Locally Homogeneous Graphs from Groups

Andrew Vince

UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN 48109

ABSTRACT

A graph is called locally homogeneous if the subgraphs induced at any two points are isomorphic. In this Note we give a method for constructing locally homogeneous graphs from groups. The graphs constructable in this way are exactly the locally homogeneous graphs with a point symmetric universal cover. As an example we characterize the graphs that are locally $n$-cycles.

1. INTRODUCTION

For a connected graph $G$ and point $v$ of $G$, let $G_v$ be the subgraph induced by the points adjacent to $v$. This $G$ is called locally $G_0$ if $G_v = G_0$ for all points $v$ of $G$. A graph is called locally homogeneous if it is locally $G_0$ for some $G_0$. Recent work concerning local homogeneity has focused on two broad questions. First, for which $G_0$ does there exist a $G$ that is locally $G_0$? This has been settled for $G_0$ a cycle, linear forest, and certain trees [1, 2]. Second is the question of characterization. For a specific graph $G_0$, characterize all graphs $G$ that are locally $G_0$. For example, let $K(n; t)$ denote the complete multipartite graph $K(n, n, \ldots, n)$, where there are $t$ parts. When $G_0 = K(n; t)$ there is a unique $G$, namely $G = K(n; t + 1)$. A recent paper of Hall [4] classifies the graphs that are locally the Petersen graph. There are exactly three.

In this Note, a large class of locally homogeneous graphs are obtained using groups. As a special case we characterize the graphs that are locally $n$-cycles.

2. HOMOGENEOUS GRAPHS FROM GROUPS

We begin with the construction of a graph $G(\Gamma, A, B, T)$. Later in this section we relate this construction to the concept of locally homogeneous graph. Let $\Gamma$ be an arbitrary group, $A$ and $B$ subgroups of $\Gamma$ and $T$ a subset of $\Gamma$ with...
the property that $T^{-1} = T$. Define $G(\Gamma, A, B, T)$ to be the graph whose points are the two sided cosets $\{BgA \mid g \in \Gamma\}$, where two distinct points $U$ and $V$ of $G(\Gamma, A, B, T)$ are defined to be adjacent if $U^{-1}V \cap T \neq \emptyset$.

For subsets $X, Y \subseteq \Gamma$, the set $XY$ is $\{xy \mid x \in X, y \in Y\}$, and let $\langle X \rangle$ denote the subgroup of $\Gamma$ generated by the elements of $X$. The following proposition is routinely verified.

**Proposition 1.** The graph $G(\Gamma, A, B, T)$ is connected if and only if $B\langle T \cup A \rangle = \Gamma$. $\blacksquare$

When $A = B = \{1\}$, and $T$ generates $\Gamma$, the definition of $G(\Gamma, A, B, T)$ coincides with that of the Cayley graph of $\Gamma$ with respect to the generating set $T$. Recall that the points of the Cayley graph are the elements of $\Gamma$ with two points $u, v$ adjacent whenever $ut = v$ for some $t \in T$.

When $A = \{1\}$, $G(\Gamma, A, B, T)$ is the Schreier coset graph of $\Gamma$ with respect to the subgroup $B$. The Schreier coset graph is a generalization of the Cayley graph, the points of the Cayley graph being regarded as cosets of the trivial subgroup $\{1\}$. The points of the Schreier coset graph are the right cosets of $B$, where two points $U, V$ are joined by a line whenever $Ut = V$ for some $t \in T$.

Next consider the case $B = \{1\}$. As an abbreviation we denote $G(\Gamma, A, \{1\}, T)$ by $G(\Gamma, A, T)$. Recall that a graph is point symmetric if its automorphism group is transitive on points. The graphs $G(\Gamma, A, T)$ are exactly the point symmetric graphs.

**Theorem 2.** A graph $G$ is point symmetric if and only if $G = G(\Gamma, A, T)$ for some group $\Gamma$, subgroup $A$ and subset $T = T^{-1}$.

**Proof:** For arbitrary points $U = gA$ and $V = hA$ of $G(\Gamma, A, T)$ the mapping $G(\Gamma, A, T) \to G(\Gamma, A, T)$ given by $X \to hg^{-1}X$ defines an automorphism of $G(\Gamma, A, T)$ taking $U$ to $V$. Conversely, given a point symmetric graph $G$, let $\Gamma$ be its automorphism group, $A$ the stabilizer subgroup of some point $v_0$ and $T = \{t \in \Gamma \mid t(v_0) \text{ or } t^{-1}(v_0) \text{ is adjacent to } v_0\}$. Consider the map $F$ from the points of $G$ to those of $G(\Gamma, A, T)$ given by $F : v \to g_vA$, where $g_v \in \Gamma$ is any automorphism taking $v_0$ to $v$. We claim that $F$ is an isomorphism. Now $g_A$ consists of all elements of $\Gamma$ taking $v_0$ to $v$. Hence $u$ and $v$ are adjacent in $G$ if and only if $g_u^{-1}g_v(v_0)$ is adjacent to $v_0$. Equivalently, $g_u^{-1}g_v \in T$. Thus $F(u)$ and $F(v)$ are adjacent in $G(\Gamma, A, T)$. $\blacksquare$

