Nonlinearity

PAPER

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To cite this article: Krzysztof Leśniak *et al* 2022 *Nonlinearity* **35** 5396

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **35** (2022) 5396–5426 https://doi.org/10.1088/1361-6544/ac8af1

Transition phenomena for the attractor of an iterated function system*

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Received 31 August 2021, revised 3 August 2022
Accepted for publication 18 August 2022
Published 23 September 2022

Abstract

Iterated function systems (IFSs) and their attractors have been central in fractal geometry. If the functions in the IFS are contractions, then the IFS is guaranteed to have a unique attractor. Two natural questions concerning contractivity arise. First, whether an IFS needs to be contractive to admit an attractor? Second, what occurs to the attractor at the boundary between contractivity and expansion of an IFS? The first question is addressed in the paper by providing examples of highly noncontractive IFSs with attractors. The second question leads to the study of two types of transition phenomena associated with an IFS family that depend on a real parameter. These are called lower and upper transition attractors. Their existence and properties are the main topic of this paper. Lower transition attractors are related to the semiattractors, introduced by Lasota and Myjak in 1990s. Upper transition attractors are related to the problem of continuous dependence of an attractor upon the IFS. A main result states that, for a wide class of IFS families, there is a threshold such that the IFSs in the one-parameter family have an attractor for parameters below the threshold and they have no attractor for parameters above the threshold. At the threshold there exists a unique upper transition attractor.

*This work was partially supported by a grant from the Simons Foundation (322515 to Andrew Vince).

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Keywords: iterated function system, attractor, threshold
Mathematics Subject Classification numbers: 28A80.

(Some figures may appear in colour only in the online journal)

1. Introduction

According to Hutchinson's seminal theorem 1.1 stated below, if the functions in an IFS on a complete metric space are contractions, then the IFS is guaranteed to have a unique attractor. Recently, however, there has been an interest in the exact relationship between contractivity and the existence of an attractor and, in particular, what occurs to the attractor at the boundary between contractivity and expansion of the IFS. These are the subjects of this paper.

Let $\mathbb{X}$ denote a complete metric space with metric $d(\cdot, \cdot)$. A finite iterated function system (IFS) is a set

$$F := \{f_1, f_2, \ldots, f_N\}$$

of $N \geqslant 2$ continuous functions from $\mathbb{X}$ to itself. An IFS is *affine* if its functions are invertible affine functions on d-dimensional Euclidean space $\mathbb{R}^d$, *projective* if its functions are non-singular projective functions on d-dimensional real projective space $\mathbb{RP}^d$, and *Möbius* if its functions are Möbius transformations on the extended complex plane $\mathbb{C} \cup \{\infty\}$, i.e., on the Riemann sphere. An affine IFS all of whose functions are similarities is referred to as a *similarity IFS*. An affine IFS all of whose functions are non-singular linear maps is refer to as a *linear IFS*.

For a function $f : \mathbb{X} \to \mathbb{X}$, let

$$\mathrm{Lip}(f, d) := \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}$$

denote the *Lipschitz constant* of f with respect to the metric d. For an IFS F, let

$$\mathrm{Lip}(F, d) := \max_{f \in F} \mathrm{Lip}(f, d).$$

A function f is *Lipschitz* if $\mathrm{Lip}(f, d) < \infty$, and an IFS F is *Lipschitz* if $\mathrm{Lip}(F, d) < \infty$. A function f is a *contraction* with respect to d if $\mathrm{Lip}(f, d) < 1$, and is *nonexpansive* if $\mathrm{Lip}(f, d) \leqslant 1$.

Definition 1.1. An IFS F on $\mathbb{X}$ is **contractive**, if there is an **equivalent metric**, also called an **admissible metric**, d' on $\mathbb{X}$, i.e., a metric d' giving the same topology as the original metric d, such that $\mathbb{X}$ remains complete with respect to d' and $\mathrm{Lip}(F, d') < 1$.

Allowing metrics topologically equivalent to the original metric is essential, for example, to the validity of theorem 3.1 in section 3. Also see example 3.4.

For the collection $2^{\mathbb{X}}$ of all subsets of $\mathbb{X}$, the **Hutchinson operator** $F : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ is given by

$$F(S) := \overline{\bigcup_{f\in F} f(S)}, \quad S \subseteq \mathbb{X}.$$

One can restrict the action of F to the collection $\mathcal{K}(\mathbb{X})$ of non-empty compact subsets of $\mathbb{X}$ equipped with the Hausdorff metric, denoted further by h. See, for example [13, 14], for the definition of h and its properties. Note that $F(S) = \bigcup_{f\in F} f(S)$ for compact S. By abuse of language, the same notation F is used for the IFS, the set of functions in the IFS, and for the Hutchinson operator; the meaning should be clear from the context.

A compact set $A \subseteq \mathbb{X}$ is the (strict) **attractor** of F if there is an open neighborhood $U \supseteq A$ such that

- (*Invariance*) $F(A) = A$, and
- (*Attraction*) $A = \lim_{n\to\infty} F^{(n)}(K)$,

where $F^{(n)}$ denotes the n-fold composition, the limit is with respect to the Hausdorff metric and is independent of the non-empty compact set $K \subseteq U$. So the attractor is the Banach fixed point of the Hutchinson operator on $\mathcal{K}(U)$. The largest such set U is called the *basin* of F.

Theorem 1.1 (Hutchinson [14]). *A contractive IFS on a complete metric space $\mathbb{X}$ has a unique attractor with basin $\mathbb{X}$.*

In classical IFS theory, it is assumed that the functions in the IFS are contractions, a natural assumption in light of Hutchinson's theorem. More recently, however, papers have appeared on IFS attractors assuming average contractivity (see [31] for a survey), on IFSs that are weakly contractive (see, for example, [21]), and on relaxing the definition of an attractor, see, for example [17, 18], in which the notion of a semiattractor is introduced to explain the nature of supports of invariant measures of average contractive IFSs [4]. This paper is concerned with attractor phenomena at the transition between contractivity and expansion of a one-parameter IFS family, between the existence and non-existence of an attractor. To illustrate this kind of transition phenomena, consider the following family F_t of IFSs that depends on a real parameter $t > 0$, which is based on [34, example 1.1].

