10542. *Proposed by Jean Anglesio, Garches, France.*

Let $\mathcal{C}$ be the circumcircle of a triangle $A_0B_0C_0$ and $\mathcal{I}$ the incircle. It is known that, for each point A on $\mathcal{C}$, there is a triangle ABC having $\mathcal{C}$ for circumcircle and $\mathcal{I}$ for incircle. Show that the locus of the centroid G of triangle ABC is a circle that is traversed three times by G as A traverses $\mathcal{C}$ once, and determine the center and radius of this circle.

NOTES

(**10538**) Two versions of this problem, one by the first four named authors, and one by the last two, arrived within a short time. The similarity of the statements suggested that they be combined. The proposers noted that the case $n = 2$ has appeared on various national selection tests for the International Mathematical Olympiad. (**10542**) The existence of ABC is a special case of Poncelet's theorem. Details may be found in M. Berger, *Geometry I*, Springer-Verlag, 1987, p. 316

SOLUTIONS

The Superregular Graphs

6617 [1989, 942]. *Proposed by Andrew Vince, University of Florida, Gainesville, FL.*

A graph Γ is *regular* if each vertex has the same degree. For a vertex x let Γ_x and Δ_x denote the subgraphs of $\Gamma - x$ induced by the vertices adjacent to and nonadjacent to x, respectively. Define *superregular* recursively as follow. The empty graph is superregular and Γ is superregular if Γ is regular and both Γ_x and Δ_x are superregular for all x. Characterize the superregular graphs.

Solution by Randall B. Maddox, Pepperdine University, Malibu, CA. We adopt the following notation: K_n is the complete graph on n vertices, mK_n is m disjoint copies of K_n, C_n is the cycle on n vertices, and G_n is the graph whose vertex set consists of n^2 vertices arranged in an $n \times n$ square, with two vertices adjacent if and only if they are in the same row or column of the square. (The graph G_n is otherwise known as the Cartesian product of K_n with itself.) For any graph G, we let $\overline{G}$ denote its complement.

The superregular graphs are precisely the following: C_5, mK_n $(m, n \geq 1)$, G_n $(n \geq 1)$, and the complements of these graphs. Call this class of graphs S.

Theorem 1. *Every graph in S is superregular.*

Proof. It is easy to see that if a graph Γ is superregular, then so is its complement $\overline{\Gamma}$. The graph $\Gamma = K_n$ is certainly superregular, since for any x, $\Gamma_x = K_{n-1}$, which is superregular by induction, and Δ_x is empty. The graph $\Gamma = mK_n$ is then seen to be superregular, since for any x, $\Gamma_x = K_{n-1}$, which is superregular, and $\Delta_x = (m-1)K_n$, which is superregular by induction on m. The graph $\Gamma = G_n$ is superregular, since for any x, $\Gamma_x = 2K_{n-1}$, which is superregular, and $\Delta_x = G_{n-1}$, which is superregular by induction. Finally C_5 is easily seen to be superregular.

What remains is to prove the converse. Before addressing this, we first point out two basic properties of superregular graphs.

Proposition 2. *If Γ is a connected, superregular graph, then any pair of nonadjacent vertices of Γ have a common neighbor. If Γ is superregular and not connected, then $\Gamma = mK_n$ for some m and n.*

Proof. Suppose that Γ is a superregular graph of degree r and that w, x, y, z is a path in Γ such that the distance from w to z is 3. Then $y, z \in \Delta_w$, y has no more than $r-1$ neighbors in Δ_w, and z has exactly r neighbors in Δ_w. The same conclusion holds if w, x, y is a path in Γ such that the distance from w to y is 2, and z lies in a different component. Thus, in either case, Δ_w is not regular, which is a contradiction.

The remainder of the solution is devoted to the proof of the converse of Theorem 1.

Theorem 3. *Every superregular graph is in S.*

Proof. Suppose, in order to obtain a contradiction, that Γ is a superregular graph not in S and that, among all such graphs, Γ has the fewest vertices. Then Γ_x and Δ_x must be in S. The various possibilities for Γ_x will be considered in the following propositions. Note that, by Proposition 2, we may assume that Γ is connected.

