Locally Homogeneous Graphs from Groups

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ABSTRACT

A graph is called locally homogeneous if the subgraphs induced at any two points are isomorphic. In this Note we give a method for constructing locally homogeneous graphs from groups. The graphs constructable in this way are exactly the locally homogeneous graphs with a point symmetric universal cover. As an example we characterize the graphs that are locally n-cycles.

1. INTRODUCTION

For a connected graph G and point v of G, let G_v be the subgraph induced by the points adjacent to v. This G is called *locally* G_0 if $G_v = G_0$ for all points v of G. A graph is called *locally homogeneous* if it is locally G_0 for some G_0. Recent work concerning local homogeneity has focused on two broad questions. First, for which G_0 does there exist a G that is locally G_0? This has been settled for G_0 a cycle, linear forest, and certain trees [1, 2]. Second is the question of characterization. For a specific graph G_0, characterize all graphs G that are locally G_0. For example, let $K(n; t)$ denote the complete multipartite graph $K(n, n, \ldots, n)$, where there are t parts. When $G_0 = K(n;t)$ there is a unique G, namely $G = K(n; t+1)$. A recent paper of Hall [4] classifies the graphs that are locally the Petersen graph. There are exactly three.

In this Note, a large class of locally homogeneous graphs are obtained using groups. As a special case we characterize the graphs that are locally n-cycles.

2. HOMOGENEOUS GRAPHS FROM GROUPS

We begin with the construction of a graph $G(\Gamma, A, B, T)$. Later in this section we relate this construction to the concept of locally homogeneous graph. Let Γ be an arbitrary group, A and B subgroups of Γ and T a subset of Γ with

Journal of Graph Theory, Vol. 5 (1981) 417–422
 CCCC 0364-9024/81/040417-06$01.00

the property that $T^{-1} = T$. Define $G(\Gamma, A, B, T)$ to be the graph whose points are the two sided cosets $\{BgA \mid g \in \Gamma\}$, where two distinct points U and V of $G(\Gamma, A, B, T)$ are defined to be adjacent if $U^{-1}V \cap T \neq \emptyset$.

For subsets $X, Y \subseteq \Gamma$, the set XY is $\{xy \mid x \in X, y \in Y\}$, and let $\langle X \rangle$ denote the subgroup of Γ generated by the elements of X. The following proposition is routinely verified.

Proposition 1. The graph $G(\Gamma, A, B, T)$ is connected if and only if $B\langle T \cup A \rangle = \Gamma$. ∎

When $A = B = \{1\}$, and T generates Γ, the definition of $G(\Gamma, A, B, T)$ coincides with that of the Cayley graph of Γ with respect to the generating set T. Recall that the points of the *Cayley graph* are the elements of Γ with two points u, v adjacent whenever $ut = v$ for some $t \in T$.

When $A = \{1\}$, $G(\Gamma, A, B, T)$ is the Schreier coset graph of Γ with respect to the subgroup B. The Schreier coset graph is a generalization of the Cayley graph, the points of the Cayley graph being regarded as cosets of the trivial subgroup $\{1\}$. The points of the Schreier coset graph are the right cosets of B, where two points U, V are joined by a line whenever $Ut = V$ for some $t \in T$.

Next consider the case $B = \{1\}$. As an abbreviation we denote $G(\Gamma, A, \{1\}, T)$ by $G(\Gamma, A, T)$. Recall that a graph is *point symmetric* if its automorphism group is transitive on points. The graphs $G(\Gamma, A, T)$ are exactly the point symmetric graphs.

Theorem 2. A graph G is point symmetric if and only if $G = G(\Gamma, A, T)$ for some group Γ, subgroup A and subset $T = T^{-1}$.

Proof. For arbitrary points $U = gA$ and $V = hA$ of $G(\Gamma, A, T)$ the mapping $G(\Gamma, A, T) \to G(\Gamma, A, T)$ given by $X \to hg^{-1}X$ defines an automorphism of $G(\Gamma, A, T)$ taking U to V. Conversely, given a point symmetric graph G, let Γ be its automorphism group, A the stabilizer subgroup of some point v_0 and $T = \{t \in \Gamma \mid t(v_0) \text{ or } t^{-1}(v_0) \text{ is adjacent to } v_0\}$. Consider the map F from the points of G to those of $G(\Gamma, A, T)$ given by $F: v \to g_vA$, where $g_v \in \Gamma$ is any automorphism taking v_0 to v. We claim that F is an isomorphism. Now g_vA consists of all elements of Γ taking v_0 to v. Hence u and v are adjacent in G if and only if $g_u^{-1}g_v(v_0)$ is adjacent to v_0. Equivalently, $g_u^{-1}g_v \in T$. Thus $F(u)$ and $F(v)$ are adjacent in $G(\Gamma, A, T)$. ∎