Example 1.1. In $\mathbb{R}^3$ let $F_t := \{f_{(i,t)}, 1 \leqslant i \leqslant 2\}$ be the one-parameter affine family where $f_{(i,t)}(v) = tL_i(v - q_i) + q_i$, and where

$$L_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is the rotation by $\pi/4$ about the z-axis and $q_1 = (0, 0, 2)$ is a fixed point of L_1 outside the xy-plane; $L_2 = 0.4L_1$ and $q_2 = (1, 0, 0)$. For $t \in (0, 1)$, the IFS F_t is contractive and has an attractor A_t. Figure 1 shows views of A_t for $t = 0.9$ and $t = 0.96$. For $t \geqslant 1$, the IFS F_t fails to be contractive and has no attractor. The value $t = 1$ is called a *threshold*, defined precisely in definition 1.2 below.

The question arises as to the nature of the transition at the threshold $t = 1$. In this example, intriguing F_1-invariant sets occur. We refer to such sets as **transition attractors**, and we consider two types: *lower transition attractors*, denoted $A_\bullet$, and *upper transition attractors*, denoted $A^\bullet$. Namely, $A_\bullet$ is the smallest set with $F_1(A_\bullet) = A_\bullet$ and which contains all fixed points of the maps in F_1 that are the limits of fixed points of the maps in F_t as $t \to 1^-$, and

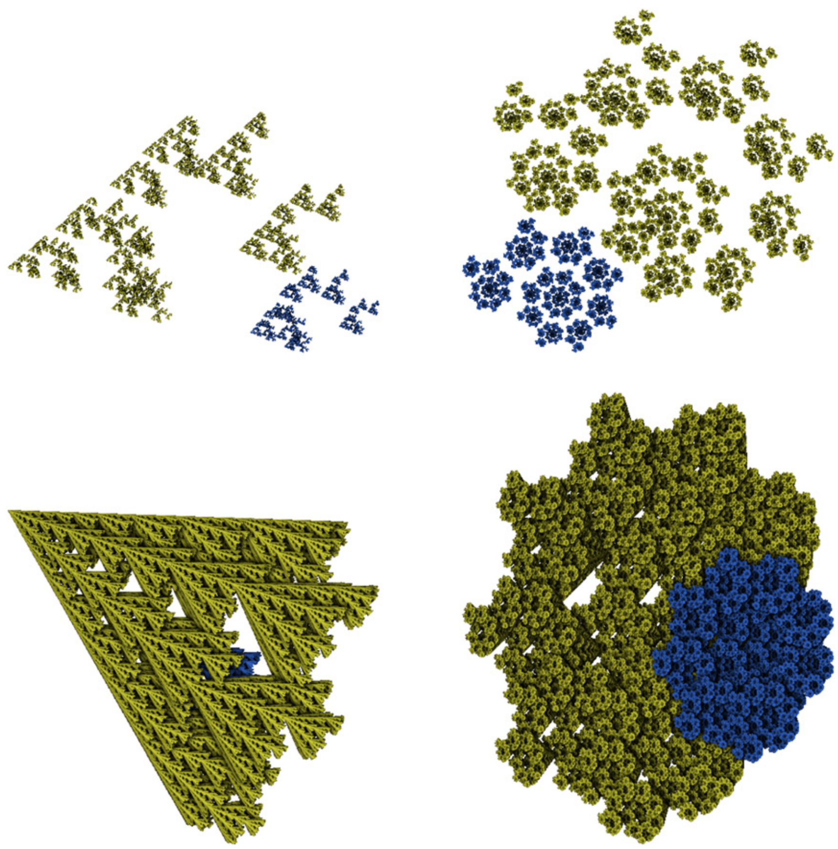

Figure 1. The attractor A_t for the one-parameter affine family F_t of example 1.1 for parameter values $t = 0.9$ (top line), $t = 0.96$ (bottom line); side and bottom view of a fractal 'cone'. The green and blue colours indicate the image of the attractor under the two maps of the IFS. Note that $f_{(1,t)}(A) \cap f_{(2,t)}(A) \neq \emptyset$.

$A^\bullet = \lim_{t\to 1^-} A_t$ (in the Hausdorff metric). Precise definitions (definitions 4.1 and 4.3) appear in section 4. The terminology 'upper' and 'lower' is due to the fact that, for appropriately defined one-parameter families, it is the case that $A_\bullet \subseteq A^\bullet$.

Figure 2 shows the lower transition attractor and figure 3 shows the upper transition attractor for the IFS family of example 1.1. The subject of transition attractors, in two guises, was introduced independently in [20, 34].

In example 1.1 it is natural to define an upper transition attractor of the family F_t as a limit of ordinary attractors of F_t as t approaches the threshold. Under mild conditions on F_t, it is not hard to prove that any sequence of attractors A_{t_n}, $t_n \to 1$, admits a convergent subsequence (theorem 4.3). Hence, potentially, there can exist several upper transition attractors $\lim_{n\to\infty} A_{t_n}$ depending on the sequence t_n. This leads to question 1.1 below about the uniqueness of an upper transition attractor. On the other hand, considering only the limits of fixed points of the maps in F_t, which necessarily belong to A_t for $t < 1$, results in a unique limit object. This leads to the concept of the lower transition attractor.

