for all $x \in [0, 1)$.

Solution by M. L. Glasser, Clarkson University, Potsdam, NY. We use the usual notation for the rising factorial $a^{(k)} = a(a+1)\cdots(a+k-1)$ and the hypergeometric function

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{a^{(k)} b^{(k)} z^k}{c^{(k)} k!}. $$

The hypergeometric series are tabulated in section 7.3 of A. P. Prudnikov et al., *Integrals and Series*, Vol. 3, Gordon and Breach, 1990.

The left hand side is

$$ 4(1-x^2)^{3/2} \sum_{k=0}^{\infty} \frac{1^{(k)} 1^{(k)} x^{2k}}{(1/2)^{(k)} k!} = 4(1-x^2)^{3/2} {}_2F_1(1, 1; 1/2; x^2) $$
$$ = 4\left(\sqrt{1-x^2} + x \arcsin x\right). $$

Now consider the right hand side. Letting $r = k - j$ we see that

$$ \sum_{j=0}^{k} (-1)^j \frac{2^{k-j+1}}{j!\,2^j} \frac{(k-j)!}{(2k-2j)!} = \frac{(-1)^k}{k!\,2^{k-1}} \sum_{r=0}^{k} \frac{(-k)^{(r)} 1^{(r)}}{(1/2)^{(r)} r!} $$
$$ = \frac{(-1)^k}{k!\,2^{k-1}} {}_2F_1(-k, 1; 1/2; 1) = \frac{(-1)^{k-1}}{k!\,2^{k-1}(2k-1)}. $$

Therefore the right hand side is

$$ 4 + \sum_{k=1}^{\infty} x^{2k} (2k)! \left(\sum_{j=0}^{k} (-1)^j \frac{2^{k-j+1}}{j!2^j} \frac{(k-j)!}{(2k-2j)!} \right)^2 $$
$$ = 4 \sum_{k=0}^{\infty} x^{2k} (2k)! \frac{1}{2^{2k}(2k-1)^2 (k!)^2} $$
$$ = 4 \sum_{k=0}^{\infty} \frac{x^{2k} (-1/2)^{(k)} (-1/2)^{(k)}}{(1/2)^{(k)} k!} $$
$$ = 4\, {}_2F_1(-1/2, -1/2; 1/2; x^2) = 4\left(\sqrt{1-x^2} + x \arcsin x\right). $$

Editorial comment. The equality of the left and right hand sides can also be derived less explicitly as a special case of Euler's identity

$$ (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b; c; z) = {}_2F_1(a, b; c; z). $$

The problem arose as follows. If (X, Y) is bivariate normal with mean 0, variance 1, and correlation x, then computing $E\big(|XY|\big)$ in two different ways gives the two series.

Solved also by P. Bracken (Canada), B. Bradie, R. J. Chapman (U. K.), O. P. Lossers (The Netherlands), R. Richberg, J. Tyson, and the proposer.

Almost Fixed Sets

10843 [2000, 951]. *Proposed by Andrew Vince, University of Florida, Gainesville, FL.* Define a mapping $f\colon [0, 1) \to [0, 1)$ by $f(x) = 2x \pmod 1$. Find

$$ \sup_{\lambda(A)=1/2} \lambda\left(A \cap f^{-1}(A)\right), $$

where λ denotes Lebesgue measure and the supremum is taken over all sets A that are the union of finitely many intervals and that satisfy $\lambda(A) = 1/2$.

Solution by Kenneth Schilling, University of Michigan, Flint, MI. The answer is $1/2$. For $x \in [0, 1]$, let $M(x) = \sup_{\lambda(A)=x} \lambda(A \cap f^{-1}(A))$. We show more generally that $M(x) = x$. In fact, the same arguments apply if we replace $([0, 1), \lambda)$ and f by any nonatomic measure space X and any measure-preserving function $f: X \to X$, provided the sets A range over arbitrary measurable sets.

Claim 1: If $0 \le x_0 \le x_1 \le 1$, then $0 \le M(x_1) - M(x_0) \le 2(x_1 - x_0)$. In particular, M is continuous.