Proposition 4. *For all x, Γ_x is not C_5.*

Proof. Suppose $\Gamma_x = C_5$. Then Γ is 5-regular, so every vertex in Γ_x has exactly 2 neighbors in Δ_x. Thus there are exactly 10 edges between Γ_x and Δ_x. It follows that Δ_x has a number of vertices which divides 10. Since this number must be even (the 5-regular graph Γ must have an even number of vertices), it must be 2 or 10. But if it is 2, then Γ must be $\overline{K_3}$ joined to C_5 with all possible edges, which is not superregular. So Δ_x must have 10 vertices and be 4-regular. The only such graph in S is $2K_5$, but then again Γ is not superregular.

Proposition 5. *For all x, Γ_x is not $\overline{G_n}$.*

Proof. We prove the stronger result that if Γ is any superregular graph with $\Gamma_x = \overline{G_n}$ for some x and some $n \geq 2$, then $\Gamma = \overline{G_{n+1}}$. To prove this when $n = 2$, one can follow a simple analysis analogous to the proof of Proposition 4. The choices for Δ_x are an edgeless graph on 2 vertices, a 2-regular graph on 4 vertices, or a 3-regular graph on 8 vertices. In the first and last case, the resulting graph Γ is not superregular. In the remaining case, $\Delta_x = C_4$ but there is only one way to join Δ_x to $\Gamma_x \cup \{x\}$ to obtain a superregular graph and the resulting graph is $\overline{G_3}$.

Suppose then that $\Gamma_x = \overline{G_n}$ for $n \geq 3$, and label the vertices of Γ_x with the elements of $\{1, \ldots, n\} \times \{1, \ldots, n\}$ so that

$$(i_1, j_1) \text{ is adjacent to } (i_2, j_2) \text{ if and only if } i_1 \neq i_2 \text{ and } j_1 \neq j_2. \tag{1}$$

The graph induced by neighbors of $(1, 1)$ in Γ_x is $\overline{G_{n-1}}$, so $\Gamma_{(1,1)}$ must be $\overline{G_n}$ by the induction hypothesis. Thus $(1, 1)$ has exactly $2(n-1)$ neighbors in Δ_x and these may be labeled $(i, 0)$ and $(0, j)$ $(1 \leq i, j \leq n)$ with edges between (i_1, j_1) and (i_2, j_2) according to (1), as long as none of i_1, i_2, j_1, j_2 are 1. Similarly, $\Gamma_{(2,2)}$ must be $\overline{G_n}$, and this requires two more vertices labeled $(1, 0)$ and $(0, 1)$, again with edges according to (1), as long as none of i_1, i_2, j_1, j_2 are 2. Finally label x with $(0, 0)$.

For $k \geq 3$, the neighbors of the vertex (k, k) are now all accounted for. The vertices of $\Gamma_{(k,k)}$ are $\{(i, j) : 0 \leq i, j \leq n \text{ and } i, j \neq k\}$ and the edges are given by (1). There remains to show only that (1) holds for all i_1, i_2, j_1, j_2. If $n \geq 4$ or if $n = 3$ and i_1, i_2, j_1, j_2 are not all different, this follows by consideration of $\Gamma_{(k,k)}$ where k is chosen to be different from all of i_1, i_2, j_1, j_2. If $n = 3$ and i_1, i_2, j_1, j_2 are all different, a count of neighbors of (i_1, j_1) would come up shy of the required 9 unless (i_2, j_2) were included. Thus (1) governs in all situations, and so $\Gamma = \overline{G_{n+1}}$ as claimed.

Lemma 6. *If Γ_x is mK_n for some x, then Γ_x is mK_n for all x.*

Proof. Suppose that $\Gamma_x = mK_n$. Then $\Gamma_y = mK_n$ also for every vertex y in Γ_x, since Γ_y contains K_n as one component, has mn vertices, and is in S. But then $\Gamma_z = mK_n$ for every vertex in Γ, since if $z \in \Delta_x$, then $m \geq 2$ and $z \in \Gamma_y$ for some $y \in \Gamma_x$ (by Proposition 2), and so the same argument applies.

We use Lemma 6 liberally in what follows.

Proposition 7. *For all x, Γ_x is not mK_n.*

Proof. We first address the case $m = 1$. If $\Gamma_x = K_n$, then $\Gamma_y = K_n$ for every vertex y. It follows that $\Gamma = mK_{n+1}$, but this contradicts the fact that $\Gamma \notin S$.

We next address the case $m = 2$. When $n = 2$, this was already addressed at the beginning of the proof of Proposition 5, since $\overline{G_2} = 2K_2$. We proceed with $m = 2$ and $n \geq 3$, proving the stronger result that any superregular graph Γ with $\Gamma_x = 2K_n$ for some x must be G_{n+1}.

