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Non-revisiting paths on surfaces with low genus

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Abstract

The non-revisiting path conjecture for polytopes, which is equivalent to the Hirsch conjecture, is open. However, for polyhedral maps on surfaces, we have recently proved the conjecture false for all orientable surfaces of genus $g \geqslant 2$ and all nonorientable surfaces of nonorientable genus $h \geqslant 4$. In this paper, a unified, elementary proof of the non-revisiting path conjecture is given for the sphere, projective plane, torus and Klein bottle. Only the case of the connected sum of three copies of the projective plane remains open. In connection with the notion of the representativity ρ of a surface embedding, it is shown that the non-revisiting path property holds for all surfaces of representativity $\rho \geqslant 4$, but there is a polyhedral map with representativity 3 for which the non-revisiting path property fails.

1. Introduction

One of the most well-known open problems in the combinatorial theory of polytopes is the Hirsch conjecture, which gives an upper bound on the diameter of the graph of a polytope. The *graph* of a polytope P is the 1-skeleton of P. More specifically, the Hirsch conjecture states that $\Delta(d,n) \leqslant n-d$, where $\Delta(d,n)$ is the maximum diameter among the graphs of d-dimensional polytopes with n facets. A *facet* is a $(d-1)$-dimensional face. The Hirsch conjecture was formulated by Hirsch in 1957 and reported by Dantzig in his book Linear Programming and Extensions [5]. The conjecture has implications for the complexity of linear programming algorithms like the simplex method. Since the diameter of the graph of a polytope P is an upper bound on the number of iterations of the best possible edge-following algorithm for an LP problem with feasible region P, the diameter $\Delta(d,n)$ gives the worst possible complexity of the best possible edge-following algorithm for LP problems with n constraints and d variables. A nice survey on the Hirsch conjecture is the paper by Klee and Kleindschmidt [9].

Equivalent to the Hirsch conjecture is the non-revisiting path conjecture of Klee and Wolfe [8]. If p is a path in the graph of a polytope, a *revisit* of p to a face F is

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a pair of vertices (x, y) such that $p[x, y] \cap F = \{x, y\}$, where $p[x, y]$ is the path along p from x to y. In other words, p visits F at x, leaves F and, subsequently, revisits F at y. The revisit (x, y) is said to *involve x and y*.

Non-Revisiting Path Conjecture. Any two vertices of a polytope P can be joined by a path that does not revisit any facet of P.

The non-revisiting path conjecture is known to be true for three-dimensional polytopes [1] and is open in higher dimensions. Klee and Walkup [10] showed it to be false, in general, for unbounded polyhedra. Klee [7] has asked about the validity of the non-revisiting path conjecture for more general complexes. Since the underlying topological space of the boundary complex of a polytope is a sphere, it is natural to ask whether the conjecture is true for cell complexes whose underlying space is a sphere. In this regard, the conjecture is true for 2-spheres, but there is a counterexample due to Mani and Walkup [11] for the 3-sphere.

This paper concerns the non-revisiting path conjecture for polyhedral maps. Unless otherwise stated, by a *surface S*, we mean a connected, compact 2-manifold without boundary. These comprise the orientable surfaces T_g of genus g, which are the connected sums of g tori, and the nonorientable surfaces U_h, of nonorientable genus h, which are the connected sums of h projective planes. Let G be a graph embedded on a surface S. The closure of a connected component of $S \backslash G$ is called a *face*. If the faces are all simply connected and the intersection of any two distinct faces is either a common edge, common vertex or empty, then $M = (G, S)$ is called a *polyhedral map*. Two distinct faces that satisfy the condition stated above are said to meet *properly*; otherwise the faces are said to meet *improperly*.

A surface S has the *non-revisiting path property* if, for any polyhedral map M on S and any two vertices x and y on M, there is a path joining x to y that does not revisit any face. Clearly, if faces were allowed to meet improperly, the non-revisiting path property would fail. Recent research has been directed toward the following question.

