$j(n) \geq \lceil nx \rceil = j_x(n)$. To prove j agrees with j_x through N, suppose there exists $n \leq N$ such that $j(n) > j_x(n) = \lceil nx \rceil$, so $j(n) \geq nx + 1$. When we express mn as n copies of m, the defining condition yields $j(mn) \leq nj(m) = nmx$. When we express mn as m copies of n, the defining condition yields $j(mn) \geq mj(n) - (m-1) \geq mnx + 1$, a contradiction.

Editorial comment. Frank Schmidt pointed out that these sequences are characterized in R. L. Graham, S. Lin, and C.-S. Lin, "Spectra of numbers", *Math. Mag.* 51 (1978), 174–176. Another reference, supplied by an editor, is M. Boshernitzan and A. S. Fraenkel, "Nonhomogeneous spectra of numbers", *Discr. Math.* 34 (1981), 325–327.

Solved also by V. Božin (student, Yugoslavia), H. von Eitzen (Germany), F. J. Flanigan, R. Holzsager, I. Kastanas, O. P. Lossers (The Netherlands), A. D. Melas (Greece), J. M. Santmyer, F. Schmidt, A. N. 't Woord (The Netherlands), GCHQ Problem Solving Group (U. K.), and the proposer.

A Modular Power Series

10314 [1993, 589]. *Proposed by Andrew Vince, University of Florida, Gainesville, FL.*

Let b be an integer greater than 1. Let S be a set of integers containing 0 such that no two members of S are congruent modulo b. If

$$\sum_{i=1}^{\infty} \frac{s_i}{b^i} = 0,$$

with $s_i \in S$, prove that all $s_i = 0$.

Solution by A. N. 't Woord, University of Technology, Eindhoven, The Netherlands. Suppose that $\sum_{i=1}^{\infty}(s_i/b^i) = 0$ with $s_i \in S$. We define a sequence $\{a_n\}_{n=0}^{\infty}$ such that $\sum_{i=1}^{n}(s_i/b^i) = a_n/b^n$, by setting $a_0 = 0$ and $a_n = b^{n-1}s_1 + b^{n-2}s_2 + \cdots + bs_{n-1} + s_n$ for $n \geq 1$. Notice that $a_n = ba_{n-1} + s_n$ for all $n \geq 1$.

If $a_n = 0$ for some $n \geq 1$, then $s_n \equiv a_n \equiv 0 \bmod b$. This requires $s_n = 0$ and $a_{n-1} = 0$, and hence $s_i = 0$ for all $i \leq n$, inductively. If $1 \leq m < n$ and $a_m = a_n$, then $s_m \equiv a_m \equiv a_n \equiv s_n \bmod b$. Therefore, $s_m = s_n$ and $a_{m-1} = a_{n-1}$. By induction, $a_{n-m} = a_0 = 0$. Hence $s_i = 0$ for all $i \leq n - m$. It suffices therefore to show that $a_m = a_n$ occurs with the difference $n - m$ arbitrarily large.

By the congruence condition, S is finite. Choose $M > 0$ such that $|s_i| \leq M$ for all i. Now

$$0 = \left|\sum_{i=1}^{\infty} \frac{s_i}{b^i}\right| \geq \frac{|a_n|}{b^n} - \sum_{i=n+1}^{\infty} \frac{M}{b^i} = \frac{1}{b^n}\left(|a_n| - \frac{M}{b-1}\right),$$

so $|a_n| \leq M/(b-1)$ for all n. Therefore, the set $\{a_n : n \geq 0\}$ is finite. Thus, for some k, we have $a_n = k$ infinitely often, and the result follows.

Solved also by K. L. Bernstein, W. Blumberg, S. M. Gagola Jr., R. Holzsager, N. Jensen (Germany), I. Kastanas, O. P. Lossers (The Netherlands), R. Martin (student), L. E. Mattics, the MMRS group of Oklahoma State University, and the proposer. Five incorrect solutions were received.

Integral Matrices with Integral Inverses

10315 [1993, 589]. *Proposed by Mowaffaq Hajja, Yarmouk University, Irbid, Jordan.*

Let A and B be matrices with integer entries of sizes r by n and n by r, respectively, with $r < n$. Suppose that AB is an r by r identity matrix. Show that A can be enlarged to an n by n integral matrix having an integral inverse.

Solution by Allan Pedersen, Søborg, Denmark. Let $a_1, \ldots, a_r$ be the row vectors of A in order from top to bottom, and let $b_1, \ldots, b_r$ be the column vectors of B in order from left to