hexagonal grid, and in higher-dimensional grids. That is the motivation for this paper. There has also been an interest in issues not directly addressed in this paper, for example efficient implementation of algorithms [2, 3], discrete approximation of curves [6], and alternate approaches to constructing discrete lines [7].

The discrete line problem. Given two points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^{n+1}$, the discrete line problem is to find a discrete line, in terms of cells (pixels), that is in some sense the best approximation to the Euclidean line $\overline{\mathbf{ab}}$. To precisely formulate the problem, let the points of a lattice L represent the "centers" of the cells in our $(n+1)$-dimensional

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grid. By a *lattice* in $\mathbb{R}^{n+1}$ we mean the set of all integer linear combinations of $n+1$ linear independent vectors. The cells are the Voronoi cells of the lattice, the *Voronoi cell* at lattice point $\mathbf{x}$ being the set of points at least as close to $\mathbf{x}$ as to any other lattice point in L. Each Voronoi cell is a polytope P, and the grid is obtained by translation of P by the lattice L.

Two lattice points will be considered *neighbors* if their respective Voronoi cells share a common facet. Given lattice points $\mathbf{a}$ and $\mathbf{b}$, define a *discrete line* joining $\mathbf{a}$ and $\mathbf{b}$ as a sequence $\mathbf{a} = \mathbf{u}_1, \ldots, \mathbf{u}_N = \mathbf{b}$ of lattice points (cells) such that $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$ are neighbors for $i = 1, \ldots, N-1$. This is a reasonable definition, especially in situations in which the cells can be viewed at variable resolutions—multiscale. This is the point of view taken in [4]. For a discrete line to be a "good approximation" to the geometric line $\overline{\mathbf{ab}}$, the discrete line should be as "short" as possible and as "close" as possible to the geometric line. More precisely, we impose the following requirements:

A. the length N should be minimum, and
B. of all such discrete lines $\mathbf{a} = \mathbf{u}_1, \ldots, \mathbf{u}_N = \mathbf{b}$ of minimum length, the points $\mathbf{u}_k$ should be chosen so as to minimize

$$\max_{1 \le k \le N} d(\mathbf{u}_k, \overline{\mathbf{ab}}),$$

where $d(\mathbf{u}_k, \overline{\mathbf{ab}})$ is the orthogonal distance from $\mathbf{u}_k$ to $\overline{\mathbf{ab}}$.

The above optimization problem will be referred to as the *discrete line problem*.

In section 2 a geometry problem is posed concerning minimization of the distance that a certain type of closed polygonal path wanders from the origin. This geometry problem in $\mathbb{R}^n$ is shown to be equivalent to the discrete line problem in $\mathbb{R}^{n+1}$. The remainder of the paper concerns this "wandering path problem." After definitions and preliminary results in section 3, sections 4 and 6 contain two simple and efficient algorithms whose output is close to an optimum solution of the wandering path problem. An optimum solution is found for the dimension 1 case in section 5, which implies a complete solution to the discrete line problem for any grid in dimension 2. Theorems 10, 11, and 12 provide upper bounds on the output of Algorithms 1, 1.1, and 2, respectively.

2. The wandering path problem. A geometry problem in $\mathbb{R}^n$ will be posed which is equivalent to the discrete line problem in $\mathbb{R}^{n+1}$. Consider any set V of vectors in $\mathbb{R}^n$. A *V-multiset* is a finite ordered multiset $W = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N)$ of elements from V. If we set

$$\mathbf{u}_k := \mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_k,$$

$1 \le k \le N$, then joining points $\mathbf{0} = \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N$ successively by line segments results in a polygonal path $P = (\mathbf{0} = \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N)$ in $\mathbb{R}^n$ called a *V-path* of *length* N. If $\mathbf{u}_N = \mathbf{0}$, then the V-path is called a *closed* V-path. Define

$$w(P) := \max_{1 \le k \le N} |\mathbf{u}_k|.$$

Then $w(P)$ is the furthest that path P wanders from the origin.

The case of interest for our application is where V is a set of exactly $n+1$ vectors in $\mathbb{R}^n$ satisfying the following two properties:

1. each subset of n vectors in V is linearly independent, and
2. there exists a closed V-path.

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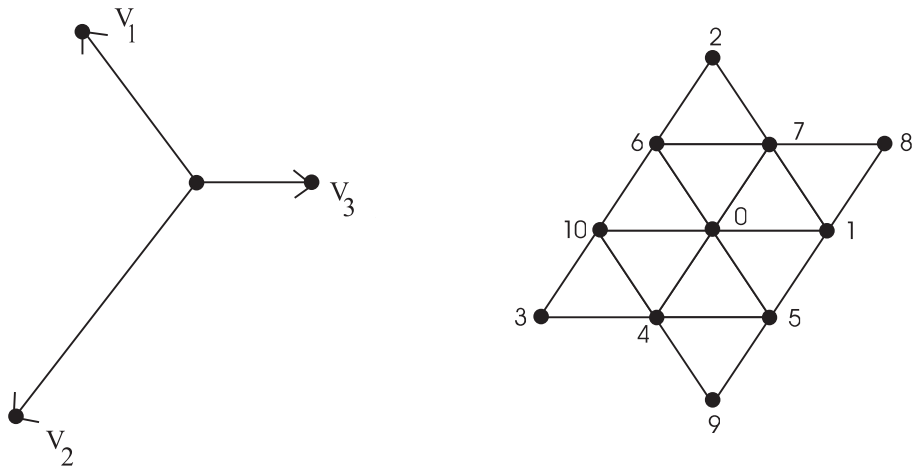

FIG. 1. *An optimum wandering path.*

Note that condition 2 is equivalent to the existence of a set $\{m_{\mathbf{v}} \mid \mathbf{v} \in V\}$ of positive integers such that

$$\sum_{\mathbf{v}\in V} m_{\mathbf{v}}\mathbf{v} = \mathbf{0}.$$

A set V satisfying properties 1 and 2 will be called a *basic set*. This paper concerns minimizing $w(P)$ over all closed V-paths P. In other words, we seek a closed V-path that stays as close as possible to the origin. Define

$$w(V) := \min\{w(P) \mid P \text{ is a closed } V\text{-path}\}.$$

Call $w(V)$ the *optimum wandering distance* for V. A closed path that realizes this distance will be called an *optimum wandering path*. The problem of finding the optimum wandering distance and optimum wandering path will be referred to as the *wandering path problem*.

A 1-dimensional example. Let $V = \{-4, 6\}$. For the closed V-path $P = (0, -4, +2, -2, 4, 0)$, we have $w(P) = 4$. In fact this is an optimum wandering path for V, so $w(V) = 4$. A complete solution to the wandering path problem for the 1-dimensional case appears in section 5. As will be shown below, this implies a complete solution to the discrete line problem in two dimensions.