Question. *Which surfaces possess the non-revisiting path property?*

It has long been known that the non-revisiting path property holds for the sphere [1,8]. Barnette gave two separate proofs for the projective plane [2] and torus [4]. Very complicated proofs were given for the Klein bottle and double torus [6]; however, at least the double torus proof has a flaw, since a result of Pulapaka and Vince [12] gives counterexamples to the non-revisiting path property for all orientable surfaces of genus $g \geqslant 2$ and all nonorientable surfaces of nonorientable genus $h \geqslant 4$. The main result of this paper is an elementary, unified proof of the fact that the surfaces U_1, U_2, T_0 and T_1 satisfy the non-revisiting path property. Therefore the only case that now remains open is U_3, the connected sum of three projective planes.

The concept of representativity of a surface embedding was developed by Robertson and Seymour [14] in connection with the subject of graph minors. A nice survey paper is by Robertson and Vitray [15]. *Circuits* in a surface S are homeomorphic images of the unit circle. If C is a circuit in a surface S, then $S\backslash C$ is obtained by cutting S along C. If one component of $S\backslash C$ is homeomorphic to an open disk, then C is *trivial*; otherwise C is *essential*. If Ψ is a graph embedding on a surface S that is not the sphere and $G(\Psi)$ is the graph of Ψ, then the *representativity* of Ψ is defined to be

$$\rho(\Psi) = \min\{|C \cap G(\Psi)| : C \text{ is an essential circuit in } S\}.$$

By elementary topology, in the definition of representativity it suffices to consider essential circuits in S that pass through only vertices and faces of Ψ and which use no vertex or face more than once. In Section 4 of this paper a short proof is given for the fact that, for polyhedral maps of representativity $\rho \geqslant 4$, the non-revisiting path property holds for all surfaces. This is best possible in the sense that there is a polyhedral map on the double torus that does not possess the non-revisiting property but has representativity 3. The proof of the result for representativity 4 is essentially the same as Richter and Vitray [13]. The proof here is shorter, probably due to the fact that the Richter–Vitray proof is in the dual form. They prove, given any two faces of a polyhedral map of representativity at least 4, there is a cycle in the embedded graph that bounds a disk and contains the two faces.

2. Preliminary results

Our proof of the non-revisiting path property for the low genus cases (Theorem 1) relies on three lemmas, the first two due to Barnette [2,3]. For the first lemma we supply a simplified proof. Let (x, y) be a revisit of a path p to a face F. If the two paths along F from x to y are denoted $F[x, y]$ and $\hat{F}[x, y]$, then the revisit (x, y) is said to be *planar* if either $F[x, y] \cup p[x, y]$ or $\hat{F}[x, y] \cup p[x, y]$ bounds a cell on the surface. (Note that if one does then so does the other.) The notation $p(u, v]$ denotes the path from the vertex succeeding u to vertex v.

Lemma 1. *Let M be a polyhedral map with vertices u and v. If there is a path in M joining u and v all of whose revisits are planar, then there is a non-revisiting path between u and v.*

Proof. Let $p[u, v]$ be a path in M all of whose revisits are planar. If $p[u, v]$ is not a non-revisiting path, then there is a vertex x on $p[u, v]$ with the following properties:
(1) There is a non-revisiting path $p_0[u,x]$ between u and x.
(2) The path $p_0[u,x] \cup p[x, v]$ has only planar revisits.
A path satisfying (1) and (2) exists; simply take $x = u$.
(3) Among all choices for x satisfying (1) and (2), choose the one which is furthest along the path $p[u, v]$.

If $x = v$, we are done; otherwise we will obtain a contradiction. There must exist a revisit (z, y) of the path $p_0[u,x] \cup p[x, v]$ to a face F of M such that $z \in p_0[u,x]$ and $y \in p(x, v]$. Otherwise, if both y and z lie in $p_0[u,x]$ then statement (1) is contradicted;

if both y and z lie in $p_0(x, v]$ then statement (3) is contradicted. Among all revisits by $p_0[u,x] \cup p[x,v]$ we choose F so that z is nearest to u along $p_0[u,x]$. Now consider the path $p_1 = p_0[u,z] \cup F[z,y] \cup p[y,v]$ from u to v and observe the following:

(i) p_1 is a path from u to v all of whose revisits are planar. To see this note that p_0 itself has no revisits. A revisit involving vertices of $p[y,v]$ alone has to be planar since p has only planar revisits. A non-planar revisit by p_1 cannot involve vertices of $F[z,y]$ since the closed path $F[z,y] \cup p[y,x] \cup p_0[x,z]$ bounds a cell. Finally, if a revisit by p_1 involves a vertex of p and a vertex of p_0, then it must be planar since $p_0[u,x] \cup p[x,v]$ admits only planar revisits.