Proof. Every set of measure x_0 is contained in a set of measure x_1, so $0 \le M(x_1) - M(x_0)$. Suppose that $\lambda(B) = x_1$. Choose $A \subset B$ with $\lambda(A) = x_0$. We have

$$A \cap f^{-1}(A) \supset \big(B \cap f^{-1}(B)\big) \setminus \big((B \setminus A) \cup f^{-1}(B \setminus A)\big),$$

so $M(x_0) \ge \lambda(A \cap f^{-1}(A)) \ge \lambda(B \cap f^{-1}(B)) - 2(x_1 - x_0)$. Take the supremum over all sets B with $\lambda(B) = x_1$ to get $M(x_0) \ge M(x_1) - 2(x_1 - x_0)$. $\square$

Claim 2: If $M(x_0) < x_0$, then $x_0 - M(x_0) \ge x - M(x)$ for all $x \in [x_0, 2x_0 - M(x_0)]$.

Proof. Suppose that $x \in (x_0, 2x_0 - M(x_0)]$ and $\lambda(A) = x_0$. Since $\lambda(A \cap f^{-1}(A)) \le M(x_0)$, we have $\lambda(f^{-1}(A) \setminus A) \ge x_0 - M(x_0)$. Since $0 < x - x_0 \le x_0 - M(x_0)$ there exists $B \subset f^{-1}(A) \setminus A$ with $\lambda(B) = x - x_0$. Thus $\lambda(A \cup B) = x$, so

$$M(x) \ge \lambda\big((A \cup B) \cap f^{-1}(A \cup B)\big) \ge \lambda(A \cap f^{-1}(A)) + \lambda(B).$$

Take the supremum over all sets A with $\lambda(A) = x_0$ to get $M(x) \ge M(x_0) + (x - x_0)$. $\square$

Main proof. Now let $N(x) = x - M(x)$. Clearly $N(0) = N(1) = 0$ and $N(x) \ge 0$. Suppose (for purposes of contradiction) that N is not identically zero. By Claim 1, N is continuous, so N takes on a maximum value $y_1 > 0$. Let x_1 be the least $x \in [0, 1]$ such that $N(x) = y_1$, and let x_0 be the greatest $x \in [0, x_1]$ such that $N(x) = y_1/2$. We then have $N(x_0) < N(x)$ for $x \in (x_0, x_1)$, which contradicts Claim 2 when $x > x_0$ but is close to x_0.

Solved also by M. Bowron, J. Feroe, A. Guetter, D. Huang, B. S. Kim (Korea), J. H. Lindsey II, O. P. Lossers (The Netherlands), R. Stong, GCHQ Problems Group (U. K.), NSA Problems Group, and the proposer.

REVIVALS

A Generalization of Hall's Theorem

10701 [1998, 956; 2001, 79]. *Proposed by Fred Galvin, University of Kansas, Lawrence, KS.* Let G be a (finite, undirected, simple) graph with vertex set V. Let $C = \{C_x : x \in V\}$ be a family of sets indexed by the vertices of G. For $X \subseteq V$, let $C_X = \cup_{x \in X} C_x$. A set $X \subseteq V$ is *C-colorable* if one can assign to each vertex $x \in X$ a "color" $c_x \in C_x$ so that $c_x \ne c_y$ whenever x and y are adjacent in G.

(a) Prove that if $|C_X| \ge |X|$ whenever X induces a connected subgraph of G, then V is C-colorable.

(b) Prove that if every proper subset of V is C-colorable and if $|C_V| \ge |V|$, then V is C-colorable.

(c) For every connected graph G, find a family $C = \{C_x : x \in V\}$ showing that the condition $|C_V| \ge |V|$ in part **(b)** cannot be weakened to $|C_V| \ge |V| - 1$.

Editorial comment. The proposer notes that part **(a)** is a corollary of part **(b)**, and part **(b)** is a restatement of Lemma 4 on p. 256 of H. A. Kierstead, On the choosability of complete multipartite graphs with part size three, *Discrete Math.* 211 (2000) 255-259.