A 2-dimensional example. Let $V = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$, where

$$\begin{aligned}\mathbf{v}_0 &= (1,\, 0),\\ \mathbf{v}_1 &= (-1,\, \sqrt{3}),\\ \mathbf{v}_2 &= (-\tfrac{3}{2},\, -\tfrac{3}{2}\sqrt{3}).\end{aligned}$$

The optimum wandering path $(\mathbf{0} = \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{11} = \mathbf{0})$ is shown in Figure 1, where the labels indicate the indices. The vectors of V are successively added in the order $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_0, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_0)$. The optimum wandering distance is $w(V) = \sqrt{3}$.

Relation between the discrete line and the optimum wandering path problems. Recall that the discrete line problem is to find a sequence $\mathbf{a} = \mathbf{u}'_1, \ldots, \mathbf{u}'_N$

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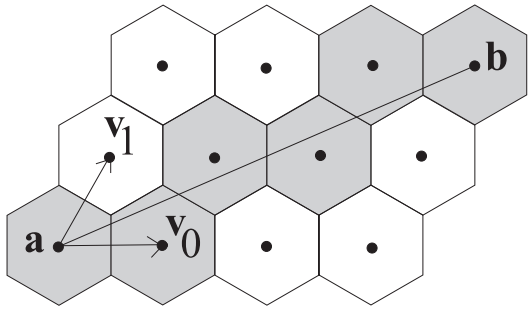

FIG. 2. *A discrete line that approximates* $\overline{\mathbf{ab}}$.

$= \mathbf{b}$ of points (cells) in an $(n+1)$-dimensional lattice L such that $\mathbf{u}'_i$ and $\mathbf{u}'_{i+1}$ are neighbors for $i = 1, \ldots, N-1$ and that satisfies conditions (A) and (B) from the Introduction. Because the cells are the Voronoi cells of L, there exists a set Y of vectors that generates L such that $Y = -Y := \{-\mathbf{y} \mid \mathbf{y} \in Y\}$ and such that two lattice points $\mathbf{x}_1, \mathbf{x}_2$ are neighbors if and only if $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{y}$ for some $\mathbf{y} \in Y$. It also follows that there is a subset V' of Y consisting of $n+1$ *direction vectors* such that $\mathbf{b} - \mathbf{a}$ lies in the $(n+1)$-dimensional polyhedral cone C spanned by V'. In Figure 2 the lattice is the hexagonal lattice, the (Voronoi) cells regular hexagons. The figure shows a line $\overline{\mathbf{ab}}$ in $\mathbb{R}^2$ and its set $V' = \{\mathbf{v}_0, \mathbf{v}_1\}$ of direction vectors.

We will assume that $\mathbf{b} - \mathbf{a}$ lies in no cone spanned by a proper subset of V'; otherwise the problem reduces to the same problem in a lower dimension. So

$$\mathbf{b} - \mathbf{a} = \sum_{\mathbf{v}' \in V'} m_{\mathbf{v}'}\, \mathbf{v}',$$

where the $m_{\mathbf{v}'}$ are uniquely determined positive integers. Let $W' = (\mathbf{w}'_1, \mathbf{w}'_2, \ldots, \mathbf{w}'_N)$ be any ordered multiset of elements from V' such that the vector $\mathbf{v}'$ appears exactly $m_{\mathbf{v}'}$ times in W'. Let

$$\mathbf{u}'_k = \mathbf{a} + \mathbf{w}'_1 + \mathbf{w}'_2 + \cdots + \mathbf{w}'_k.$$

Then the lattice points $\mathbf{a}$, $\mathbf{u}'_1$, $\mathbf{u}'_2$, $\ldots$, $\mathbf{u}'_N = \mathbf{b}$ form a discrete line joining $\mathbf{a}$ and $\mathbf{b}$. Moreover, the number $N = \sum_{\mathbf{v}' \in V'} m_{\mathbf{v}'}$ is the length of a shortest discrete line joining $\mathbf{a}$ and $\mathbf{b}$.

To solve the discrete line problem it remains to satisfy condition (B). To find the orthogonal distance $d(\mathbf{u}'_k, \overline{\mathbf{ab}})$ from each of the points $\mathbf{u}'_k$ to the line $\overline{\mathbf{ab}}$, let H be the n-dimensional hyperplane orthogonal to vector $\mathbf{b} - \mathbf{a}$, and let $\mathtt{proj}_H$ denote the orthogonal projection onto H. Further let $V = \{\mathtt{proj}_H(\mathbf{v}') \mid \mathbf{v}' \in V'\}$. By the assumption that $\mathbf{b} - \mathbf{a}$ lies in no cone spanned by a proper subset of V', every set of n vectors from V is linearly independent. Moreover, defining $m_{\mathbf{v}} := m_{\mathbf{v}'}$ if $\mathbf{v} = \mathtt{proj}_H(\mathbf{v}')$, note that

$$\sum_{\mathbf{v} \in V} m_{\mathbf{v}} = N \qquad \text{and} \qquad \sum_{\mathbf{v} \in V} m_{\mathbf{v}} \mathbf{v} = \mathbf{0}, \tag{1}$$

the latter because $\sum_{\mathbf{v}' \in V'} m_{\mathbf{v}'} \mathbf{v}' = \mathbf{b} - \mathbf{a}$ and H is orthogonal to $\mathbf{b} - \mathbf{a}$. Hence V is a basic set. Let $\mathbf{w}_i = \mathtt{proj}_H(\mathbf{w}'_i)$ for each i and let

$$\mathbf{u}_k := \mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_k.$$

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Note that $P := (\mathbf{0} = \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N = \mathbf{0})$ is a closed V-path and that $d(\mathbf{u}'_k, \overline{\mathbf{ab}}) = \mathbf{u}_k$. Hence the problem of satisfying condition (B) is exactly the problem of minimizing

$$\max_{1 \le k \le N} |\mathbf{u}_k|$$

over all closed V-paths $P = (\mathbf{0} = \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N = \mathbf{0})$ of length N. In other words, solving the discrete line problem in $\mathbb{R}^{n+1}$ reduces to solving the wandering path problem in $\mathbb{R}^n$. The requirement that the closed V-path P have the same length N as the discrete line is not a serious restriction, as explained in Remark 3 of section 4.

In the example of Figure 2 the basic set is $V = \{3c, -2c\}$, where $c = \sqrt{57}/38$. The solution to the wandering path problem for basic set V is $P = (0, -2c, c, -c, 2c, 0)$, obtained by successive additions $(-2c, +3c, -2c, 3c, -2c)$. The corresponding discrete line is

$$\begin{aligned}
&\mathbf{a}\\
&\mathbf{a} + \mathbf{v}_0\\
&\mathbf{a} + \mathbf{v}_0 + \mathbf{v}_1\\
&\mathbf{a} + \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_0\\
&\mathbf{a} + \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_0 + \mathbf{v}_1\\
&\mathbf{a} + \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_0 = \mathbf{b},
\end{aligned}$$

indicated by the shaded hexagons in Figure 2.