(ii) $p_1[u,y]$ does not revisit any face of M. A revisit by $p_1[u,y]$ to a face F_1 must involve y and a vertex $\bar{z}$ of $p_0[u,z)$. Note that $\bar{z} \neq z$; otherwise F and F_1 meet improperly at y and z. Now $(\bar{z}, y)$ is a revisit of the path $p_0[u,x] \cup p[x,v]$. This contradicts the choice of F with z nearest to u on $p_0[u,x]$.

The existence of y contradicts the choice of x as the vertex that was furthest along $p[u,v]$ satisfying conditions (1) and (2). □

Similar to the notion of a path in a polyhedral map having a disconnected intersection with a face of the polyhedral map, one may consider a cycle of a polyhedral map that has a disconnected intersection with a face of the polyhedral map. A cycle of a polyhedral map refers to a cycle in the underlying graph of the polyhedral map. Let $M = (G, S)$ be a polyhedral map and C a cycle in M. Then C is said to be *non-planar* if it does not bound a cell on S. A cycle is *non-revisiting* if it does not have any revisits; in other words, for each face F of M, $C \cap F$ is either empty, or connected.

Lemma 2 (Barnette [3]). *Every polyhedral map M on the projective plane, torus or Klein bottle has a non-planar, non-revisiting cycle C. In the case of the torus or Klein bottle, cutting M along C yields an annulus.*

In the definition of a polyhedral map $M = (G, S)$, in the case that the surface has a non-empty boundary ∂S, we require that $G \cap \partial S = \partial S$.

Lemma 3. *Let S be a surface with boundary ∂S and $M = (G, S)$ a polyhedral map on S such that the intersection of any face of M with ∂S is either empty or connected. Then any two vertices of M that lie in the interior of S can be joined by a path in M that is contained in the interior of S.*

Proof. Let p be the path from u to v with the least number of vertices on ∂S. Let x be the first vertex of p on ∂S. Further let w be the vertex preceding x on p and y the point of p succeeding x on p. Let R be the union of the faces of M incident with x such that $p[w, y]$ is part of the boundary of R. By the assumption of properly meeting faces and proper intersection of any face of M with ∂S, R is topologically a disk. Therefore there are two paths from w to y along the boundary of R. Consider the path γ not containing x. Again, under the assumptions of the lemma, γ does not meet ∂S. Let a be the first vertex of p on γ and z the last vertex of p on γ. Now the path $p[u,a] \cup \gamma[a,z] \cup p[z,v]$ meets ∂S in fewer vertices than p, contradicting the minimality of p. $\square$

3. The main theorem

Theorem 1. *The sphere, torus, projective plane and Klein bottle satisfy the non-revisiting path property.*

Proof. Let M be a polyhedral map on the sphere, projective plane, torus or Klein bottle and u and v be vertices of M. We will show that u and v can be joined by a path in M, all of whose revisits are planar. Consequently, by Lemma 1, there is a non-revisiting path joining u and v. That all revisits are planar is automatically true for the sphere. For the remaining three surfaces, Lemma 2 implies that M has a non-revisiting cycle C such that cutting M along C yields a cell H in the case of a projective plane and an annulus A in the case of the torus or Klein bottle. The proof consists of three main cases and several subcases. The main cases are classified according to whether both, one, or neither of u and v lie on C.

Case 1. *u and v lie on C.* Since C is non-revisiting, either of the two paths along C from u to v must be non-revisiting.

Case 2. *u lies on C and v does not lie on C.* Since every vertex of M has degree at least three, there must be a vertex u_1 of M that lies in the interior of H or A, respectively, such that uu_1 is an edge of M. Since the cycle C is non-revisiting, by Lemma 3, there is a path p_0 from u_1 to v that lies in the interior of H or A, respectively. Define $p = p_0 \cup uu_1$. Thus p is a path joining u and v that meets the boundary of H or A at u only. If p has only planar revisits, we are done. So assume that p has a non-planar revisit (s,t) to a face F with the vertex s closer to v than the vertex t is to v. Among all non-planar revisits of p, choose F so that s is nearest to v along p. Of all such non-planar revisits with s nearest to v, choose one with t nearest to u. The strategy from here on will be to alter the path p, perhaps several times, until it can be shown that all revisits are planar.