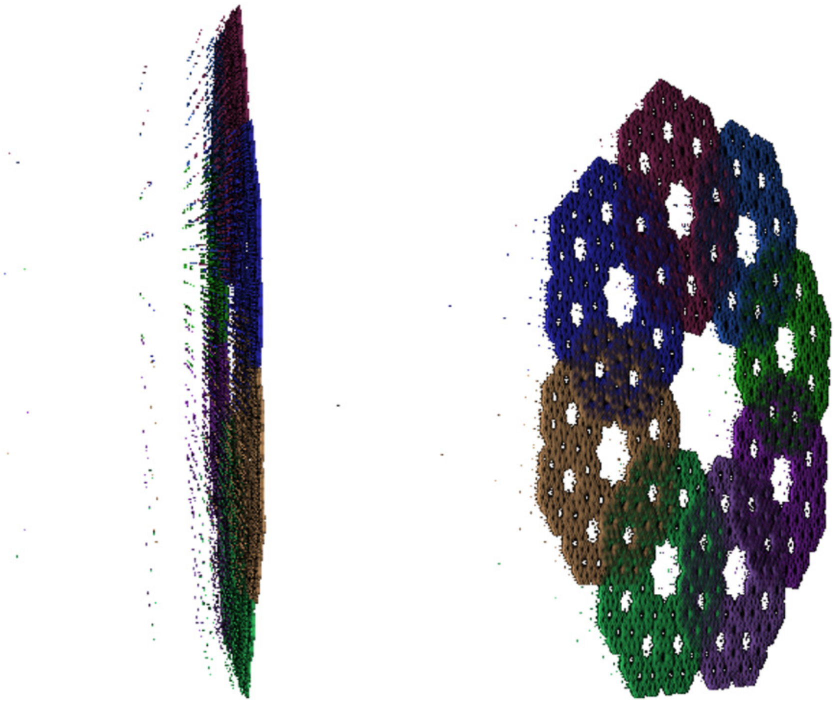

Figure 2. The lower transition attractor of example 1.1—side and bottom view of a fractal 'cone'. Note the thin dotted layers in the left panel that make up a 'sliced cone'.

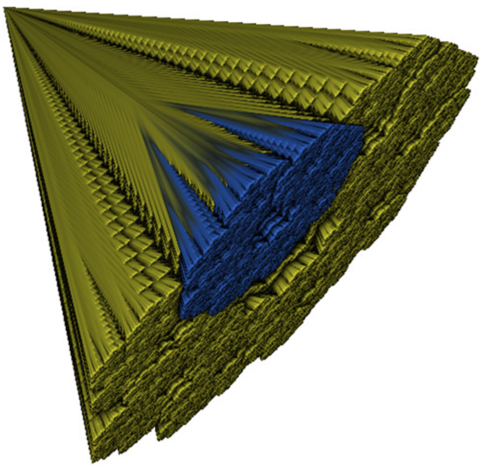

Figure 3. The upper transition attractor of example 1.1.

Definition 1.2. A **one-parameter family** is an IFS family

$$F_t := \{f_{(1,t)}, f_{(2,t)} \ldots, f_{(N,t)}\}$$

parametrized by a real number $t \in [0, \infty)$. The intuition is that, the nearer the parameter t is to 0, the more contractive the functions in the IFS, and as t increases, the functions in F_t become less contractive. A real number t_0 is called the **threshold** for the existence of an attractor of F_t if F_t has an attractor for $t < t_0$ but fails to have an attractor for $t > t_0$.

Under mild conditions (see theorems 4.2 and 4.3), a one-parameter family has a unique lower transition attractor and at least one upper transition attractor. The main open question in [34] was the following.

Question 1.1. If A_t, $t \in (0, t_0)$, denotes the attractor of a one-parameter family F_t of affine IFSs with threshold t_0, what conditions on F_t guarantee the existence of a unique upper transition attractor, i.e., a compact F_{t_0}-invariant set $A^\bullet$ such that

$$A^\bullet = \lim_{t \to t_0} A_t.$$

In [34] certain conditions on a one-parameter family of affine functions were conjectured to guarantee such a unique upper transition attractor $A^\bullet$. Roughly speaking these conditions were:

(a) F_t consists of similarity maps for each $t \in [0, t_0]$;
(b) Each function $f_{(i,t)}$ is conjugated to a linear function and this conjugation is independent of the parameter t or equivalently, for each i, all maps $f_{(i,t)}$, $t \in [0, t_0)$, have a common fixed point [34, proposition 5.4];
(c) There exists $i_* \in \{1, \ldots, N\}$ such that:
  1. $f_{(i_*,t_0)}$ is an isometry;
  2. The maps $f_{(i,t_0)}$ for $i \neq i_*$ have a smaller scaling ratio than $f_{(i_*,t_0)}$.

A main result of this paper, theorem 5.2 and its corollary 5.1, establishes uniqueness in the more general setting of a real Banach spaces instead of Euclidean space as in [34], and we allow for maps that are not necessarily affine. The theorem and corollary assume:

(a) Pointwise continuity of the maps in F_t with respect to the parameter t.
(b) The existence of a special index i_* so that all maps $f_{(i_*,t)}$, $t \in [0, t_0)$, have a common fixed point (no common fixed point is required for $i \neq i_*$);
(c)
  1. $\{f_{(i_*,t)}\}$, $t \in [0, t_0]$, is an affine family of the form considered in [34] (see definition 3.1) such that $f_{(i_*,t_0)}$ is a surjective isometry with a periodic linear part;
  2. $\sup_{i \neq i_*, t \in [0,t_0]} \mathrm{Lip}\, f_{(i,t)} < 1 = \mathrm{Lip}\, f_{(i_*,t_0)}$.

2. Organization and results

The paper is organized as follows.

- (Section 3: contractivity, attractors, and thresholds)

From the origin of IFS theory, the existence of an attractor has been associated with the contractivity of the IFS. The precise relationship, however, has not been completely delineated. The issue involves the converse of Hutchinson's theorem 1.1. For an IFS F on a complete metric space, Hutchinson's theorem states that contractivity of an IFS is a sufficient condition for the existence of a unique attractor. When the IFS contains only one mapping, the converse (which is a converse to the Banach contraction mapping theorem) was proved by Janös [16] and by Leader [19]. A converse is known to hold for affine, projective and Möbius IFSs; see theorem 3.1. In general, however, there are IFSs which admit attractors, but there is no equivalent metric with respect to which the functions in the IFS are contractions.