3. The lattices L and Λ, multiplicity, and modulus. For a real number α, vector $\mathbf{y}$, and set X of vectors we use the notation $\alpha X = \{\alpha\mathbf{x} \mid \mathbf{x} \in X\}$ and $\mathbf{y} + X = \{\mathbf{y} + \mathbf{x} \mid \mathbf{x} \in X\}$. In the first lemma, the V-multiset $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N)$ is used as an alternate way to denote the V-path $(\mathbf{0} = \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N = \mathbf{0})$, where $\mathbf{u}_k = \mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_k$. The proof of Lemma 1 is clear.

LEMMA 1. *If V is a basic set and α is a positive real number, then $w(\alpha V) = \alpha\, w(V)$. Moreover, if V-multiset $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ is an optimum wandering path for V, then $(\alpha\mathbf{w}_1, \alpha\mathbf{w}_2, \ldots, \alpha\mathbf{w}_m)$ is an optimum wandering path for αV.*

LEMMA 2. *Let V be a basic set such that $\sum_{\mathbf{v}\in V} m_{\mathbf{v}}\, \mathbf{v} = \mathbf{0}$, with the $m_{\mathbf{v}}$ relatively prime integers. Then $\sum_{\mathbf{v}\in V} m'_{\mathbf{v}}\, \mathbf{v} = \mathbf{0}$ if and only if there exists an integer c such that $m'_{\mathbf{v}} = c\, m_{\mathbf{v}}$ for all $\mathbf{v} \in V$.*

Proof. Letting $V = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$, we have

$$\sum_{i=1}^{n} \frac{m_i}{m_0}\, \mathbf{v}_i = -\mathbf{v}_0 = \sum_{i=1}^{n} \frac{m'_i}{m'_0}\, \mathbf{v}_i,$$

which implies by the linear independence of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ that $m_i/m_0 = m'_i/m'_0$ or $m'_i = (m'_0/m_0)m_i$ for all i. Since the m_i have no nontrivial common divisor, $c := m'_0/m_0$ is an integer. ∎

If V is a basic set, then, by definition, there exist positive integers $m_{\mathbf{v}}$ such that $\sum_{\mathbf{v}\in V} m_{\mathbf{v}}\mathbf{v} = \mathbf{0}$. In light of Lemma 2, call $m_{\mathbf{v}} := m_{\mathbf{v}}(V)$ the *multiplicity* of $\mathbf{v}$ in V if the $m_{\mathbf{v}}$ are relatively prime. Further denote

$$m := m(V) = \sum_{\mathbf{v}\in V} m_{\mathbf{v}},$$

where $m_{\mathbf{v}}$ is the multiplicity of $\mathbf{v}$ in V. Call $m := m(V)$ the *modulus* of V.

COROLLARY 3. *The length of any closed wandering V-path is divisible by the modulus $m(V)$.*

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Proof. If P is a closed V-path of length N and $m_{\mathbf{v}}(P)$ is the number of occurrences of $\mathbf{v}$ in P, then $\sum_{\mathbf{v}\in V} m_{\mathbf{v}}(P)\mathbf{v} = \mathbf{0}$. By Lemma 2 there is a constant c such that $m_{\mathbf{v}}(P) = c\, m_{\mathbf{v}}(V)$. Therefore $N = \sum_{\mathbf{v}\in V} m_{\mathbf{v}}(P) = c \sum_{\mathbf{v}\in V} m_{\mathbf{v}}(V) = c\, m(V)$. ∎

CONJECTURE 4. *The length of an optimum wandering path for a basic set V is equal to $m(V)$.*

Comments relevant to Conjecture 4 appear in Remark 3.

LEMMA 5. *If V is a basic set in $\mathbb{R}^n$, then the set of all integer linear combinations of elements from V is an n-dimensional lattice.*

Proof. Let L be the set of all integer linear combinations of elements from $V = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$, and let $\{m_0, m_1, \ldots, m_n\}$ be a set of integers such that $\sum_{i=0}^n m_i\, \mathbf{v}_i = \mathbf{0}$. Then by eliminating $\mathbf{v}_0$,

$$\sum_{i=0}^{n} a_i\, \mathbf{v}_i = \sum_{i=1}^{n} \frac{m_0 a_i - m_i a_0}{m_0}\, \mathbf{v}_i$$

for any integers a_i. Therefore $\mathbf{x} \in L$ if and only if $\mathbf{x} = \sum_{i=1}^n \beta_i \mathbf{v}_i$, where $\beta_i = b_i/m_0$ and $(b_1, \ldots, b_n)$ is a multiple of $(m_1, m_2, \ldots, m_n)$ modulo m_0. It readily follows that L is a sublattice of the lattice generated by the vectors $\frac{1}{m_0}\, \mathbf{v}_i,\ 1 \le i \le n$. ∎

The following is a converse of Lemma 5.

LEMMA 6. *Let V be any set of $n+1$ points of an n-dimensional lattice, every n of which are linearly independent. Then V is a basic set if and only if V is not contained in any closed half-space determined by a hyperplane through the origin.*

Proof. Clearly, if V is contained in some half-space, then $\sum_{i=0}^n m_i\, \mathbf{v}_i = \mathbf{0}$ is impossible for positive integers m_i. Conversely, assume that V is not a basic set. Since V is a dependent set of lattice points, $\sum_{i=0}^n a_i\, \mathbf{v}_i = \mathbf{0}$ for some integers a_i. It is not possible that all the a_i are positive, since there exists no closed V-path. Thus $\mathbf{v}_0 = \sum_{i=1}^n b_i\, \mathbf{v}_i$, where at least one of the b_i, say b_n without loss of generality, is positive. Let $\mathbf{x}$ be a vector orthogonal to $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$. Then $\langle \mathbf{x}, \mathbf{v}_i \rangle = 0$ for $i = 1, 2, \ldots, n-1$ and $\langle \mathbf{x}, \mathbf{v}_0 \rangle = \sum_{i=1}^n b_i \langle \mathbf{x}, \mathbf{v}_i \rangle = b_n \langle \mathbf{x}, \mathbf{v}_n \rangle$. This shows that V is contained in the closed positive half-space determined by $\mathbf{x}$. ∎

In light of Lemma 5, let $L := L(V)$ denote the lattice of all integer linear combinations of elements from V:

$$L(V) := \left\{ \sum_{\mathbf{v}\in V} a_{\mathbf{v}} \mathbf{v} \mid a_{\mathbf{v}} \in \mathbb{Z} \right\}.$$

The sublattice $\Lambda := \Lambda(V) \subset L(V)$ defined by

$$\Lambda(V) := \left\{ \sum_{\mathbf{v}\in V} a_{\mathbf{v}} \mathbf{v} \mid a_{\mathbf{v}} \in \mathbb{Z},\ \sum_{\mathbf{v}\in V} a_{\mathbf{v}} = 0 \right\}$$

also plays an important role in the wandering path problem. For any $\mathbf{v}_0 \in V$ the lattice $\Lambda(V)$ is generated by the set $\{\mathbf{v} - \mathbf{v}_0, \mid \mathbf{v} \in V \setminus \{\mathbf{v}_0\}\}$ of vectors. For $\mathbf{x}, \mathbf{y} \in L$ define

$$\mathbf{x} \equiv \mathbf{y} \pmod{\Lambda} \quad \text{if} \quad \mathbf{x} - \mathbf{y} \in \Lambda.$$