First consider the case of the projective plane. For the revisit to be non-planar, F must contain u; thus $t = u$. Replace p by the path $p_1 = F[u,s] \cup p[s,v]$. If p_1 has only planar revisits, we are done. On the other hand, if p_1 has a non-planar revisit to a face F_1, then it must involve a vertex s_1 of $p_1(s,v]$ and a vertex of $p_1 \cap \partial H$. Among all choices for F_1, choose the one for which s_1 is nearest to v along p_1. Since C is non-revisiting, $F_1 \cap \partial H$ is connected. Let t_1 be the vertex on $F_1 \cap p_1 \cap \partial H$ that is nearest to u. Replace p_1 by the path $p_2 = p_1[u,t_1] \cup F_1[t_1,s_1] \cup p_1[s_1,v]$. Now p_2 can have only planar revisits.

Next consider the case where M is a polyhedral map on the torus or Klein bottle. Our arguments will pertain to the annulus A. Let C_1 and C_2 denote the bounding cycles of A. Without loss of generality, assume that p meets C_1 at only u and avoids C_2. Since C is non-revisiting, F cannot meet both C_1 and C_2. The proof is now divided into two cases, denoted (A) and (B), depending on whether or not F contains u.

Case A. *F contains u.* Since the vertex u lies on the non-revisiting cycle C, the face F must meet either C_1 or C_2, but not both. Fig. 1(a) shows a situation where F meets u at C_2. Cases where F meets u at C_1 are shown in Figs. 1(b) and (c). This classification into two cases is based on the fact that the removal of the cycle $p[s,t] \cup F[t,s]$ divides the annulus into two components, one containing C_1, the other containing C_2. The two possbilities are that vertex v lies in the C_1 component or the C_2 component. Since C is non-revisiting, $F \cap C_2$ is connected. We now consider separately the subcases that (a) F meets u on C_2 and (b) F meets u on C_1.

Case a. *F meets u on C_2* (Fig. 1(a)). Replace p by the path from u (on C_2) to an endpoint t_0 of $F \cap C_2$; then along the face F from t_0 to the first point s_0 on the path p; then from s_0 to v along p. Such a path is not unique. Choose the one, denoted p_1, for which the point s_0 is closest to v along the path p. So s_0 is at least as close to v along the path p as is s.

If p_1 has only planar revisits, we are done; so assume that p_1 has a non-planar revisit to a face F_1. Such a revisit cannot involve two vertices on $p_1[u,s_0]$ since this would mean that the faces F and F_1 meet improperly. Otherwise, the only possibility for a non-planar revisit (that does not contradict the minimality of s for the revisit to F) is for it to involve a vertex s_1 of $p_1(s_0,v]$ and a vertex of $p_1[u,t_0]$. (Note that, by the minimality of s, this latter vertex is located on C_1.) Among all choices for F_1, choose the one for which s_1 is nearest to v along p_1. Let t_1 be the vertex on $F_1 \cap C_1$ that is nearest to u (on C_1). Replace p_1 by the path $p_2 = p_1[u,t_1] \cup F_1[t_1,s_1] \cup p_1[s_1,v]$. Using the minimality of s, s_0 and s_1, the path p_2 can have only planar revisits.

Case b. *F meets u on C_1 (Figs.* 1(b) *and* (c)). Replace p by the path $p_1 = F[u,s] \cup p[s,v]$ (the dotted path in Fig. 1(b)). If p_1 has only planar revisits, we are done; so assume that p_1 has a non-planar revisit to a face F_1. Such a revisit cannot involve a vertex of $p_1(s,v]$ and u (on C_2) since this would contradict the minimality of s.

First consider the case in Fig. 1(b). The only possibility is for the revisit to involve a vertex s_1 of $p_1(s,v]$ and a vertex t_1 of $p_1[u,t]$. Among all choices for F_1, choose the one for which s_1 is nearest to v along p_1. Replace p_1 by the path $p_2 = p_1[u,t_1] \cup F_1[t_1,s_1] \cup p_1[s_1,v]$. (If there are two possibilities for $F_1[t_1,s_1]$, choose one such that p_2 is a path that meets ∂A only at u.) By the minimality of s and s_1, the path p_2 can have only planar revisits.

