10314
Author(s): Andrew Vince and A. N. 't Woord
Reviewed work(s):
Source: The American Mathematical Monthly, Vol. 103, No. 4 (Apr., 1996), p. 349
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2975202
Accessed: 25/04/2012 13:51

Your use of the JSTOR archive indicates your acceptance of the Terms \& Conditions of Use, available at http://www.jstor.org/page/info/about/policies/terms.jsp

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Mathematical Association of America is collaborating with JSTOR to digitize, preserve and extend access to The American Mathematical Monthly.
$j(n) \geq\lceil n x\rceil=j_{x}(n)$. To prove $j$ agrees with $j_{x}$ through $N$, suppose there exists $n \leq N$ such that $j(n)>j_{x}(n)=\lceil n x\rceil$, so $j(n) \geq n x+1$. When we express $m n$ as $n$ copies of $m$, the defining condition yields $j(m n) \leq n j(m)=n m x$. When we express $m n$ as $m$ copies of $n$, the defining condition yields $j(m n) \geq m j(n)-(m-1) \geq m n x+1$, a contradiction.

Editorial comment. Frank Schmidt pointed out that these sequences are characterized in R. L. Graham, S. Lin, and C.-S. Lin, "Spectra of numbers", Math. Mag. 51 (1978), 174-176. Another reference, supplied by an editor, is M. Boshernitzan and A. S. Fraenkel, "Nonhomogeneous spectra of numbers", Discr. Math. 34 (1981), 325-327.

Solved also by V. Božin (student, Yugoslavia), H. von Eitzen (Germany), F. J. Flanigan, R. Holzsager, I. Kastanas, O. P. Lossers (The Netherlands), A. D. Melas (Greece), J. M. Santmyer, F. Schmidt, A. N. 't Woord (The Netherlands), GCHQ Problem Solving Group (U. K.), and the proposer.

## A Modular Power Series

10314 [1993, 589]. Proposed by Andrew Vince, University of Florida, Gainesville, FL.
Let $b$ be an integer greater than 1 . Let $S$ be a set of integers containing 0 such that no two members of $S$ are congruent modulo $b$. If

$$
\sum_{i=1}^{\infty} \frac{s_{i}}{b^{i}}=0
$$

with $s_{i} \in S$, prove that all $s_{i}=0$.
Solution by A. N. 't Woord, University of Technology, Eindhoven, The Netherlands. Suppose that $\sum_{i=1}^{\infty}\left(s_{i} / b^{i}\right)=0$ with $s_{i} \in S$. We define a sequence $\left\{a_{n}\right\}_{n=0}^{\infty}$ such that $\sum_{i=1}^{n}\left(s_{i} / b^{i}\right)=a_{n} / b^{n}$, by setting $a_{0}=0$ and $a_{n}=b^{n-1} s_{1}+b^{n-2} s_{2}+\cdots+b s_{n-1}+s_{n}$ for $n \geq 1$. Notice that $a_{n}=b a_{n-1}+s_{n}$ for all $n \geq 1$.

If $a_{n}=0$ for some $n \geq 1$, then $s_{n} \equiv a_{n} \equiv 0 \bmod b$. This requires $s_{n}=0$ and $a_{n-1}=0$, and hence $s_{i}=0$ for all $i \leq n$, inductively. If $1 \leq m<n$ and $a_{m}=a_{n}$, then $s_{m} \equiv a_{m} \equiv a_{n} \equiv s_{n} \bmod b$. Therefore, $s_{m}=s_{n}$ and $a_{m-1}=a_{n-1}$. By induction, $a_{n-m}=a_{0}=0$. Hence $s_{i}=0$ for all $i \leq n-m$. It suffices therefore to show that $a_{m}=a_{n}$ occurs with the difference $n-m$ arbitrarily large.

By the congruence condition, $S$ is finite. Choose $M>0$ such that $\left|s_{i}\right| \leq M$ for all $i$. Now

$$
0=\left|\sum_{i=1}^{\infty} \frac{s_{i}}{b^{i}}\right| \geq \frac{\left|a_{n}\right|}{b^{n}}-\sum_{i=n+1}^{\infty} \frac{M}{b^{i}}=\frac{1}{b^{n}}\left(\left|a_{n}\right|-\frac{M}{b-1}\right)
$$

so $\left|a_{n}\right| \leq M /(b-1)$ for all $n$. Therefore, the set $\left\{a_{n}: n \geq 0\right\}$ is finite. Thus, for some $k$, we have $a_{n}=k$ infinitely often, and the result follows.

Solved also by K. L. Bernstein, W. Blumberg, S. M. Gagola Jr., R. Holzsager, N. Jensen (Germany), I. Kastanas, O. P. Lossers (The Netherlands), R. Martin (student), L. E. Mattics, the MMRS group of Oklahoma State University, gnd the proposer. Five incorrect solutions were received.

## Integral Matrices with Integral Inverses

## 10315 [1993, 589]. Proposed by Mowaffaq Hajja, Yarmouk University, Irbid, Jordan.

Let $A$ and $B$ be matrices with integer entries of sizes $r$ by $n$ and $n$ by $r$, respectively, with $r<n$. Suppose that $A B$ is an $r$ by $r$ identity matrix. Show that $A$ can be enlarged to an $n$ by $n$ integral matrix having an integral inverse.

Solution by Allan Pedersen, Søborg, Denmark. Let $a_{1}, \ldots, a_{r}$ be the row vectors of $A$ in order from top to bottom, and let $b_{1}, \ldots, b_{r}$ be the column vectors of $B$ in order from left to