LEMMA 7. *If V is a basic set with modulus m, then*

$$\sum_{\mathbf{v}\in V} a_{\mathbf{v}} \mathbf{v} \equiv \mathbf{0}\ (\textit{mod}\ \Lambda) \quad \textit{if and only if} \quad \sum_{\mathbf{v}\in V} a_{\mathbf{v}} \equiv 0\ (\textit{mod}\ m).$$

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Proof. By definition, $\sum_{\mathbf{v}\in V} a_{\mathbf{v}}\mathbf{v} \equiv \mathbf{0}$ if and only if $\sum_{\mathbf{v}\in V} a_{\mathbf{v}}\mathbf{v} \in \Lambda$ if and only if there exist integers $b_{\mathbf{v}}$ such that $\sum_{\mathbf{v}\in V} a_{\mathbf{v}}\mathbf{v} = \sum_{\mathbf{v}\in V} b_{\mathbf{v}}\mathbf{v}$, where $\sum_{\mathbf{v}\in V} b_{\mathbf{v}} = 0$. This occurs if and only if $\sum_{\mathbf{v}\in V}(a_{\mathbf{v}}\mathbf{v} - b_{\mathbf{v}}\mathbf{v}) = \mathbf{0}$, which, by Lemma 2, occurs if and only if there is an integer c such that $a_{\mathbf{v}} - b_{\mathbf{v}} = c\,m_{\mathbf{v}}$ for all $\mathbf{v} \in V$. In one direction this implies that $\sum_{\mathbf{v}\in V} a_{\mathbf{v}} = \sum_{\mathbf{v}\in V} b_{\mathbf{v}} + c\sum_{\mathbf{v}\in V} m_{\mathbf{v}} = c\,m$. So $\sum_{\mathbf{v}\in V} a_{\mathbf{v}} \equiv 0 \pmod{m}$. In the other direction, if $\sum_{\mathbf{v}\in V} a_{\mathbf{v}} \equiv 0 \pmod{m}$, then take $b_{\mathbf{v}} = a_{\mathbf{v}} - c\,m_{\mathbf{v}}$, in which case $\sum_{\mathbf{v}\in V} b_{\mathbf{v}} = \sum_{\mathbf{v}\in V} a_{\mathbf{v}} - c\sum_{\mathbf{v}\in V} m_{\mathbf{v}} = 0$. ∎

COROLLARY 8. *If V is a basic set with modulus $m(V)$, then the order of the quotient group L/Λ is $m(V)$.*

4. First algorithm. The first result in this section is a lower bound on the optimum wandering distance. The first of two algorithms for the wandering path problem is then presented. The input is a basic set V, the output a closed V-path that is near optimum.

PROPOSITION 9. *If V is a basic set, then*

$$w(V) \geq \frac{1}{2}\max_{\mathbf{v}\in V}|\mathbf{v}|\,.$$

Proof. Let $\mathbf{v} \in V$ be an element that realizes the maximum in $\max_{\mathbf{v}\in V}|\mathbf{v}|$, and let $\mathbf{u}$ and $\mathbf{u}'$ be two consecutive points in an optimum wandering path such that $\mathbf{v} = \mathbf{u} - \mathbf{u}'$. By the triangle inequality $2\,w(V) \geq |\mathbf{u}| + |\mathbf{u}'| \geq |\mathbf{v}|$. ∎

Although this lower bound is somewhat trivial, it is, in a sense, best possible. For the following family in $\mathbb{R}^2$, for example, the bound is achieved: $V = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$,

$$\begin{aligned}\mathbf{v}_0 &= (1, \beta),\\ \mathbf{v}_1 &= (1, -\beta),\\ \mathbf{v}_2 &= (-4k, 0),\end{aligned}$$

where $k \geq 2$ is an integer and $0 < \beta \leq \sqrt{3}$. In this case the optimum wandering path is

$$(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_0, \mathbf{v}_1),$$

where $\mathbf{v}_0, \mathbf{v}_1$ is repeated k time on each side of $\mathbf{v}_2$. Then

$$w(V) = w(P) = 2k = \frac{1}{2}|\mathbf{v}_2| = \frac{1}{2}\max_{\mathbf{v}\in V}|\mathbf{v}|.$$

In Algorithm 1 below, the notation $C'_{\mathbf{v}}$ stands for the polyhedral cone spanned by the vectors in $V \setminus \{\mathbf{v}\}$:

$$C'_{\mathbf{v}} := \left\{ \sum_{\mathbf{u}\in V\setminus\{\mathbf{v}\}} \alpha_{\mathbf{u}}\,\mathbf{u} \;\middle|\; \alpha_{\mathbf{u}} \geq 0 \right\}.$$

Let

$$\mathbf{x}_0 = \frac{1}{2}\sum_{\mathbf{v}\in V}\mathbf{v} \qquad \text{and} \qquad C_{\mathbf{v}} = -\mathbf{x}_0 + C'_{\mathbf{v}}.$$

Then $C_{\mathbf{v}}$ is a copy of $C'_{\mathbf{v}}$ translated by $-\mathbf{x}_0$. It follows from Lemma 6 that $\bigcup_{\mathbf{v}\in V} C'_{\mathbf{v}} = \mathbb{R}^n$, and therefore

$$\bigcup_{\mathbf{v}\in V} C_{\mathbf{v}} = \mathbb{R}^n. \tag{2}$$

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ALGORITHM 1.
`Input`: *A basic set V in $\mathbb{R}^n$.*
`Output`: *A closed V-path P in $\mathbb{R}^n$.*
`initialize`: $i = 0$, $\mathbf{u}_0 = \mathbf{0}$
`until` $i = m(V)$ `do`
  find a $\mathbf{v} \in V$ *such that* $\mathbf{u}_i \in C_{\mathbf{v}}$
  $\mathbf{u}_{i+1} \leftarrow \mathbf{u}_i + \mathbf{v}$
  $i \leftarrow i + 1$
`end`
`return`: *path* $P = (\mathbf{0}, \mathbf{u}_1, \ldots, \mathbf{u}_{m(V)} = \mathbf{0})$.

THEOREM 10. *If V is a basic set with modulus $m(V)$, then Algorithm 1 finds a closed V-path P of length $m(V)$ with*

$$w(P) \leq \frac{1}{2} \max \left| \sum_{\mathbf{v} \in V} \pm \mathbf{v} \right|,$$

where the maximum is taken over all choices of signs $\pm$.

Proof. Note that (2) insures that the main step in the algorithm (find a $\mathbf{v} \in V$ such that $\mathbf{u}_i \in C_{\mathbf{v}}$) is always possible.