Suppose that $\Gamma_x = 2K_n$. Let the two components of Γ_x be denoted A and B and let $A = \{y_1, \ldots, y_n\}$. Then, inside Δ_x, Γ_{y_i} has a component K_n which we denote B_i. The B_i must be disjoint, since $\Gamma_{y_i} = 2K_n$. Now suppose that z_i is a vertex in B_i. By repeatedly applying $\Gamma_z = 2K_n$ for various vertices z, one can infer that z_i has exactly $n-1$ neighbors in $\Delta_x - \Gamma_{y_i}$, none of which are adjacent to any vertices in $\Gamma_{y_i} - \{z_i\}$. Since Δ_x is superregular, the neighbors of z_i induce a $2K_{n-1}$. By the induction hypothesis, $\Delta_x = G_n$. Finally, every vertex in B_i must have exactly one neighbor v in B, and the fact that $\Gamma_v = 2K_n$ forces $\Gamma = G_{n+1}$.

Last, we address the case $m \geq 3$. Assume that $\Gamma_x = mK_n$. We consider the possibilities for Δ_x. Applying Proposition 4 to $\overline{\Gamma}$ rules out $\Delta_x = C_5$ and applying Proposition 5 to $\overline{\Gamma}$ rules out $\Delta_x = G_n$.

Suppose that $\Delta_x = \overline{G_t}$ for some $t \geq 3$ and let y be a vertex in Δ_x, which we now view as $\overline{\Gamma}_x$. Then the neighbors of y in this G_t induce a $2K_{t-1}$, so $\Delta_y = \overline{G_t}$ also. This Δ_y has $(t-1)^2$ vertices in Γ_x and these vertices do not form a clique. But Γ_x has no induced subgraphs whose components are not complete. So $\Delta_x \neq G_t$.

Suppose that $\Delta_x = sK_t$ for some $s, t \geq 1$. Let y be a vertex in Γ_x and z a neighbor of y in Δ_x. Since $\Gamma_y = mK_n$, there are at least $m-1$ neighbors of y in Δ_x that induce a graph with no edges. Thus $s \geq m-1$. Since $\Gamma_z = mK_n$, the neighbors of z in Δ_x must form one copy of K_n, so $t = n+1$. Now the number of edges from Γ_x to Δ_x is $mn^2(m-1)$, while the number of edges from Δ_x to Γ_x is $st(mn-(t-1)) = s(n+1)(mn-n)$. Equating these two counts yields $mn = s(n+1)$, which together with $s \geq m-1$ implies that $s = m-1 = n$. Thus $\Gamma_x = mK_{m-1}$ and $\Delta_x = (m-1)K_m$. But since $m \geq 3$, y has a neighbor y' in Γ_x, and both y and y' have $m-1$ neighbors in each copy of K_m in Δ_x. Hence y and y' have a common neighbor, contradicting the fact that $\Gamma_y = mK_{m-1}$.

Finally, suppose that $\Delta_x = \overline{sK_t}$. The number of edges from Γ_x to Δ_x is $mn^2(m-1)$, while the number from Δ_x to Γ_x is $st(mn-(s-1)t)$. Equating these yields $mn^2(m-1) = st(mn-(s-1)t)$. The fact that $\Gamma_y = mK_n$ and $\Gamma_z = mK_n$ implies that $n \leq s \leq n+1$ and $m-1 \leq t \leq m$, but it is then easy to rule out all the possibilities.

Proposition 8. *For all x, Γ_x is not $\overline{mK_n}$.*

Proof. Since $\overline{mK_1} = K_m$, the case $n = 1$ is handled by Proposition 7. So take $n \geq 2$ and assume that $\Gamma_x = \overline{mK_n}$. An argument similar to Lemma 6 shows that all $\Gamma_y = \overline{mK_n}$. As in Proposition 7, one can then reduce to the case where $\Delta_x = sK_t$ or $\Delta_x = \overline{sK_t}$. The latter case is ruled out by applying Proposition 7 to $\overline{\Gamma}$, so we may assume $\Delta_x = sK_t$ with $t \geq 2$.