A map $\pi : \hat{G} \to G$ from the points of graph $\hat{G}$ to the points of graph G is called a *covering map* if $\hat{G}_v$ is mapped isomorphically onto $G_{\pi v}$ for all points v of $\hat{G}$. Let B be a subgroup of Aut G, the automorphism group of G. The points of the *quotient graph* G/B are the orbits $\langle v \rangle$ of points v in G under the action of B, where distinct points U and V are adjacent in G/B whenever there are points $u \in U$ and $v \in V$ adjacent in G. As an example, consider a group Γ, subgroups A and B and subset $T = T^{-1}$. Then B can be regarded as a subgroup of Aut $G(\Gamma, A, T)$ by defining $b(V) = bV$ for $b \in B$, in which case $G(\Gamma, A, T)/B \cong G(\Gamma, A, B, T)$. In general, if the map

$$\begin{aligned} f\colon G &\to G/B \\ v &\to \langle v \rangle \end{aligned}$$

given by

is a covering, then B is called *properly discontinuous* with respect to G. An equivalent formulation is the following: B is properly discontinuous if, for all points $u, v, f(u) = f(v)$ implies that u and v are at a distance greater than 3 in G. The next proposition follows from the fact that a point symmetric graph is locally homogeneous.

Proposition 3. If B is a properly discontinuous group of automorphism with respect to a point symmetric graph G, then G/B is locally homogeneous. ∎

A sufficient condition for $G(\Gamma, A, B, T)$ to be locally homogeneous follows from Theorem 2 and Proposition 3.

Corollary 4. If $B \leq \Gamma$, considered as a subgroup of Aut $G(\Gamma, A, T)$, is properly discontinuous, then $G(\Gamma, A, B, T)$ is locally homogeneous. ∎

Moreover, every locally homogeneous graph that is a quotient G/B of a point symmetric graph G by a properly discontinuous subgroup B is obtained by th $G(\Gamma, A, B, T)$ construction.

Theorem 5. If B is properly discontinuous with respect to a point symmetric graph G, then $G/B \cong G(\Gamma, A, B, T)$, where $\Gamma = \text{Aut}\, G, A$ is the stabilizer in Γ of any point v_0 of G and $T = \{t \in \Gamma \mid t(v_0) \text{ or } t^{-1}(v_0) \text{ is adjacent to } v_0\}$.

Proof. For a point $\langle v \rangle$ of G/B let $g_{\langle v \rangle}$ be any automorphism of G taking v_0 to v. It is easy to show that $Bg_{\langle v \rangle}A$ depends only on the orbit $\langle v \rangle$ and not on the representative v, so that the function $F\colon G/B \to G(\Gamma, A, B, T)$ given by $\langle v \rangle \to Bg_{\langle v \rangle}A$ is well defined. The proof that F is an isomorphism proceeds exactly as in Theorem 2. ∎

3. UNIVERSAL COVER

The main result of this section utilizes a notion due to Ronan [6]. For a graph G let ΔG be the 2-dimensional simplicial complex formed from G by adding a 2-simplex for each triple of mutually adjacent points of G. Let $\Delta\pi\colon \widetilde{\Delta G} \to \Delta G$ be the universal topological covering of ΔG and $\pi\colon \widetilde{G} \to G$ the restriction of $\Delta\pi$ to the 1-skeleton of $\widetilde{\Delta G}$. The graph $\widetilde{G}$ will be called the Δ-*universal* *cover* of G. (We remark that $\widetilde{G}$ should not be confused with the universal cover of G regarded as a 1-dimensional simplicial complex.) The graph $\widetilde{G}$ is universal in the sense that if φ: $\widehat{G} \to G$ is any cover, then there is a cover ψ: $\widetilde{G} \to \widehat{G}$ yielding a commutative diagram:

$$\begin{tikzcd} & \widetilde{G} \arrow[dl, dashed, "\psi"'] \arrow[d, "\pi"] \\ \widehat{G} \arrow[r, "\varphi"'] & G \end{tikzcd}$$

Denote by $\Gamma(\widetilde{G}, G)$ the group of covering transformations of π:$\widetilde{G} \to G$, ie. the elements of Aut $\widetilde{G}$ preserving fibers. If $\Delta\Gamma(\Delta G, \Delta G)$ denotes the group of covering transformations of $\Delta\pi$:$\Delta \widetilde{G} \to \Delta G$, then certainly $\Delta\Gamma(\Delta G, \Delta G)$ and $\Gamma(\widetilde{G}, G)$ are equal as permutation groups acting on vertices. Hence, by standard results on covering spaces [7], $\Gamma(\widetilde{G}, G)$ is transitive on the points of each fiber and is isomorphic to the fundamental group $\pi_1(\Delta G)$.

Lemma 6. If a graph G has a point symmetric cover, then $\widetilde{G}$ is point symmetric.