Let D' be the zonotope generated by V; in other words,

$$D' = \left\{ \sum_{\mathbf{v} \in V} \alpha_{\mathbf{v}} \mathbf{v} \mid 0 \leq \alpha_{\mathbf{v}} \leq 1 \right\},$$

and let $D = -\mathbf{x}_0 + D'$ be the translate of D' by $-\mathbf{x}_0$. We first show by induction that, at each iteration of Algorithm 1, the point $\mathbf{u}_i \in D$. The point $\mathbf{0} \in D$; in fact, it is the barycenter of D because $\mathbf{x}_0$ is the barycenter of D'. If $\mathbf{u}_i \in D \cap C_{\mathbf{v}}$, then $\mathbf{x}_0 + \mathbf{u}_i \in D' \cap C'_{\mathbf{v}}$, which, by the definition of D', implies that $\mathbf{x}_0 + \mathbf{u}_i + \mathbf{v} \in D'$. Therefore $\mathbf{u}_{i+1} = \mathbf{u}_i + \mathbf{v} \in D$.

Let G be the group consisting of translations of $\mathbb{R}^n$ by the vectors in $\Lambda(V)$. The quotient space $\mathbb{R}^n/G$ is often referred to as a *fundamental domain*. Such a fundamental domain contains exactly one representative from each coset of L/Λ. Note that D', and therefore D, is such a fundamental domain.

The points in the path constructed by Algorithm 1 are of the form $\mathbf{u}_i = \sum_{\mathbf{v} \in V} a_{i,\mathbf{v}} \mathbf{v}$, where the $a_{i,\mathbf{v}}$ are positive integers such that $\sum_{\mathbf{v} \in V} a_{i,\mathbf{v}} = i$. Letting m be the modulus, $\sum_{\mathbf{v} \in V} a_{m,\mathbf{v}} \equiv 0 \pmod{m}$, which by Lemma 7 implies that $\mathbf{u}_m \in \Lambda$. By the facts proved above, $\mathbf{0} \in D$ and $\mathbf{u}_m \in D$. But D can contain only one representative from each coset of L/Λ. Therefore $\mathbf{u}_m = \mathbf{0}$. Thus Algorithm 1 returns to $\mathbf{0}$ in $m(V)$ steps; it cannot, by Lemma 2, return sooner.

Also $w(P)$ is bounded above by

$$\max_{\mathbf{x} \in D} |\mathbf{x}| = \max_{\mathbf{x} \in D'} |-\mathbf{x}_0 + \mathbf{x}| = \left| -\frac{1}{2} \sum_{\mathbf{v} \in V} \mathbf{v} + \max_{Y \subsetneq V} \sum_{\mathbf{y} \in Y} \mathbf{y} \right| = \frac{1}{2} \max \left| \sum_{\mathbf{v} \in V} \pm \mathbf{v} \right|. \qquad \square$$

Remark 1. In dimension n, the difference between the upper bound in Theorem 10 and the lower bound in Proposition 9 is

$$\frac{1}{2} \max \left| \sum_{\mathbf{v} \in V} \pm \mathbf{v} \right| - \frac{1}{2} \max_{\mathbf{v} \in V} |\mathbf{v}| \leq \frac{n}{2} \max_{\mathbf{v} \in V} |\mathbf{v}|.$$

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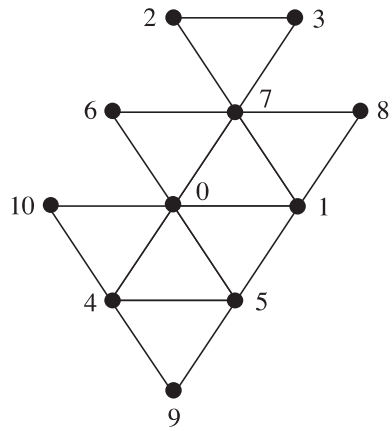

FIG. 3. *A closed V-path using Algorithm* 1.

Remark 2. In section 5 it will be shown that Algorithm 1 always finds an optimum wandering path in the 1-dimensional case. This is not necessarily true in higher dimensions, although finding examples is not completely trivial. In dimension 2 consider the basic set consisting of the following three vectors:

$$\begin{aligned}\mathbf{v}_0 &= (4, 0),\\ \mathbf{v}_1 &= (0, 3),\\ \mathbf{v}_2 &= (-7, -8).\end{aligned}$$

Algorithm 1 produces a closed wandering path P of length $m = 65$ with $w(P) = 5\sqrt{2} \approx 7.07$, where the point of P furthest from the origin is $(5, 5)$. However, the optimum wandering path Q (of the same length 65) has $w(Q) = \sqrt{41} \approx 6.40$, where the point furthest from the origin is $(5, 4)$. Another example is the 2-dimensional example

$$\begin{aligned}\mathbf{v}_0 &= (1,\, 0),\\ \mathbf{v}_1 &= (-1,\, \sqrt{3}),\\ \mathbf{v}_2 &= (-\tfrac{3}{2},\, -\tfrac{3}{2}\sqrt{3}),\end{aligned}$$

mentioned in section 2. The closed V-path $P = (\mathbf{0} = \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{11} = \mathbf{0})$ found by Algorithm 1 is shown in Figure 3, where the labels indicate the indices. The vectors of V are successively added in the order $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_0)$. The optimum wandering path for V is shown in Figure 1. Note that $w(P) = 2 > \sqrt{3} = w(V)$.

Remark 3. Algorithm 1 for the wandering path problem implies a corresponding algorithm for the discrete line problem. To obtain a solution to the discrete line problem from the wandering path problem, what is required is a closed V-path P of the same length N as the discrete line. By (1), the length N of the discrete line is also the length of some closed wandering V-path. Corollary 3 then implies that N is a multiple of $m(V)$. According to Theorem 10, the output of Algorithm 1 is a closed V-path P_0 of length $m(V)$. Therefore it suffices to take for P the path P_0 concatenated with itself $N/m(V)$ times, in which case $w(P) = w(P_0)$.

A discussion of the upper bound on $w(V)$ given in Theorem 10 is postponed until Algorithm 2 is introduced in section 6. In that section the bounds for the two

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algorithms are compared. The output of Algorithm 1 is a closed V-path P_0 of length $m(V)$. If Conjecture 4 is true, then the same can be said of the optimum wandering path.

5. The wandering path problem in dimension 1. In one dimension a basic set is just a pair $\{\alpha, \beta\}$ of real numbers for which there exist integers m_α and m_β such that $m_\alpha\alpha + m_\beta\beta = 0$. If α, β is such a basic set, then it is easy to find a relatively prime pair of integers a, b and a real number λ such that $\alpha = \lambda a$ and $\beta = \lambda b$. Therefore, by Lemma 1, there is no loss of generality in assuming, in the 1-dimensional case, that the basic set is $V = \{a, b\}$, where a and b are a relatively prime pair of integers.

THEOREM 11. *Let $V = \{a, b\}$ be a basic set in $\mathbb{R}$ with a and b relatively prime. Then $w(V) = \lfloor(|a| + |b|)/2\rfloor$. Moreover, Algorithm* 1.1 *below finds an optimum wandering path of length $|a| + |b|$.*

ALGORITHM 1.1.