A map $\pi : \hat{G} \to G$ from the points of graph $\hat{G}$ to the points of graph $G$ is called a covering map if $\hat{G}_v$ is mapped isomorphically onto $G_{g_v}$ for all points $v$ of $\hat{G}$. Let $B$ be a subgroup of Aut $G$, the automorphism group of $G$. The points of the quotient graph $G/B$ are the orbits $\langle v \rangle$ of points $v$ in $G$ under the action of $B$, where distinct points $U$ and $V$ are adjacent in $G/B$ whenever there are points $u \in U$ and $v \in V$ adjacent in $G$. As an example, consider a group $\Gamma$, subgroups $A$ and $B$ and subset $T = T^{-1}$. Then $B$ can be regarded as a
subgroup of Aut $G(\Gamma, A, T)$ by defining $b(V) = bV$ for $b \in B$, in which case $G(\Gamma, A, T)/B \cong G(\Gamma, A, B, T)$. In general, if the map

$$f: G \to G/B$$

given by

$$v \mapsto \langle v \rangle$$

is a covering, then $B$ is called *properly discontinuous* with respect to $G$. An equivalent formulation is the following: $B$ is properly discontinuous if, for all points $u, v, f(u) = f(v)$ implies that $u$ and $v$ are at a distance greater than 3 in $G$. The next proposition follows from the fact that a point symmetric graph is locally homogeneous.

**Proposition 3.** If $B$ is a properly discontinuous group of automorphism with respect to a point symmetric graph $G$, then $G/B$ is locally homogeneous.

A sufficient condition for $G(\Gamma, A, B, T)$ to be locally homogeneous follows from Theorem 2 and Proposition 3.

**Corollary 4.** If $B \leq \Gamma$, considered as a subgroup of Aut $G(\Gamma, A, B, T)$, is properly discontinuous, then $G(\Gamma, A, B, T)$ is locally homogeneous.

Moreover, every locally homogeneous graph that is a quotient $G/B$ of a point symmetric graph $G$ by a properly discontinuous subgroup $B$ is obtained by the $G(\Gamma, A, B, T)$ construction.

**Theorem 5.** If $B$ is properly discontinuous with respect to a point symmetric graph $G$, then $G/B \cong G(\Gamma, A, B, T)$, where $\Gamma = \text{Aut } G, A$ is the stabilizer in $\Gamma$ of any point $v_0$ of $G$ and $T = \{ t \in \Gamma | t(v_0) \text{ or } t^{-1}(v_0) \text{ is adjacent to } v_0 \}$.

**Proof.** For a point $\langle v \rangle$ of $G/B$ let $g_{\langle v \rangle}$ be any automorphism of $G$ taking $v_0$ to $v$. It is easy to show that $B_{g_{\langle v \rangle}}A$ depends only on the orbit $\langle v \rangle$ and not on the representative $v$, so that the function $F: G/B \to G(\Gamma, A, B, T)$ given by $\langle v \rangle \mapsto B_{g_{\langle v \rangle}}A$ is well defined. The proof that $F$ is an isomorphism proceeds exactly as in Theorem 2.

**3. UNIVERSAL COVER**

The main result of this section utilizes a notion due to Ronan [6]. For a graph $G$ let $\Delta G$ be the 2-dimensional simplicial complex formed from $G$ by adding a 2-simplex for each triple of mutually adjacent points of $G$. Let $\Delta \pi: \Delta \tilde{G} \to \Delta G$ be the universal topological covering of $\Delta G$ and $\pi: \tilde{G} \to G$ the restriction of $\Delta \pi$ to the 1-skeleton of $\Delta \tilde{G}$. The graph $\tilde{G}$ will be called the $\Delta$-universal
cover of $G$. (We remark that $\tilde{G}$ should not be confused with the universal cover of $G$ regarded as a 1-dimensional simplicial complex.) The graph $\tilde{G}$ is universal in the sense that if $\varphi: \tilde{G} \to G$ is any cover, then there is a cover $\psi: \tilde{G} \to \tilde{G}$ yielding a commutative diagram:

$$
\begin{array}{ccc}
\psi & \tilde{G} \\
\downarrow \pi & \downarrow \\
G & \to & G \\
\varphi
\end{array}
$$

Denote by $\Gamma(\tilde{G}, G)$ the group of covering transformations of $\pi: \tilde{G} \to G$, i.e., the elements of Aut $\tilde{G}$ preserving fibers. If $\Delta \Gamma(\Delta G, \Delta G)$ denotes the group of covering transformations of $\Delta \pi: \Delta \tilde{G} \to \Delta G$, then certainly $\Delta \Gamma(\Delta G, \Delta G)$ and $\Gamma(\tilde{G}, G)$ are equal as permutation groups acting on vertices. Hence, by standard results on covering spaces [7], $\Gamma(\tilde{G}, G)$ is transitive on the points of each fiber and is isomorphic to the fundamental group $\pi_1(\Delta G)$.