In the first part of section 3 we construct IFSs which admit attractors but are non-contractive (examples 3.1, 3.2, and 3.3). Two of those IFSs F are defined on the circle $\mathbb{X} = \mathbb{S}^1$ and have the property that $\mathrm{Lip}(F, d) > 1$ for all equivalent metrics d on $\mathbb{X}$, while the third one is defined on the Euclidean space $\mathbb{X} = \mathbb{R}^n$ and has the property that $\mathrm{Lip}(F, d) \geqslant 1$ for all equivalent metrics d on $\mathbb{X}$.

In the second part of section 3 the theory of affine one-parameter families of IFSs (definition 3.1) is reviewed. In particular, for a one-parameter affine IFS family F_t, there is a single threshold t_0 between contractivity and non-contractivity of F_t, between the existence and non-existence of an attractor of F_t, where t_0 is the reciprocal of the joint spectral radius of the linear parts of the affine functions involved. See theorems 3.2 and 3.3. Example 3.4 shows that, in general, a one-parameter family may have an attractor for all positive values of the parameter, despite the fact that the functions are not contractions for all large values of the parameter.

- (Section 4: lower transition attractors, upper transition attractors, and semiattractors)

In this section we develop a general theory of, not necessarily affine, one-parameter families F_t and their thresholds. The results are based on general hypotheses about

1. Continuous dependence of the IFS F_t upon t (called (H1), (H1′)),
2. Contractivity of the maps in F_t (called (H2), (H2′)), and
3. The behaviour of fixed points of the maps in F_t (called (H3)).

Under these hypotheses, the concepts of lower and upper transition attractors are defined as phenomena which occur at the threshold of F_t.

The results in this section are as follows.

- Theorem 4.2 establishes the existence of a lower transition attractor and its relation to the Lasota–Myjak theory of semiattractors (see definition 4.2 and theorem 4.1).
- Corollary 4.1 and remark 4.4 provide conditions under which the lower transition attractor is compact.
- Theorem 4.3 establishes the existence of a not necessarily unique upper transition attractor under very general conditions.
- Theorem 4.4 shows that in general the upper transition attractor is an invariant set and it contains the lower transition attractor.
- Proposition 4.1 shows that transition attractors are, in an appropriate sense, often symmetric sets.

Examples illustrate subtleties of the developed theory. Namely, example 4.1 exhibits a lower transition attractor that is unbounded, and example 4.3 shows that the upper transition attractor may not exist, despite the fact that the attractors exist for all parameter values of the one-parameter family of IFSs.

- (Section 5: the existence of a unique upper transition attractor)

Theorem 5.2 (and its corollary 5.1), the main result of the paper, provides an answer to question 1.1 in the introduction—giving conditions that guarantee a *unique* upper transition attractor. There is a short discussion in the introduction.

Examples 5.1–5.4 show that none of the assumptions in theorem 5.2 can be eliminated. Without any one of them, theorem 5.2 may fail to hold.

- (Section 6: open problems)

There are questions and conjectures about transition attractors that remain open. Several are posed in this section. For instance, question 6.2 asks whether the 'periodicity' assumption in theorem 5.2 can be dropped in a less general setting.

3. Contractivity, attractors, and thresholds

For an IFS on a complete metric space, the converse of Hutchinson's theorem 1.1 does not, in general, hold. Examples 3.1–3.3 are provided below. These examples notwithstanding, a converse does hold in the affine, Möbius, and projective cases.

Theorem 3.1 ([2, 5, 33]). *An affine, Möbius, or projective IFS can have at most one attractor. Moreover,*

(*a*) *An affine IFS F has an attractor if and only if F is contractive on $\mathbb{R}^d$.*
(*b*) *A Möbius IFS F has an attractor $A \neq \mathbb{C} \cup \{\infty\}$ if and only if F is contractive on an open set whose closure is not $\mathbb{C} \cup \{\infty\}$.*
(*c*) *A projective IFS F has an attractor that avoids some hyperplane if and only if F is contractive on the closure of some open set.*

There exist IFSs that have an attractor but are not contractive. For the examples F in [6, 21] $\mathrm{Lip}(F, d) = 1$. Our counterexamples below are of

(a) An IFS F on the circle $\mathbb{S}^1$ that admits a unique attractor but $\mathrm{Lip}(F, d) > 1$ for all equivalent metrics d on $\mathbb{X}$ (example 3.1),
(b) A stronger counterexample of an IFS F on $\mathbb{S}^1$ that admits a unique attractor but $\mathrm{Lip}(f, d) > 1$ for all $f \in F$ and all equivalent metrics d on $\mathbb{X}$ (example 3.2), and
(c) An IFS on $\mathbb{R}^n$ that has an attractor but is not contractive (example 3.3).

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be the angle doubling map $f(z) = z^2$ (see [10]). Let $\rho : \mathbb{S}^1 \to \mathbb{S}^1$ be the rotation map $\rho(z) = \mathrm{e}^{\mathrm{i}\alpha} z$ where $\alpha/2\pi$ is irrational and let $g(z) = \rho \circ f(z)$. The following proposition, which can be obtained from the standard theory of topological dynamics, is helpful in showing the validity of the two examples.

Proposition 3.1. *If d is any metric on $\mathbb{S}^1$ inducing the standard topology on $\mathbb{S}^1$, then* $\mathrm{Lip}(f, d) > 1$ *and* $\mathrm{Lip}(g, d) > 1$.