`Input`*: A basic set $V = \{a, b\}$*
(without loss of generality, a, b are relatively prime and $a > 0$, $b < 0$).
`Output`*: An optimum wandering path for V.*
`initialize`*: $i = 0$, $u_0 = 0$*
`until` $i = |a| + |b|$ `do`
  `if` $u_i \geq -(a + b)/2$ `then`
    $u_{i+1} \leftarrow u_i + b$
  `else`
    $u_{i+1} \leftarrow u_i + a$
  $i \leftarrow i + 1$
`end`
`return`: *Path $P = (0, u_1, u_1, \ldots, u_{|a|+|b|} = 0)$.*

Proof. Algorithm 1.1 is exactly the 1-dimensional case of Algorithm 1 in the previous section. In this case the modulus $m(V) = |a| + |b|$. Theorem 10 then implies that Algorithm 1.1 finds a closed V-path P of length $|a|+|b|$ with $w(P) \leq \frac{1}{2}(|a|+|b|)$. Therefore

$$w(V) \leq \left\lfloor \frac{|a| + |b|}{2} \right\rfloor.$$

However, in the closed interval between $-\lfloor\frac{|a|+|b|}{2}\rfloor$ and $+\lfloor\frac{|a|+|b|}{2}\rfloor$ there are at most $|a|+|b|+1$ integers. By Corollary 3, any closed V-path P has at least $m(V) = |a|+|b|$ distinct points. Hence one of these points must be either $-\lfloor\frac{|a|+|b|}{2}\rfloor$, $+\lfloor\frac{|a|+|b|}{2}\rfloor$, or a point even further from the origin . Thus

$$w(V) \geq \left\lfloor \frac{|a| + |b|}{2} \right\rfloor. \qquad \square$$

6. Second algorithm. Our second algorithm for the wandering path problem is a "greedy" algorithm, choosing at each step the vector that brings the path closest to the origin.

ALGORITHM 2.

`Input`*: A basic set V in $\mathbb{R}^n$.*
`Output`*: A closed V-path P in $\mathbb{R}^n$.*
`initialize`*: $i = 0$, $\mathbf{u}_0 = \mathbf{0}$*
`until` $\mathbf{u}_i = \mathbf{u}_{i_0}$ *for some $i_0 < i$* `do`

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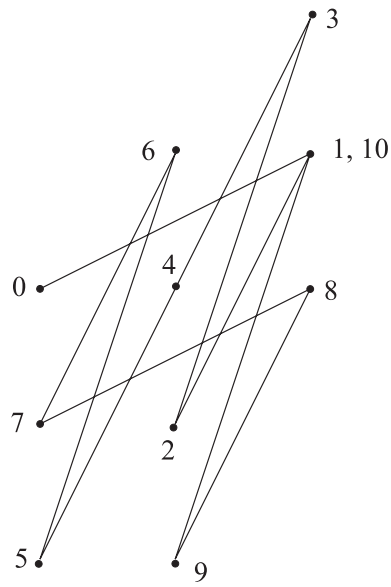

FIG. 4. *Closed V-path found using Algorithm* 2.

choose $\mathbf{v} \in V$ *such that* $|\mathbf{u}_i + \mathbf{v}| \leq |\mathbf{u}_i + \mathbf{v}'|$ *for all* $\mathbf{v}' \in V$
$\mathbf{u}_{i+1} \leftarrow \mathbf{u}_i + \mathbf{v}$
$i \leftarrow i + 1$
`end`
`return:` *path* $P = (0, \mathbf{u}_{i_0+1} - \mathbf{u}_{i_0}, \mathbf{u}_{i_0+2} - \mathbf{u}_{i_0}, \ldots, \mathbf{0})$.

Remark 4. The translation by $-\mathbf{u}_{i_0}$ in Algorithm 2 is sometimes necessary. An example in dimension 2 is $V = \{(1,3), (2,1), (-1,-2)\}$. The V-path ($\mathbf{0} = \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{10} = \mathbf{u}_1$) found by Algorithm 2 is shown in Figure 4. The labels indicate the order in which the points of the path are found by Algorithm 2. In this case $i_0 = 1$. The output from Algorithm 2 is the closed V-path that starts and ends at $\mathbf{u}_1$ (labeled 1 in the figure). This V-path is translated so that $\mathbf{u}_1$ sits at the origin.

If V is a basic set, then by Lemma 6 the convex hull of V is an n-simplex $\Delta := \Delta(V)$ containing the origin. For $\mathbf{v} \in V$ let $\Delta_{\mathbf{v}}$ denote the n-simplex with vertex set $V \cup \{\mathbf{0}\} \setminus \{\mathbf{v}\}$. For any n-simplex Δ, let $R(\Delta)$ denote its circumradius.

THEOREM 12. *If V is a basic set, then Algorithm* 2 *finds a closed V-path that is contained in a ball of radius*

$$r(V) \leq \max\{R(\Delta), R(\Delta_{\mathbf{v}}) \,|\, \mathbf{v} \in V\}.$$

Before proving Theorem 12 several more remarks are in order.

Remark 5. Unlike Theorem 10, Theorem 12 does not state the length of the constructed path. We conjecture that the length is the modulus $m(V)$.

Remark 6. It is possible to find an explicit formula for the upper bound in Theorem 12. Let $V = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$. The circumcenter $(x_1, \ldots, x_n)$ of $\Delta_k := \Delta_{\mathbf{v}_k}$ can be found by solving a system of linear equations as follows. Because each vertex of Δ_k is equidistant from the circumcenter of Δ_k,

$$\sum_j x_j^2 = \sum_j (x_j - v_{ij})^2$$

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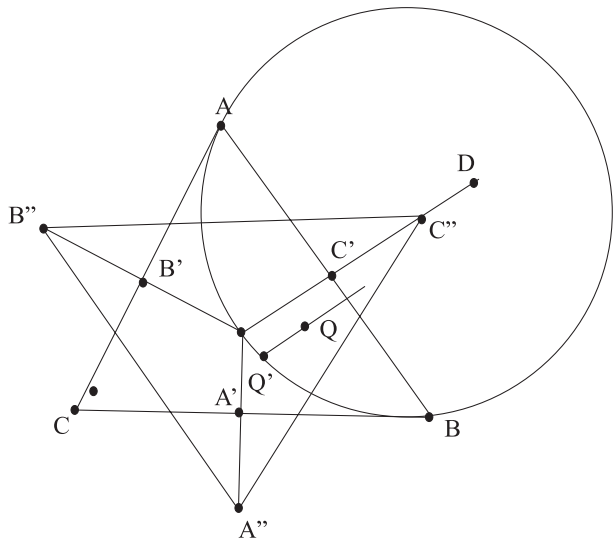

FIG. 5. *Proof of Theorem* 13.

for $i = 0, 1, 2, \ldots, k-1, k+1, \ldots, n$, where $\mathbf{v}_i = (v_{i1}, \ldots, v_{in})$. This simplifies to the $n \times n$ linear system

$$\sum_j v_{ij}\, x_j \;=\; \frac{1}{2} \sum_j v_{ij}^2.$$

Cramer's rule provides a determinantal formula for the circumradius $R(\Delta_k)$. Let $M_k = (v_{ij})$, and let M_{kjt} denote the result of replacing the jth column of M_k by the coordinatewise square of the tth column of M_k. Then

$$\begin{aligned} x_j \;&=\; (\textstyle\sum_{t=1}^n \det M_{kjt})/(2 \det M_k), \\ R(\Delta_k) \;&=\; (\textstyle\sum_{j=1}^n x_j^2)^{\frac{1}{2}}. \end{aligned}$$

A similar formula is easily obtained for $R(\Delta)$ by translating one vertex to the origin.