**Lemma 6.** If a graph $G$ has a point symmetric cover, then $\tilde{G}$ is point symmetric.

**Proof.** Assume $\tilde{G}$ is a point symmetric cover of $G$. Let $\tilde{u}, \tilde{v}$ be arbitrary points of $\tilde{G}$ and $\tilde{u}, \tilde{v}$ their images under the map $\psi: \tilde{G} \to \tilde{G}$. Let $f: \tilde{G} \to \tilde{G}$ be an automorphism such that $f(\tilde{u}) = \tilde{v}$. Since $\psi: \tilde{G} \to \tilde{G}$ and $f \circ \psi: \tilde{G} \to \tilde{G}$ are both $\Delta$-universal covers, there is a covering transformation $F$ yielding a commutative diagram:

$$
\begin{array}{ccc}
\tilde{G} \\
\downarrow f \circ \psi \\
\tilde{G} \to \tilde{G} \\
\psi
\end{array}
$$

Because $\psi \circ F = f \circ \psi$, $F(\tilde{u})$ lies in the same fiber as $\tilde{v}$ with respect to the covering map $\psi$. Since $\Gamma(\tilde{G}, \tilde{G})$ is transitive on the points of each fiber, there is a map $F': \tilde{G} \to \tilde{G}$ such that $F' \circ F(\tilde{u}) = (\tilde{v})$.

**Theorem 7.** If $G$ has a point symmetric cover, then $G \cong G(\Gamma, A, B, T)$, where $\Gamma = \text{Aut} \tilde{G}$, $B = \Gamma(G, G)$, $A$ is the stabilizer of any point $v_0$ of $G$, and $T = \{ t \in \Gamma | t(v_0) \text{ or } t^{-1}(v_0) \text{ is adjacent to } v_0 \}$.

**Proof.** Consider the group $\Delta B$ of covering transformations of $\Delta \pi: \Delta \tilde{G} \to \Delta G$. A standard result in the theory of covering spaces is $\Delta G \cong \Delta G/\Delta B$. This implies $G \cong \tilde{G}/B$. The theorem then follows from Theorem 5 and Lemma 6.
Remark. It is not always true that if $G$ is locally homogeneous, then $\tilde{G}$ is point symmetric. Blass, Harary, and Miller [1] state the existence of a graph that is locally the tree $G_0$ in Figure 1. However, their results imply that no point symmetric graph is locally $G_0$.

Example. Let $C_n$ be an $n$-cycle, $n \geq 3$, and $G$ a connected graph that is locally $C_n$. Then $\Delta G$ is a connected 2-manifold without boundary and $\Delta G$ is the regular tessellation of a simply connected surface $S$ into triangles with $n$ triangles incident at each vertex. Coxeter [3] discusses these tessellations in detail. The surface $S$ is the sphere, plane or unit disk (hyperbolic plane) depending on whether $n < 6$, $n = 6$ or $n > 6$, respectively. The $\Delta$-universal cover $\tilde{G}_n$ is the graph underlying the tessellation. Figure 2 shows $\tilde{G}_3$ and part of the infinite graph $\tilde{G}_6$.

$\text{Aut} \tilde{G}_n$ is the well known triangle group having the presentation

$$\Gamma_n = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^3 = (yz)^n = (zx)^2 = 1 \rangle$$

The graphs that are locally $C_n$ are exactly the quotient graphs $\tilde{G}_n/B$, where $B$ is a properly discontinuous subgroup of $\Gamma_n$. By Theorem 7 these are the graphs

$$G(\Gamma_n, \langle y, z \rangle, B, \{x\}).$$

It is not difficult to show that this representation is unique up to conjugacy of $B$ in $\Gamma_n$. The only locally $C_3$, $C_4$, and $C_5$ graphs are the 1-skeletons of the tetrahedron, octahedron and icosahedron, respectively. By a result of Ronan [6] there are infinitely many graphs locally $C_n$ for $n \geq 6$. 

![Figure 1: No point symmetric graph is locally $G_0$.](image1)

![Figure 2: $\Delta$-Universal covers.](image2)
References