Proof. Assume $\widehat{G}$ is a point symmetric cover of G. Let $\tilde{u}$, $\tilde{v}$ be arbitrary points of $\widetilde{G}$ and $\hat{u}, \hat{v}$ their images under the map ψ:$\widetilde{G} \to \widehat{G}$. Let f:$\widehat{G} \to \widehat{G}$ be an automorphism such that $f(\hat{u}) = \hat{v}$. Since ψ:$\widetilde{G} \to \widehat{G}$ and $f^\circ\psi$:$\widetilde{G} \to \widehat{G}$ are both Δ-universal covers, there is a covering transformation F yielding a commutative diagram:

$$\begin{tikzcd} & \widetilde{G} \arrow[dl, dashed] \arrow[d, "f_0\psi"] \\ \widetilde{G} \arrow[r, "\psi"'] & \widehat{G} \end{tikzcd}$$

Because $\psi \circ F = f \circ \psi$, $F(\tilde{u})$ lies in the same fiber as $\tilde{v}$ with respect to the covering map ψ. Since $\Gamma(\widetilde{G}, \widehat{G})$ is transitive on the points of each fiber, there is a map F':$\widetilde{G} \to \widetilde{G}$ such that $F' \circ F(\tilde{u}) = (\tilde{v})$. ∎

Theorem 7. If G has a point symmetric cover, then $G \cong G(\Gamma, A, B, T)$, where $\Gamma =$ Aut $\widetilde{G}$, $B = \Gamma(\widetilde{G}, G)$, A is the stabilizer of any point v_0 of G, and $T = \{t \in \Gamma \mid t(v_0)$ or $t^{-1}(v_0)$ is adjacent to $v_0\}$.

Proof. Consider the group ΔB of covering transformations of $\Delta\pi$:$\Delta\widetilde{G} \to \Delta G$. A standard result in the theory of covering spaces is $\Delta G \cong \Delta\widetilde{G}/\Delta B$. This implies $G \cong \widetilde{G}/B$. The theorem then follows from Theorem 5 and Lemma 6. ∎

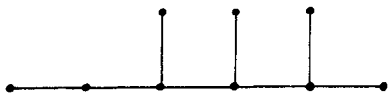

FIGURE 1. No point symmetric graph is locally G_0.

REMARK. It is not always true that if G is locally homogeneous, then $\widetilde{G}$ is point symmetric. Blass, Harary, and Miller [1] state the existence of a graph that is locally the tree G_0 in Figure 1. However, their results imply that no point symmetric graph is locally G_0.

Example. Let C_n be an n-cycle, $n \geq 3$, and G a connected graph that is locally C_n. Then ΔG is a connected 2-manifold without boundary and $\widetilde{\Delta G}$ is the regular tessellation of a simply connected surface S into triangles with n triangles incident at each vertex. Coxeter [3] discusses these tessellations in detail. The surface S is the sphere, plane or unit disk (hyperbolic plane) depending on whether $n < 6$, $n = 6$ or $n > 6$, respectively. The Δ-universal cover $\widetilde{G}_n$ is the graph underlying the tessellation. Figure 2 shows $\widetilde{G}_3$ and part of the infinite graph $\widetilde{G}_6$.

Aut $\widetilde{G}_n$ is the well known triangle group having the presentation

$$\Gamma_n = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^3 = (yz)^n = (zx)^2 = 1 \rangle$$

The graphs that are locally C_n are exactly the quotient graphs $\widetilde{G}_n/B$, where B is a properly discontinuous subgroup of Γ_n. By Theorem 7 these are the graphs

$$G(\Gamma_n, \langle y, z \rangle, B, \{x\}).$$

It is not difficult to show that this representation is unique up to conjugacy of B in Γ_n. The only locally C_3, C_4, and C_5 graphs are the 1-skeletons of the tetrahedron, octahedron and icosahedron, respectively. By a result of Ronan [6] there are infinitely many graphs locally C_n for $n \geq 6$.

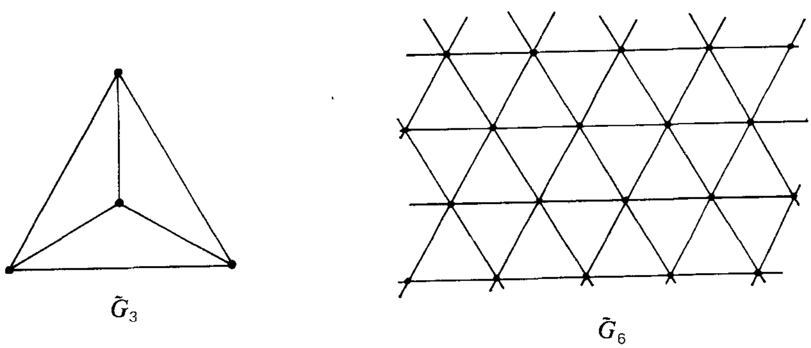

FIGURE 2. Δ-Universal covers.

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