Proof. One can easily see that f and g are locally distance doubling with respect to the arc metric on $\mathbb{S}^1$. Therefore they are topologically expanding ([1], chapter 2.2 and [29]). Since the notion of a topologically expanding map on a compact space does not depend on the choice of metric, this proves proposition 3.1. Moreover, neither f nor g are locally nonexpansive at any point under any equivalent metric d on $\mathbb{S}^1$. □

Example 3.1 (An IFS on the circle $\mathbb{S}^1$ having an attractor, but with Lip(*F*, *d*) >1 for any admissible metric *d* on $\mathbb{S}^1$). With f and ρ as defined above, let $F := \{f, \rho, \mathrm{id}\}$, where id is the identity map on $\mathbb{S}^1$. That $\mathrm{Lip}(F, d) > 1$ follows from proposition 3.1. That $\mathbb{S}^1$ is the attractor of F is seen as follows. The invariance $F(\mathbb{S}^1) = \mathbb{S}^1$ is clear since ρ is a rotation. That $\lim_{n\to\infty} F^{(n)}(z) = \mathbb{S}^1$ for any $z \in \mathbb{S}^1$ can be seen as follows. We have $\{\rho^{(m)}(z) : 0 \leqslant m \leqslant n\} \subseteq F^{(n)}(z)$ and $\{\rho^{(m)}(z)\}_{m=0}^{\infty}$ is dense in $\mathbb{S}^1$, since ρ is an irrational rotation.

Note that one could remove the identity map from the IFS in example 3.1 and achieve the same outcome. The justification, however, becomes more complicated as in example 3.2 below.

Example 3.2 (An IFS on the circle $\mathbb{S}^1$ having an attractor, but with Lip(f, d) > 1 for all $f \in F$ under any admissible metric d on $\mathbb{S}^1$). With f and g as defined above, let $F := \{f, g\}$. Again, that $\mathrm{Lip}(f, d) > 1$ and $\mathrm{Lip}(g, d) > 1$ follows from proposition 3.1. That $\mathbb{S}^1$ is the attractor of F is seen as follows. The invariance $F(\mathbb{S}^1) = \mathbb{S}^1$ is clear since f maps $\mathbb{S}^1$ onto itself. That $\lim_{n\to\infty} F^{(n)}(z) = \mathbb{S}^1$ for any $z \in \mathbb{S}^1$ can be seen as follows. Abbreviate the point $\mathrm{e}^{\mathrm{i}\theta}$ by z_θ. For $(a_1, a_2, \ldots, a_n) \in (\mathbb{Z}_2)^n$, denote by $f_{(a_1,a_2,\ldots,a_n)} : \mathbb{S}^1 \to \mathbb{S}^1$ the map given by

$$f_{(a_1,a_2,\ldots,a_n)}(z_\theta) = z_{\left(2^n\theta + \sum_{k=1}^{n} 2^{k-1} a_k \alpha\right)}.$$

(That is $f_{(a_1,a_2,\ldots,a_n)} = f_{a_n} \circ \cdots \circ f_{a_2} \circ f_{a_1}$, under identification $f_0 := f$, $f_1 := g$.) Then for any $z = z_\theta$ we have

$$F^{(n)}(z) = \{f_{(a_1,\ldots,a_n)}(z_\theta) : (a_1, \ldots, a_n) \in (\mathbb{Z}_2)^n\} = \{\rho^{(m)}(z_{2^n\theta}) : 0 \leqslant m < 2^n\}.$$

Therefore, for any $\epsilon > 0$ there is an n such that there is no arc on $\mathbb{S}^1$ of length ϵ not containing a point of $F^{(n)}(z)$. Therefore $\lim_{n\to\infty} F^{(n)}(z) = \mathbb{S}^1$.

Example 3.3 (An IFS on $\mathbb{R}^n$ that has an attractor but is not contractive). Let A be a unit cube in $\mathbb{R}^n$, or any other convex compact set in $\mathbb{R}^n$, other than a single point, that is the attractor of an IFS F. Then A is a retract of $\mathbb{R}^n$, i.e., there exists a continuous map $r : \mathbb{R}^n \to A$ such that $r(\mathbb{R}^n) = A$ and r restricted to A is the identity map. (In fact, any set homeomorphic to a convex compact subset of a Banach space $\mathbb{X}$ is a retract of $\mathbb{X}$, see [12, chapter I, corollary 1.4, definition 1.7 and theorem 1.9.1].) Since A contains more than one point, the map r cannot be a contraction with respect to any metric equivalent to the Euclidean metric. Now let $G = F \cup \{r\}$. Then G is an IFS with attractor A. Indeed, $F^{(k)}(S) \subseteq G^{(k)}(S) \subseteq F^{(k)}(S) \cup A$ for any non-empty $S \subseteq \mathbb{R}^n$. Since r cannot be a contraction with respect to any metric equivalent to the Euclidean metric, the IFS G is not contractive.

Remark 3.1. The possibility of remetrization of a given IFS F by a metric making each map weakly contractive is equivalent to the existence of a coding map [3, 25].

Definition 3.1. A one-parameter family

$$F_t := \{f_{(1,t)}, f_{(2,t)} \ldots, f_{(N,t)}\}$$

whose functions have the form

$$f_{(i,t)}(x) = t\, f_i(x) + q_i, \quad x \in \mathbb{R}^d$$

where

$$F := \{f_1, f_2, \ldots, f_N\} \quad \text{and} \quad Q := \{q_1, q_2, \ldots, q_N\}$$

are a set of invertible affine transformations on $\mathbb{R}^d$ and a set of vectors in $\mathbb{R}^d$, respectively, is called a **one-parameter affine family**.

Theorem 3.2 below states that a one-parameter affine family has a threshold for the existence of an attractor. The threshold in example 1.1 is $t_0 = 1$. See [8, 9, 30] for background on the joint spectral radius.

Theorem 3.2 ([34]). *For a one-parameter affine family F_t, let $t_0 = 1/\rho(F)$, where $\rho(F)$ is the joint spectral radius of the linear parts of the functions in F. Then F_t has an attractor for $t < t_0$ and fails to have an attractor for $t > t_0$.*

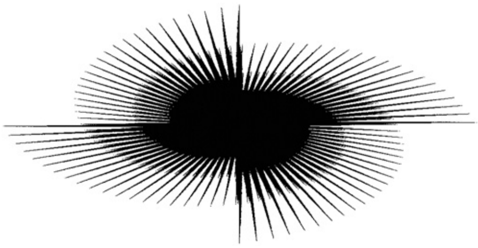

Figure 4. A transition attractor for a linear one-parameter family.