Remark 7. The following theorem allows the upper bound $\max\{R(\Delta), R(\Delta_{\mathbf{v}}) \mid \mathbf{v} \in V\}$ in Theorem 12 to be simplified to $\max\{R(\Delta_{\mathbf{v}}) \mid \mathbf{v} \in V\}$ in the 2-dimensional case. We conjecture that the statement is true for a simplex Δ of arbitrary dimension $n \geq 2$, but the proof given below for a triangle does not seem to extend to higher dimensions.

THEOREM 13. *Let V denote the set of vertices of a 2-simplex Δ. If $\mathbf{v} \in V$ and $\mathbf{x}$ is any point in Δ, let $\Delta_{\mathbf{v}}^{\mathbf{x}}$ denote the n-simplex with vertex set $V \cup \{\mathbf{x}\} \setminus \{\mathbf{v}\}$. Then*

$$R(\Delta) \leq \max_{\mathbf{v} \in V} R(\Delta_{\mathbf{v}}^{\mathbf{x}}).$$

Proof. Denote the vertices of triangle Δ by A, B, C and its circumradius by R. We will prove the theorem in the case that Δ is an acute triangle, leaving the easy case of Δ obtuse to the reader. Let A', B', C' be the midpoints of sides BC, CA, AB, respectively. (See Figure 5.) Then triangle $\Delta' = \triangle(A'B'C')$ is similar to triangle Δ with ratio 1/2, and with the same circumcenter O as triangle Δ. Since Δ is acute, O lies within triangle Δ. Let A'' be a point on the line OA' such that the distance

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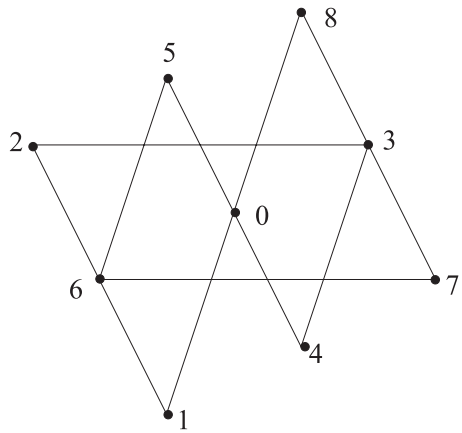

FIG. 6. *An optimum wandering path.*

OA'' is twice the distance OA' and such that A' lies between O and A'', similarly for B'' and C''. Note that $d(C'', A) = d(O, A) = R$ and $d(C'', B) = d(O, B) = R$, where d denotes Euclidean distance. Triangles Δ' and $\Delta'' = \triangle(A'', B''C'')$ are similar with ratio 2 implying that triangles Δ and Δ'' are congruent. Let O'' be the circumcenter of Δ''. Since $\Delta'' \cong \Delta$ we have $d(O'', C'') = R$. (Although not necessary for this proof, it is not hard to show that O'' is, in fact, the orthocenter of Δ, the intersection of the three altitudes.) Let S be the circumscribed circle of $\triangle(AO''B)$, which we have proved has center C''.

Let Q be an arbitrary point inside Δ. The three line segments AO'', BO'', CO'' subdivide Δ into three (some perhaps degenerate) triangles. Without loss of generality, assume that Q lies in triangle $AO''B$. Let R_Q be the circumradius of $\triangle(AQB)$, and let Q' be the point of intersection (inside Δ) of S with the line $\mathcal{L}$ through Q perpendicular to AB. The intersection of the perpendicular bisectors of the three line segments $AB, AQ', Q'B$ is C'', the center of the circle S. The circumcenter D of AQB lies on the ray OC'' because D is the intersection of the perpendicular bisectors of the three sides of $\triangle(AQB)$, and OC'' is the perpendicular bisector of side AB. On line $\mathcal{L}$, the point Q is closer to line AB than is Q'. It easily follows that $d(O, D) \geq d(O, C'')$, which in turn implies that $R_Q = d(D, A) \geq d(C'', A) = R$. This completes the proof. $\square$

Remark 8. There is the question of which of the two algorithms, Algorithm 1 or 2, gives the better result. Often they both find the optimum wandering path. Consider the following examples in $\mathbb{R}^2$, where the basic set is $V = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$:

$$\begin{aligned} \mathbf{v}_0 &= (5, 0), & \mathbf{v}_0 &= (5, 0), \\ \mathbf{v}_1 &= (-2, 4), & \mathbf{v}_1 &= (-2, 4), \\ \mathbf{v}_2 &= (-1, -3), & \mathbf{v}_2 &= (-2, -3). \end{aligned}$$

For the first example both algorithms find the same optimum wandering path,

$$P = (0,0),\ (-1,-3),\ (-3,1),\ (2,1),\ (1,-2),\ (-1,2),\ (-2,-1),\ (3,-1),\ (1,3),\ (0,0),$$

of length 9 with $w(V) = \sqrt{10} \approx 3.16$. This optimum wandering path, is shown in Figure 6.

For the second example both algorithms find the same optimum wandering path of length 49 with $w(V) = \sqrt{17} \approx 4.12$. For this example the upper bound provided

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for Algorithm 1 in Theorem 10 is $\sqrt{82}/2 \approx 4.53$, while the upper bound provided for Algorithm 2 in Theorem 12 is $\sqrt{389}/4 \approx 4.93$. In all the examples we have tried, the upper bound from Theorem 10 is less than or equal to the upper bound from Theorem 12.

It is not always the case that both algorithms find an optimum wandering path. It was noted in Remark 2 of section 4 that in neither of the following two examples does Algorithm 1 find an optimum wandering path. However, Algorithm 2 does find an optimum wandering path in both cases:

$$\begin{aligned} \mathbf{v}_0 &= (1,\, 0), & \mathbf{v}_0 &= (4, 0), \\ \mathbf{v}_1 &= (-1,\, \sqrt{3}), & \mathbf{v}_1 &= (0, 3), \\ \mathbf{v}_2 &= (-\tfrac{3}{2},\, -\tfrac{3}{2}\sqrt{3}), & \mathbf{v}_2 &= (-7, -8). \end{aligned}$$

For the first example, the optimum wandering path found using Algorithm 2 is shown in Figure 1, while the nonoptimum path found by Algorithm 1 is shown in Figure 3. For the second example, Algorithm 2 finds an optimum wandering path P_2 of length 65 with $w(P_2) = w(V) = \sqrt{41} \approx 6.40$, less than $w(P_1) = 5\sqrt{2} \approx 7.07$ for the closed V-path P_1 of the same length found by Algorithm 1.