Next consider the case in Fig. 1(c). By the minimality of s, the only possibility for a non-planar revisit of p_1 is for it to involve a vertex of $p_1(s,v]$ and a vertex of $p_1(s,t)$. Let t_1 be the vertex on $F_1 \cap p_1$ that is nearest to u (on C_1) and let s_2 be the vertex on $F_1 \cap p_1$ that is nearest to v (see Fig. 2). Replace p_1 by the path $p_2 = p_1[u,t_1] \cup F_1[t_1,s_2] \cup p_1[s_2,v]$. By the minimality of s, s_1 and s_2, the path p_2 can have only planar revisits.

Now return to case (B).

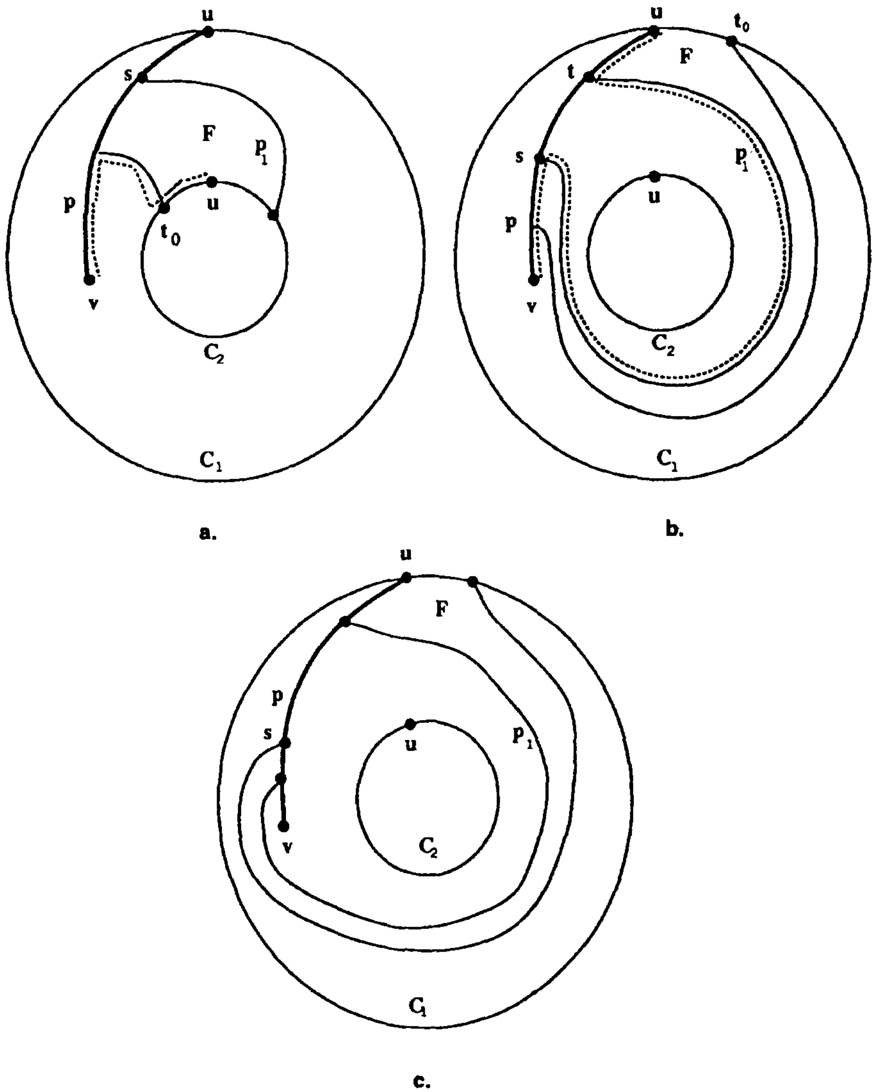

Fig. 1.

Case B. *F does not contain u.* Recall that F can meet at most one of C_1 or C_2. Again, there are two possibilities for F, depending on whether (a) F does not meet C_2 or (b) F does not meet C_1.

