Let $b$ be an integer greater than 1. Let $S$ be a set of integers containing 0 such that no two members of $S$ are congruent modulo $b$. If

$$\sum_{i=1}^{\infty} \frac{s_i}{b^i} = 0,$$

with $s_i \in S$, prove that all $s_i = 0$.

**Solution by A. N. 't Woord, University of Technology, Eindhoven, The Netherlands.**

Suppose that $\sum_{i=1}^{\infty} (s_i / b^i) = 0$ with $s_i \in S$. We define a sequence $\{a_n\}_{n=0}^{\infty}$ such that $\sum_{i=1}^{n} (s_i / b^i) = a_n / b^n$, by setting $a_0 = 0$ and $a_n = b^{n-1}s_1 + b^{n-2}s_2 + \cdots + bs_{n-1} + s_n$ for $n \geq 1$. Notice that $a_n = ba_{n-1} + s_n$ for all $n \geq 1$.

If $a_n = 0$ for some $n \geq 1$, then $s_n \equiv a_n \equiv 0 \mod b$. This requires $s_n = 0$ and $a_{n-1} = 0$, and hence $s_i = 0$ for all $i \leq n$, inductively. If $1 \leq m < n$ and $a_m = a_n$, then $s_m \equiv a_m \equiv a_n \equiv s_n \mod b$. Therefore, $s_m = s_n$ and $a_{m-1} = a_{n-1}$. By induction, $a_{n-m} = a_0 = 0$. Hence $s_i = 0$ for all $i \leq n - m$. It suffices therefore to show that $a_m = a_n$ occurs with the difference $n - m$ arbitrarily large.

By the congruence condition, $S$ is finite. Choose $M > 0$ such that $|s_i| \leq M$ for all $i$. Now

$$0 = \left| \sum_{i=1}^{\infty} \frac{s_i}{b^i} \right| \leq \frac{|a_n|}{b^n} - \sum_{i=1}^{\infty} \frac{M}{b^i} = \frac{1}{b^n} \left( |a_n| - \frac{M}{b - 1} \right),$$

so $|a_n| \leq M/(b - 1)$ for all $n$. Therefore, the set $\{a_n : n \geq 0\}$ is finite. Thus, for some $k$, we have $a_n = k$ infinitely often, and the result follows.

Solved also by K. L. Bernstein, W. Blumberg, S. M. Gagola Jr., R. Holzsager, N. Jensen (Germany), I. Kastanas, O. P. Lossers (The Netherlands), R. Martin (student), L. E. Maticos, the MMRS group of Oklahoma State University, and the proposer. Five incorrect solutions were received.

**A Modular Power Series**

**10314 [1993, 589]. Proposed by Andrew Vince, University of Florida, Gainesville, FL.**

Let $A$ and $B$ be matrices with integer entries of sizes $r$ by $n$ and $n$ by $r$, respectively, with $r < n$. Suppose that $AB$ is an $r$ by $r$ identity matrix. Show that $A$ can be enlarged to an $n$ by $n$ integral matrix having an integral inverse.

**Solution by Allan Pedersen, Søborg, Denmark.** Let $a_1, \ldots, a_r$ be the row vectors of $A$ in order from top to bottom, and let $b_1, \ldots, b_r$ be the column vectors of $B$ in order from left to right.