On the other hand, there are examples for which Algorithm 1 finds an optimum wandering path while Algorithm 2 does not. For example, let $V = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$, where

$$\begin{aligned} \mathbf{v}_0 &= (3,\, 1), \\ \mathbf{v}_1 &= (4,\, 1), \\ \mathbf{v}_2 &= (-14,\, -4). \end{aligned}$$

Algorithm 2 finds the closed V-path

$$P_2 = (0,0), (3,1), (6,2), (-8,-2), (-4,-1), (0,0),$$

with $w(P_2) = 2\sqrt{17} \approx 8.25$, while Algorithm 1 finds an optimum wandering path

$$P_1 = (0,0), (4,1), (7,2), (-7,2), (-3,-1), (0,0),$$

with $w(P_1) = w(V) = \sqrt{53} \approx 7.28$.

Proof of Theorem 12. The strategy is to find a ball B of radius $\max\{R(\Delta),\, R(\Delta_{\mathbf{v}}) \mid \mathbf{v} \in V\}$ such that, for all $\mathbf{x} \in B$, there is a $\mathbf{v} \in V$ such that $\mathbf{x} + \mathbf{v} \in B$. For distinct $\mathbf{u}, \mathbf{v} \in V$ let $H_{\mathbf{uv}}$ denote the hyperplane that is the perpendicular bisector of the segment joining $-\mathbf{u}$ and $-\mathbf{v}$. The intersection of all the $H_{\mathbf{uv}}$ is the circumcenter O of the simplex $-\Delta$, which is the convex hull of $-V$. Let $H^+_{\mathbf{uv}}$ be the closed half-space that contains $-\mathbf{v}$ determined by $H_{\mathbf{uv}}$, and let

$$P_{\mathbf{v}} = \bigcap_{\mathbf{u} \in V \setminus \{\mathbf{v}\}} H^+_{\mathbf{uv}}.$$

Note that

$$P_{\mathbf{v}} = \{\mathbf{x} \,:\, |\mathbf{x} + \mathbf{v}| \le |\mathbf{x} + \mathbf{u}| \text{ for all } \mathbf{u} \in V \setminus \{\mathbf{v}\}\}.$$

The $P_{\mathbf{v}}$, $\mathbf{v} \in V$, partition $\mathbb{R}^n$ into $n+1$ polyhedral cones with vertex at O.

For $\mathbf{v} \in V$, let $H_{\mathbf{v}}$ denote the hyperplane that is the perpendicular bisector of the segment joining $-\mathbf{v}$ and $\mathbf{0}$. Then

$$O_{\mathbf{v}} = \bigcap_{\mathbf{u} \in V \setminus \{\mathbf{v}\}} H_{\mathbf{u}}$$

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is the circumcenter of $-\Delta_{\mathbf{v}}$. Therefore $O_{\mathbf{v}}$ also lies on each hyperplane $H_{\mathbf{u}\mathbf{u}'}$ with $\mathbf{u}, \mathbf{u}' \in V \setminus \{\mathbf{v}\}$. Let $H_{\mathbf{v}}^+$ denote the closed half-space that contains O determined by the hyperplane $H_{\mathbf{v}}$, and let $H_{\mathbf{v}}^-$ denote the complementary closed half-space. Further partition each $P_{\mathbf{v}}$ as follows. Let

$$P_{\mathbf{v}}^+ = P_{\mathbf{v}} \cap H_{\mathbf{v}}^+, \qquad P_{\mathbf{v}}^- = P_{\mathbf{v}} \cap H_{\mathbf{v}}^-.$$

Note that $P_{\mathbf{v}}^+$ is a simplex with vertex set $\{O\} \cup \bigcup_{\mathbf{u} \in V \setminus \{\mathbf{v}\}} O_{\mathbf{u}}$. Any point $\mathbf{x}$ in $P_{\mathbf{v}}^-$ satisfies the properties

a. $|\mathbf{x} + \mathbf{v} \leq |\mathbf{x} + \mathbf{u}|$ for all $\mathbf{u} \in V \setminus \{\mathbf{v}\}$, and
b. $|\mathbf{x} + \mathbf{v}| \leq |\mathbf{x}|$.

Moreover, any point $\mathbf{x}$ in $P_{\mathbf{v}}^+$ satisfies the properties

c. $|\mathbf{x} + \mathbf{v}| \leq |\mathbf{x} + \mathbf{u}|$ for all $\mathbf{u} \in V \setminus \{\mathbf{v}\}$, and
d. $|\mathbf{x} + \mathbf{v}| \leq \max_{\mathbf{u} \in V \setminus \{\mathbf{v}\}} \{|O + \mathbf{v}|, |O_{\mathbf{u}} + \mathbf{v}|\}$,

inequality (d) following from the fact that $\mathbf{x}$ lies within the simplex $P_{\mathbf{v}}^+$ with vertex set $\{O\} \cup \bigcup_{\mathbf{u} \in V \setminus \{\mathbf{v}\}} O_{\mathbf{u}}$. But for $\mathbf{u} \in V \setminus \{\mathbf{v}\}$ we have $|O_{\mathbf{u}} + \mathbf{v}| = |O_{\mathbf{u}}|$, because $O_{\mathbf{u}}$ lies on the hyperplane $H_{\mathbf{v}}$, which is the perpendicular bisector of the line segment joining $\mathbf{0}$ and $-\mathbf{v}$. Also $|O_{\mathbf{u}}|$ is the circumradius of $-\Delta_{\mathbf{u}}$, and $|O + \mathbf{v}|$ is the circumradius of $-\Delta$. From property (d) it follows that for any $\mathbf{x} \in P_{\mathbf{v}}^+$ we have

e. $|\mathbf{x} + \mathbf{v}| \leq \max\{R(-\Delta), R(-\Delta_{\mathbf{v}}) \mid \mathbf{v} \in V\} = \max\{R(\Delta), R(\Delta_{\mathbf{v}}) \mid \mathbf{v} \in V\}$.

Properties (a) and (c) are the greedy property, and (b) and (e) insure that at each iteration i of Algorithm 2,

$$|\mathbf{u}_i| \leq \max\{R(\Delta), R(\Delta_{\mathbf{v}}) \mid \mathbf{v} \in V\}.$$

Because $\{\mathbf{0} = \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots\}$ is a bounded set and is a set of lattice points in L (by Lemma 5), it follows that there is a point of the constructed V-path that appears at least twice in this sequence. If $\mathbf{u}_{i_0}$ is the first such point, then the closed V-path $P = (\mathbf{u}_{i_0}, \mathbf{u}_{i_0+1}, \mathbf{u}_{i_0+2}, \ldots, \mathbf{u}_{i_0})$ is contained in a ball or radius $\max_{\mathbf{v} \in V} R(\Delta_{\mathbf{v}})$. ∎

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