Case a. *F may meet* C_1 *but not* C_2. Replace p by the path $p_1 = p[u,t] \cup F[t,s] \cup p[s,v]$ where $F[t,s]$ avoids ∂A. If p_1 has a non-planar revisit to a face F_1, let s_1 be the vertex of the revisit that is closer to v along p_1 than the other vertex of the revisit. Among all choices for F_1, choose the one for which s_1 is nearest is v along p_1. Now there are three possibilities for such a non-planar revisit. The revisit either involves

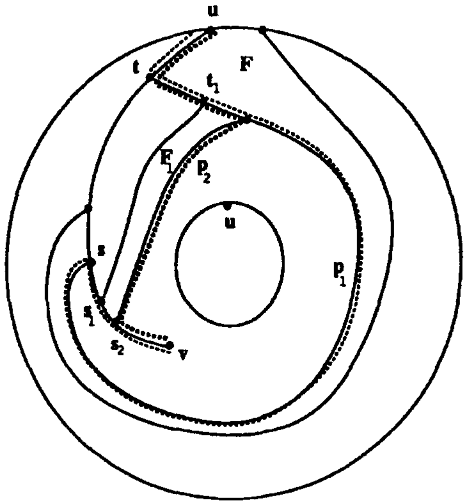

Fig. 2.

(i) a vertex t_1 on $p_1[u,t]$ and the vertex s_1 on $p_1(s,v]$, (ii) the vertex s_1 on $p_1[s,t]$ and u (on C_2), or (iii) the vertex s_1 on $p_1(s,v]$ and a vertex on $p_1[s,t]$.

In case (i), replace p_1 by the path $p_2 = p_1[u,t_1] \cup F_1[t_1,s_1] \cup p_1[s_1,v]$ that avoids ∂A. Now the only possbility for a non-planar revisit of p_2 to a face F_2 is if F does not meet C_1 and s coincides with the vertex s_1. In this case the revisit would involve s_1 and a vertex t_2 of $p_2(t_1,u]$. Now, for the first time, we use the minimality of t being nearest to u for the original revisit (s,t). The involvment of the vertex t_2 in the revisit (s_1,t_2) would contradict this minimality of t.

In case (ii), note that s_1 is the vertex on $p_1 \cap F_1$ that is nearest to u along p_1. Replace p_1 by the path $p_2 = F_1[u,s_1] \cup p_1[s_1,v]$. (There are two choices for the path $F_1[u,s_1]$; chose the one for which p_2 is a path.) Since C is non-revisiting, $F_1 \cap C_2$ is connected. Let t_0 be the vertex of $F_1 \cap C_2$ that is nearest to s_1 along p_2. Now there are two possibilities for a non-planar revisit of p_2 to a face F_2. The first is for the revisit to involve a vertex s_3 of $p_2(s_1,v]$ and a vertex t_2 of $p_2[t_0,s_1]$, while the second is for the non-planar revisit to involve a vertex s_2 of $p_2(s_1,v]$ and a vertex t_2 (on C_1) of $p_2(u,t_0]$. In either case, among all choices for F_2, choose the one for which s_2 is nearest to v on p_2. Replace p_2 by the path $p_3 = p_2[u,t_2] \cup F_2[t_2,s_2] \cup p_2[s_2,v]$, so that p_3 avoids ∂A. By the minimality of s, s_1 and s_2, the path p_3 can have only planar revisits.

Case (iii) is identical to the case in Fig. 2.

Case b. *F may meet* C_2, *but not* C_1. Let s_0 and t_0 be the vertices on $F \cap p$ that are nearest to v and u, respectively. Replace p by the path $p_1 = p[u,t_0] \cup F[t_0,s_0] \cup p[s_0,v]$. Note that p_1 avoids ∂A. There are now two possibilities for a non-planar revisit of p_1 to a face F_1. Such a revisit must either involve (i) a vertex t_1 of $p_1[u,t_0)$ and a vertex s_1 of $p_1[s_0,v]$, or (ii) a vertex t_1 of $p_1[u,t_0)$ and a vertex s_1 of $p_1[t_0,s_0]$. In either case, among all choices for F_1, choose the one for which s_1 is nearest to v along p_1, and among all such choices, choose the one for which t_1 is nearest to u (on C_1).

