NOTES

(**10286**) In the innermost sum, we employ the convention that a binomial coefficient $\binom{h}{i}$ is zero unless $0 \le i \le h$. (**10287**) For a similar result in the traditional peg solitaire, see John D. Beasley, ***The Ins & Outs of Peg Solitaire***, Oxford University Press, 1985, chapter 12. (**10288**) The number of balls, b, is fixed so its value may appear in the answer. For each n, X_n is a random variable whose expectation is denoted by $\mathbf{E}(X_n)$. (**10289**) The case $a = 1/2$ appeared as problem 1365 in *Mathematics Magazine*.

SOLUTIONS

Scrambling Points on the Unit Circle

6647 [1991, 63]. *Proposed by Andrew Vince, University of Florida, Gainesville, FL.*

Let $S_n = \{1, \zeta, \zeta^2, \dots, \zeta^{n-1}\}$ be the set of n-th roots of unity and suppose f is any function on S_n into the set of complex numbers of absolute value one. For every positive integer k less than $n/2$ prove that there exist integers i and j such that

$$|\zeta^i - \zeta^j| \ge |1 - \zeta^k| \ge \left|f(\zeta^i) - f(\zeta^j)\right|.$$

Note: The proposer had been confused with a different A. Vince in the original presentation of the problem.

The solution will be divided into two parts, called lemmas 1 and 2. In these proofs, distance will be the shortest distance measured along the unit circle in units of one-n-th of a circle, so that all the distances between points in S_n are integers. Since Euclidean distance is a monotonic function of this distance, the desired results can be obtained from the corresponding results for this distance. Also, the set of all images of points in S_n under f will be referred to as $f(S_n)$.

Lemma 1. *There exists a closed arc of the unit circle of length k which contains at least $k + 1$ points of $f(S_n)$.*

Solution I by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. Let the elements of $a_j \in f(S_n)$ be ordered by their arguments

$$0 \le a_0 \le \cdots \le a_{n-1} < n$$

and extend the indexing to all $j \in \mathbb{Z}$ so that $a_{j+n} = a_j$. Then,

$$(a_k - a_0) + \cdots + (a_{nk} - a_{(n-1)k}) = nk$$

so at least one of the terms, say $(a_{lk} - a_{(l-1)k})$, has a value not exceeding the average of the n terms which is k. Now, the $k + 1$ points a_j with $(l - 1)k \leq j \leq lk$ all lie in the closed arc from $a_{(l-1)k}$ to a_{lk}, which was chosen to have length at most k.

Solution II by Sharad Kanetkar, University of Massachusetts, Boston, MA. Consider n closed arcs on the unit circle, each of length k, chosen so that their beginning points are equally spaced (at distance 1) and so that at least one such beginning point coincides with one of the points of $f(S_n)$.

Each point of the circle lies in at least k such arcs, and the endpoints lie in $k + 1$ arcs. In particular, each point of $f(S_n)$ lies in at least k arcs and at least one of them lies in $k + 1$ arcs. Thus the total number of incidences of points of $f(S_n)$ with these arcs is at least $nk + 1$. For Lemma 1 to be false, however, each of these arcs would contain at most k elements of $f(S_n)$ and the total number of incidents would be at most nk.

Lemma 2. *If $A \subset S_n$ has $k + 1$ elements, then there is a pair of elements in A whose distance is at least k.*

Solution by the Editors, based on an idea of Sharad Kanetkar. For each element $a_i \in A$, there is a set $A_i \subset S_n$ consisting of a_i together with the $k - 1$ consecutive elements of S_n clockwise from a_i and the $k - 1$ consecutive elements of S_n counterclockwise from a_i.

If, for any i, some element a of A lies outside A_i, then a_i and a are the desired elements. Otherwise, A lies wholly within each A_i. By DeMorgan's laws, this is equivalent to saying that the union of complements C_i of the A_i is a subset of the complement of A in S_n. However, the complement of A contains $n - k - 1$ elements and we shall show that the union of the C_i must contain at least $n - k + 1$ elements.

To prove the latter claim, note that each A_i contains $2k - 1$ consecutive points of S_n, so that C_i contains $n - 2k + 1$ consecutive points of S_n. Start from a point not in the union of the C_i (if no such point exists, the claim is clearly true) and look at the C_i in clockwise order starting from this point. The first C_i gives us $n - 2k + 1$ points and each of the k subsequent C_i gives at least one point beyond (in the clockwise sense) the previous C_i since the C_i are distinct intervals.

Editorial comment. The result clearly follows from the lemmas: Lemma 1 gives a set A of $k + 1$ of the ζ^i satisfying the condition on the $f(\zeta^i)$ and Lemma 2 allows a pair of these elements to be selected to satisfy

$$|\zeta^i - \zeta^j| \geq |1 - \zeta^k|$$

as well.

All successful solvers except the proposer followed the outline presented here, although Lemma 2 seemed rather elusive.

Lemma 1 was obtained also by L. E. Mattics and R. Stong. The proposer's solution and one other were judged to be unconvincing.