More can be said for a **linear family** F_t, all of whose maps are of the form $f_t(x) = t\,L(x)$, where L is a non-singular linear map. In this case, it immediately follows from theorem 3.2 that the attractor A_t of F_t is the origin, a single point, for all $t < t_0$, and there is no attractor for all $t > t_0$. However, the following holds.

Theorem 3.3 ([7]). *Let F_t be an irreducible (F admits no non-trivial invariant subspace), one-parameter linear IFS family on $\mathbb{R}^d$ with threshold t_0. Then there exists a compact F_{t_0}-invariant set that is centrally symmetric, star-shaped, and whose affine span is $\mathbb{R}^d$.*

In other words, the attractor evolves with the parameter t from trival to non-existent, blowing up only at the single threshold value $t = t_0$. An example in $\mathbb{R}^2$ is shown in figure 4 for $F := \{L_1, L_2\}$ where

$$L_1 = \begin{pmatrix} 0.02 & 0 \\ 0 & 1 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0.0594 & -1.98 \\ 0.495 & 0.01547 \end{pmatrix}.$$

In general, an IFS F_t may admit an attractor for all $t \geqslant 0$, despite the fact that the functions in F_t are not contractions with respect to the Euclidean metric for large values of t. See example 3.4 below.

Example 3.4 (A family of IFSs F_t on $\mathbb{R}^2$ that admits an attractor for all $t \geqslant 0$, but the functions in F_t are contractions with respect to the Euclidean metric for only values of t in a finite interval, cf [21], example 6.3). Define $F_t := \{f_{(1,t)}, f_{(2,t)}\}$, where

$$f_{(1,t)}(v) = \begin{pmatrix} 0 & \kappa_1 t \\ \lambda_1/t & 0 \end{pmatrix} v, \quad f_{(2,t)}(v) = \begin{pmatrix} 0 & \kappa_2 t \\ \lambda_2/t & 0 \end{pmatrix} v + \begin{pmatrix} 1/\lambda_2 - t \\ 1/\kappa_2 - 1/t \end{pmatrix},$$

where $\lambda_i, \kappa_j > 0$, $\lambda_i \kappa_j < 1$ for $i, j \in \{1, 2\}$.

Note that the functions in F_t are affine, but F_t is not an affine one-parameter family as in definition 3.1. The functions in F_t are contractions with respect to the Euclidean metric only for $\min\{1/\kappa_1, 1/\kappa_2\} > t > \max\{\lambda_1, \lambda_2\}$. We claim, however, that F_t admits an attractor A_t for all $t > 0$. Since F_t consists of affine functions for each $t > 0$, it would then follow that the IFS F_t is contractive for each $t > 0$ by theorem 3.1 part (a).

To see that F_t has an attractor, consider the second iterate $F_t^2 := \{f_{(i,t)} \circ f_{(j,t)} : 1 \leqslant i, j \leqslant 2\}$ of F_t given by

$$f_{(i,t)} \circ f_{(j,t)}\,(v) = \begin{pmatrix} \kappa_i \lambda_j & 0 \\ 0 & \kappa_j \lambda_i \end{pmatrix} v + a_{i,j}(t),$$

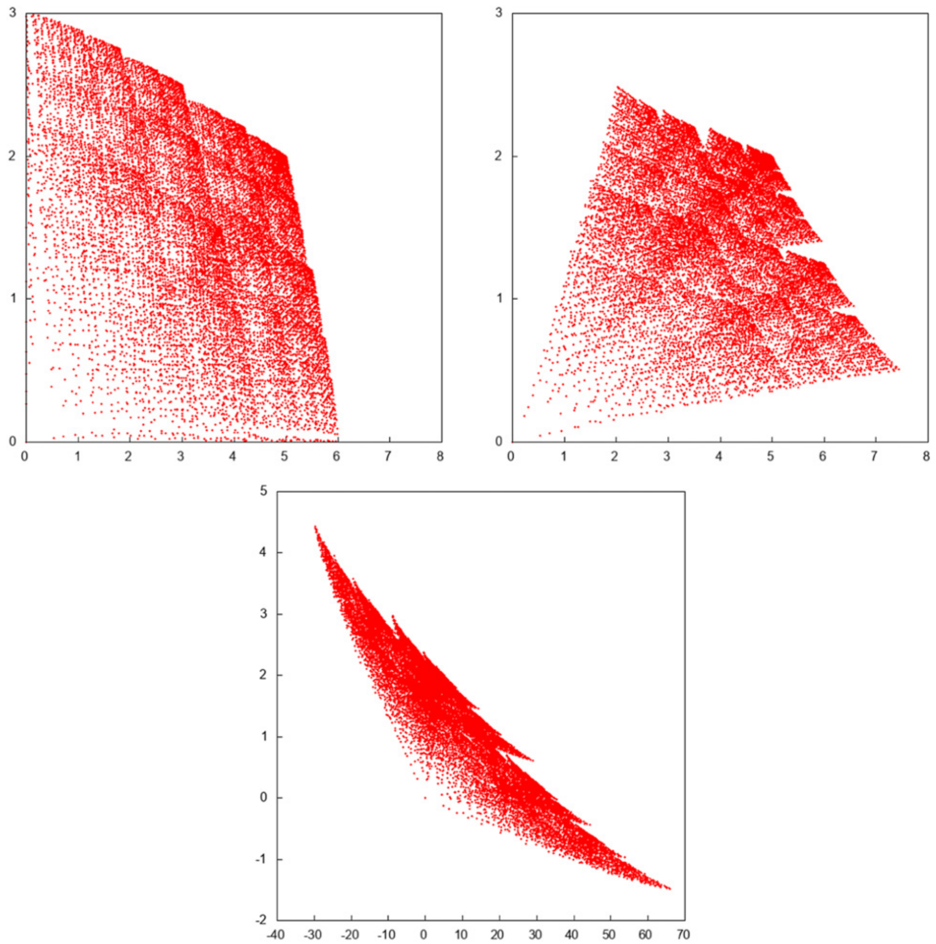

Figure 5. The attractor A_t for the one-parameter affine family F_t of example 3.4 for successive parameter values $t = 0.5, 1, 5$.