In both cases, replace p_1 by the path $p_2 = p_1[u,t_1] \cup F_1[t_1,s_1] \cup p_1[s_1,v]$. Of the two possibilities for $F_1[t_1,s_1]$, choose the one for which there can be no face of M that meets both $F_1(t_1,s_1)$ and $p_1[u,t_1)$.

In case (i), by the minimality of s_0, s_1 and t_1 and the assumption that F does not contain u, the path p_2 can have only planar revisits.

In case (ii), the only possibility for a non-planar revisit of p_2 to a face F_2 is for it to involve s_1 on $p_2(s_0,t_0)$ and a vertex t_2 on $p_2[u,t_1)$. Among all choices for F_2, choose the one for which t_2 is nearest to u (on C_1) along p_2. Replace p_2 by the path $p_3 = p_2[u,t_2] \cup F_2[t_2,s_1] \cup p_2[s_1,v]$. Of the two choices for $F_2[t_2,s_1]$, choose one in such a way that there can be no face of M that meets both $F_2(t_2,s_1)$ and $p_3[u,t_2)$. By the minimality of s_0, s_1, t_1 and t_2, the path p_3 can have only planar revisits.

We now return to the third main case.

Case 3. *Neither u nor v lie on C.* By Lemma 3, there is a path p between u and v that is contained in the interior of H or A. In the case of the projective plane, H is a cell and hence p can have only planar revisits. Next consider the case of the torus or Klein bottle. If p has only planar revisits, we are done; so assume that p has a non-planar revisit (s,t) to a face F. Among all choices for F, choose one for which s is nearest to v along p. Again consider the two cases, where (a) F does not meet C_1 and where (b) F does not meet C_2.

Case a. *F may meet C_2 but not C_1.* Let s_0 and t_0 be the vertices on $F \cap p$ that are nearest to v and u, respectively. Replace p by the path $p_1 = p[u,t_0] \cup F[t_0,s_0] \cup p[s_0,v]$. Now there are three possibilities for a non-planar revisit of p_1 to a face F_1. Either the revisit involves a vertex t_1 of $p_1[u,t_0]$ and a vertex s_1 of $p_1[s_0,v]$; or it involves a vertex s_1 of $p_1(s_0,v]$ and a vertex of $p_1[s_0,t_0]$; or it involves a vertex t_1 of $p_1[u,t_0]$ and a vertex of $p_1[s_0,t_0]$. The first case is identical to subcase (b) in case (B). The argument for the second and third cases are identical because the situation is symmetric. So consider only the second case. Among all choices for F_1, choose one for which the vertex s_1 is nearest to v on $F_1 \cap p_1$. Let t_1 be the vertex on $F_1 \cap p_1$ that is nearest to u along p_1. Replace p_1 by the path $p_2 = p_1[u,t_1] \cup F_1[t_1,s_1] \cup p_1[s_1,v]$, so that p_2 avoids ∂A. Now the only possibility for a non-planar revisit of p_2 to a face F_2 is for it to involve a vertex s_2 of $p_2(s_1,v]$ and the vertex t_1. Replace the path p_2 by a path $p_3 = p_2[u,t_1] \cup F_2[t_1,s_2] \cup p_2[s_2,v]$ such that p_3 is contained in the interior of the annulus A. By the minimality of s_1, the path p_3 can have only planar revisits.

Case b. *F may meet C_1 but not C_2.* Replace p by the path $p_1 = p[u,t] \cup F[t,s] \cup p[s,v]$. Notice that the situation now is identical to case (a) above, with the annulus A inverted, i.e., the role of C_1 and C_2 interchanged.

Thus in each of the cases 1, 2 and 3, the vertices u and v can be joined by a path in M that has only planar revisits and, by Lemma 1, we are done. □

4. Representativity and the non-revisiting path property

By definition, the requirement that faces of a polyhedral map meet properly implies that the representativity is at least three.