If $t = 2$, then the number of edges from Γ_x to Δ_x is $mn(n-1)$ while the number of edges from Δ_x to Γ_x is $2s(mn-1)$. Equating these yields $mn(n-1) = 2s(mn-1)$. This

requires that $2s$ be a multiple of mn, but then the right-hand side becomes larger than the left. If $t \geq 3$, then let y be a vertex in Γ_x and z a neighbor of y in Δ_x. Since $\Gamma_y = \overline{mK_n}$, z must be adjacent to the $m-1$ copies of K_n in $\overline{\Gamma_x - \Gamma_y}$. Since $\Gamma_z = \overline{mK_n}$, the remaining neighbors of z must induce a subgraph with no edges. But since $t \geq 3$, z has a pair of adjacent neighbors in Δ_x, a contradiction.

Proposition 9. *For all x, Γ_x is not G_n.*

Proof. Since $\overline{G_3} = G_3$, we may assume $n \geq 4$. Assume that $\Gamma_x = G_n$. Then Γ is n^2-regular, and every vertex in Γ_x has exactly $(n-1)^2$ neighbors in Δ_x. So the number of edges from Γ_x to Δ_x is $n^2(n-1)^2$. Also, since Δ_x must be $\overline{G_t}$ for some $t \geq 4$, every vertex in Δ_x has exactly $n^2-(t-1)^2$ neighbors in Γ_x, so the number of edges from Δ_x to Γ_x is $t^2(n^2-(t-1)^2)$. Setting these equal yields

$$\left(\frac{n-1}{t}\right)^2 + \left(\frac{t-1}{n}\right)^2 = 1.$$

It is easy to see that this has no solutions in integers greater than 2.

Editorial comment. Five claimed solutions to this problem were available when the deadline for solutions was reached, but none turned out to be correct. Later, Douglas B. West solved the problem, but as a report on his solution was being prepared, a solution from Randall B. Maddox was received. What we have presented above is a digest of Maddox's proof. In order to illustrate the organization of the solution in limited space, many details are left to the reader. West's solution has been accepted for publication in *J. Graph Theory*.

A Square Crossing

10322 [1993, 688]. *Proposed by Jiang Huanxin, student, FuDan University, ShangHai, China.*

Let $ABCD$ and $AEFG$ be squares with the common vertex A and *different* edge lengths. Let $\theta = \angle EAD$ $(0 < \theta < \pi/2)$. Suppose that EF and CD intersect at the point P. For which value of θ will $AP \perp CF$?

Solution by H. Sunil Gunaratne, Universiti Brunei Darussalam, Gadong, Brunei. Assume $|AE| : |AD| = \lambda : 1$, with $\lambda > 0$ and $\lambda \neq 1$. Then there are two cases. **Case (i)**: $AEFG$ has the same orientation as $ABCD$. Then $\theta = \pi/4$ is the unique solution, independent of λ. **Case (ii)**: $AEFG$ has the orientation opposite to that of $ABCD$. Then the solutions are of the form $\theta = \alpha \pm \beta$ where $-\pi/2 < \alpha, \beta < \pi/2$ with $\cos\alpha = (\lambda^2+1)(2+\lambda^4)^{-1/2}$, $\cos\beta = 2\lambda(2+\lambda^4)^{-1/2}$. In addition, α and β should have the same sign, with $0 < \alpha < \pi/2$ if $\lambda > 1$ and $-\pi/2 < \alpha < 0$ if $0 < \lambda < 1$. It is also necessary that $\sqrt{2}-1 < \lambda < \sqrt{2}+1$ in order to have $-\pi/2 < \theta < \pi/2$.

To show this, assume without loss of generality that $|AD| = 1$, $|AE| = \lambda$, and use vectors based at A. Thus, we write $\overrightarrow{AB} = \mathbf{i}$, $\overrightarrow{AD} = \mathbf{j}$, $\overrightarrow{AE} = \lambda\mathbf{e}$, $\overrightarrow{AG} = \lambda\mathbf{g}$, where $\mathbf{i}, \mathbf{j}, \mathbf{e}, \mathbf{g}$ are unit vectors and $\mathbf{i}\cdot\mathbf{j} = 0$. Then we have

$$\mathbf{e} = \mathbf{i}\sin\theta + \mathbf{j}\cos\theta$$
$$\pm\mathbf{g} = -\mathbf{i}\cos\theta + \mathbf{j}\sin\theta$$

with $-\pi/2 < \theta < \pi/2, \theta \neq 0$. The plus sign is taken in Case (i) and the minus sign in Case (ii). Then $\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP} = \mathbf{j} + t\mathbf{i} = \overrightarrow{AE} + \overrightarrow{EP} = \lambda\mathbf{e} + r\lambda\mathbf{g}$ for scalars r and t to be