where the vectors $a_{i,j}(t)$, $i, j \in \{1, 2\}$, are readily calculated. The two functions in F_t^2 are contractions for all $t \in (0, \infty)$ when $0 < \kappa_i \lambda_j < 1$ for $i, j \in \{1, 2\}$, and therefore have an attractor for all $t \in (0, \infty)$. If an attractor exists for one of them, then F_t and F_t^2 have the same attractor. Therefore F_t admits an attractor A_t for all t. The attractor of F_t^2 is shown in figure 5 for three values of t in the case that $\lambda_1 = 1/4, \kappa_1 = 3, \lambda_2 = 1/5$ and $\kappa_2 = 2$. The functions in F_t are contractions with respect to the Euclidean metric only for $t \in (1/4, 1/3)$, yet the functions in the second iterate F_t^2 are contractions for all $t \in (0, \infty)$.

4. Lower transition attractors, upper transition attractors, and semiattractors

Consider a one-parameter family

$$F_t := \{f_{(i,t)} : 1 \leqslant i \leqslant N\},\ \ t \in [0, \infty),$$

consisting of Lipschitz maps defined on a complete metric space $(\mathbb{X}, d)$. Let t_0 be the threshold for the existence of an attractor as given in definition 1.2. We say that $\widehat{t_0}$ is a **threshold for contractivity** if F_t is contractive for all $0 < t < \widehat{t_0}$ and

$$\widehat{t_0} = \sup \{t : F_t \text{ is contractive}\}.$$

We assume throughout this section that F_t has a finite contractivity threshold. Note that for $t < \widehat{t_0}$, the IFS F_t has an attractor; hence

$$\widehat{t_0} \leqslant t_0$$

if a threshold t_0 exists. It is often the case and it is the interesting case when $\widehat{t_0} = t_0$, but we know from the examples in section 3 that this is not always true.

Various subsequent results in the paper will concern an IFS family F_t for values of t in an interval $[0, t^*]$, where $0 < t^* \leqslant t_0$. Often, but not always, t^* will be a threshold value. It can be assumed, without loss of generality, that $t^* = 1$. Indeed, we can reparametrize

$$\widetilde{f_{(i,t)}} := f_{(i,tt_*)}$$

and

$$\widetilde{F}_t := \{\widetilde{f_{(i,t)}} : 1 \leqslant i \leqslant N\}.$$

Therefore, throughout the rest of the paper, the parameter t is restricted to the closed interval $[0, 1]$. This allows for ease of exposition and a greater generality of the results.

The following conditions on the one-parameter family F_t on the metric space $(\mathbb{X}, d)$ appear in the hypotheses of the results in this paper:

(H1) The map $t \mapsto f_{(i,t)}(x) \in \mathbb{X}$ is continuous for every $x \in \mathbb{X}$ and every $i = 1, \ldots, N$;

(H1′) The map $t \mapsto f_{(i,t)} \in \mathcal{C}(\mathbb{X})$ is continuous for every $i = 1, \ldots, N$, where the space $\mathcal{C}(\mathbb{X})$ of continuous selfmaps of $\mathbb{X}$ is endowed with the topology of uniform convergence;

(H2) $\mathrm{Lip}(F_t, d) < 1$ for all $t \in [0, 1)$;

(H2′) $\sup_{t\in[0,1)} \mathrm{Lip}(F_t, d) < 1$;

(H3) $q_i := \lim_{t\to 1^-} q_{i,t}$ exist for each $1 \leqslant i \leqslant N$, where $q_{i,t}$ denotes the unique fixed point of $f_{(i,t)}, t \in [0, 1)$. Define $Q := \{q_i : 1 \leqslant i \leqslant N\}$.

Clearly conditions (H1)–(H3) are satisfied within the framework of affine one-parameter families considered in [34], see definition 3.1.

Whenever the IFS F_t has an attractor, we denote it by A_t.

Remark 4.1. If (H1) and (H2) hold, then

(a) Each $F_t : \mathcal{K}(\mathbb{X}) \to \mathcal{K}(\mathbb{X})$, $t < 1$, is a Banach contraction in the Hausdorff metric.

(b) $\mathrm{Lip}(F_1, d) \leqslant 1$. In particular, the Hutchinson operator $F_1 : \mathcal{K}(\mathbb{X}) \to \mathcal{K}(\mathbb{X})$ is nonexpansive in the Hausdorff metric.

(c) The limit point $q_i \in Q$ in (H3) is a fixed point of $f_{(i,1)}$ as easily follows from (d) below. One should be aware, however, that q_i is not necessarily a unique fixed point of $f_{(i,1)}$ (just think of the affine one-parameter family $f_{(i,t)}(x) = tx$; see also example 4.2).

(d) The following weakening of condition (H1′) is fulfilled: the map $t \mapsto f_{(i,t)} \in \mathcal{C}(\mathbb{X})$ is continuous for every $i = 1, \ldots, N$, where the space $\mathcal{C}(\mathbb{X})$ of continuous selfmaps of $\mathbb{X}$ is endowed with the topology of uniform convergence on compacta. It is worth pointing out that the above weakening of condition (H1′) holds when the condition (H2) is relaxed to the following one: $\mathrm{Lip}(F_t, d) \leqslant 1$ for all $t \in [0, 1)$.