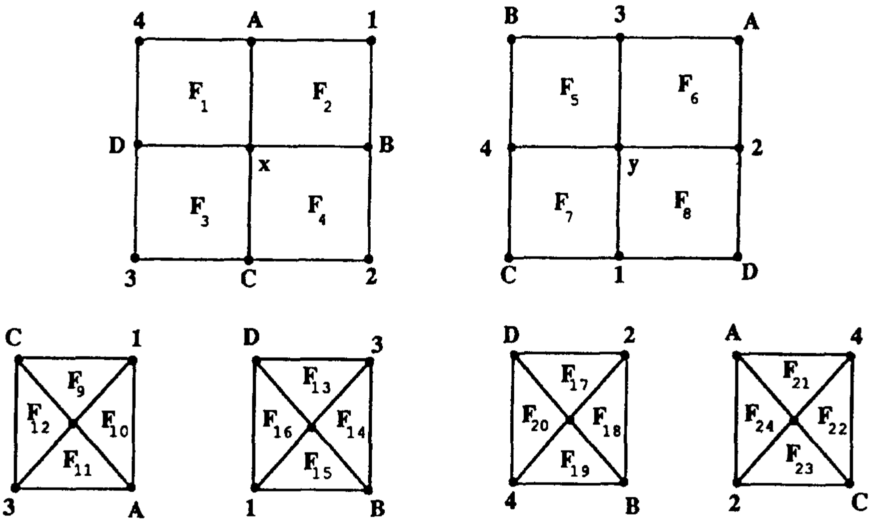

Fig. 3. Counterexample on the surface T_2 with $\rho = 3$.

Theorem 2. (1) *The non-revisiting path property holds for every polyhedral map of representativity* $\rho \geqslant 4$.

(2) *There exist polyhedral maps of representativity* $\rho = 3$ *for which the non-revisiting path property fails.*

Proof. Concerning statement (1), let M be a polyhedral map of representativity $\rho \geqslant 4$. If u and v are vertices of M, consider a sequence $\mathscr{F} = \{F_1, F_2, \dots, F_n\}$ of faces such that $u \in F_1$, $v \in F_n$, and $F_i \cap F_{i+1} \neq \emptyset$ for $i = 1, 2, \dots, n-1$. Further consider a sequence $\mathscr{U} = \{u = u_0, u_1, \dots, u_n = v\}$ of vertices such that $u_i \in F_i \cap F_{i+1}$, $i = 1, 2, \dots, n-1$, and let s_i be the length of a shortest path p_i from u_{i-1} to u_i along the boundary of F_i. Now choose the sequences $\mathscr{F}$ and $\mathscr{U}$ minimum in the following sense. Consider $\mathscr{F}$ so that n is minimum; of all such $\mathscr{F}$, choose an $\mathscr{F}$ and $\mathscr{U}$ such that $S = \{s_0, s_1, \dots, s_n\}$ is lexicographically minimum.

It now suffices to prove that the path $p = p_1 \cup p_2 \cup \cdots \cup p_n$ is a non-revisiting path from u to v. If not, let F be a face revisited by p. By Lemma 1 it may be assumed that this revisit is not planar. Let x be the first vertex of p incident with F and y the last vertex of p incident with F. Assume j and k are such that $x \in p[u_j, u_{j+1}]$ and $y \in p[u_k, u_{k+1}]$, but $x \neq u_j$ and $y \neq u_{k+1}$. Then $k - j \geqslant 2$ since $\rho \geqslant 4$. If $k - j \geqslant 3$, then the sequence of faces $F_1, \dots, F_{j+1}, F, F_{k+1}, \dots, F_n$ contradicts the minimality of n. If $k - j = 2$, then $\rho = 4$, which implies that $x \neq u_{j+1}$. Now the sequence of vertices $\{u_0, u_1, \dots, u_j, x, y, u_{k+1}, \dots, u_n\}$ yields a sequence of lengths $S' = \{s'_0, s'_1, \dots, s'_n\}$ such that $s'_i = s_i$ for $i < j$, but $s'_j < s_j$, because the length of the shortest path s'_j from u_j to x along the boundary of F_{j+1} is less than the length s_j of the shortest path from u_j to u_{j+1} along the boundary of F_{j+1}. This contradicts the minimality of S.

The example in Fig. 3 shows that there is a polyhedral map on the double torus that does not possess the non-revisiting path property but has representativity $\rho = 3$. This example is essentially the same as the one we used in [12] to show that T_2 does not satisfy the non-revisiting path property. Consider the 24 faces given in Fig. 3. Identify like labeled edges. It is shown in [12] that the result is a polyhedral map M on the surface T_2 not satisfying the non-revisiting path property. In particular, there can be no non-revisiting path between the vertices labeled x and y. It is easily checked that M has representativity 3. □

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