4.1. The lower transition attractor and semiattractor

Definition 4.1. Let a one-parameter family F_t satisfy condition (H3). The **lower transition attractor** of F_t is the smallest (with respect to inclusion) set $A_\bullet$ which is **($\mathbf{F_1}$, Q)-invariant**, i.e., $F_1(A_\bullet) = A_\bullet$ and $A_\bullet \supseteq Q$. (Equivalently, $A_\bullet$ is the smallest set with $F_1(A_\bullet) \cup Q = A_\bullet$; see the first part of proof of theorem 4.2.)

Definition 4.2 ([17, 27]). Let F be an IFS on a metric space $\mathbb{X}$. If the intersection is nonempty, then the **semiattractor** of F is

$$A_* := \bigcap_{x\in\mathbb{X}} \mathrm{Li}(F^{(n)}(\{x\})),$$

where $\mathrm{Li}(S_n)$ is the lower Kuratowski limit ([13]) of a sequence of sets $S_n \subseteq \mathbb{X}$, i.e.,

$$\mathrm{Li}(S_n) := \{y \in \mathbb{X} : \text{there exist points} \quad x_n \in S_n \quad \text{such that} \quad x_n \to y\}.$$

Note that a semiattractor can be unbounded, e.g. [17]. The following properties of an IFS with semiattractor A_* hold.

Theorem 4.1 ([27]). *If F is an IFS on a complete metric space with semiattractor A_*, then*

(*a*) *$F(A_*) = A_*$; moreover A_* is the smallest F-invariant set.*
(*b*) *If F admits an attractor A with a full basin $\mathbb{X}$, then $A_* = A$.*

Originally the notion of a semiattractor appeared in the works of Lasota and Myjak; see [17, 18]. This framework was intended to give a systematic approach to IFSs where noncontractive maps are added to an IFS consisting of contractions. Interest in such systems stems from computer graphics and dates back to Barnsley and Elton [4]. In a particular case, where an isometry is added to a contractive IFS, such a system is related to another contractive IFS. This allows one to use the standard methods for computer drawing of attractors of contractive IFSs, as is done in example 4.2 below; see [20, 32]. However, in general, when an IFS contains noncontractive maps or maps with parabolic fixed points, specific methods of drawing attractors need to be used; see [26].

Theorem 4.2 below is a significantly more general version of those parts of [34, theorem 8.2] pertaining to the lower transition attractor. In addition, part (d) of theorem 4.2 relates the lower transition attractor of a one-parameter family F_t to the semiattractor of an associated IFS.

Theorem 4.2. *Let F_t be a one-parameter family $F_t, t \in [0, 1]$, on a complete metric space $(\mathbb{X}, d)$ that satisfies (H1)–(H3). Then the lower transition attractor $A_\bullet$ exists.*

Furthermore, let $Q' = \{q_i : i \in J\}$, where $J \neq \emptyset$ is such that

$$\{i \in \{1, \ldots, N\} : \mathrm{Lip}(f_{(i,1)}) = 1\} \subseteq J \subseteq \{1, \ldots, N\}.$$

In other words, $Q' \subseteq Q$ contains at least the limits of fixed points q_i that correspond to those functions $f_{(i,1)}$ that are not contractions. Then $A_\bullet$ satisfies the following properties:

(*a*) *$A_\bullet$ is the smallest (F_1, Q')-invariant set and*

$$A_\bullet = \bigcap \{A \in 2^{\mathbb{X}} : F_1(A) = A \quad \text{and} \quad Q' \subseteq A\};$$

(*b*) *$A_\bullet = \overline{\bigcup_{n\geqslant 0} F_1^n(Q')}$;*
(*c*) *The lower transition attractor $A_\bullet$ is the semiattractor of any IFS of the form $F_1^\flat := F_1 \cup \{\check{q}(x) : q \in Q'\}$, where $\check{q}(x) := q$ is the constant map on $\mathbb{X}$.*

Proof. Clearly, $F_1^\flat(S) = F_1(S) \cup Q'$ for any nonempty $S \subseteq \mathbb{X}$, and $F_1(Q') \supseteq Q'$. First note that the set A is the smallest $F_1^\flat$-invariant set if and only if A is the smallest F_1-invariant set which contains Q'. Indeed, $A = F_1^\flat(A) = F_1(A) \cup Q'$ implies $F_1(A) \subseteq A$ and $A = A \cup Q' \supseteq Q'$. Hence

$$F_1(A) = F_1(A \cup Q') = F_1(A) \cup F_1(Q') \supseteq F_1(A) \cup Q' = A.$$

In the reverse direction, if $A = F_1(A)$ and $A \supseteq Q'$, then $F_1^\flat(A) = F_1(A) \cup Q' = A \cup Q' = A$.

Second, observe that the subsystem $\{\check{q} : q \in Q'\} \subseteq F_1^\flat$ consists of contractions and admits a semiattractor (even attractor), which is Q'. Hence, by the Lasota–Myjak criterion ([27], theorem 6.3), $F_1^\flat$ admits a semiattractor, denoted A'_*. By the equivalence shown in the first paragraph, we have

$$A'_* = \bigcap \{A \in 2^{\mathbb{X}} : F_1(A) = A \quad \text{and} \quad Q' \subseteq A\}. \tag{4.1}$$

Furthermore, since $A'_* \supseteq Q'$ and $(F_1^\flat)^n(Q') = F_1^n(Q')$, we have

$$A'_* = \overline{\bigcup_{n \geqslant 0} F_1^n(Q')} \tag{4.2}$$

due to the self-regeneration formula in the Lasota–Myjak criterion ([27], theorem 6.3, equation (6.9)). In particular, the above is true for $Q' = Q$, in which case we write A_* for the semiattractor. We have established the existence of a lower transition attractor, which is $A_\bullet = A_*$.

Third, we shall establish that all A'_* are equal to A_*. This will give property (c) and, in turn, (b) (due to (4.2)) and (a) (due to (4.1)). Of course $A'_* \subseteq A_*$. Consider $q_i = f_{(i,1)}(q_i)$ with $i \notin J$. Since $\{q_i\}$ is the attractor of the subsystem $\{f_{(i,1)}\} \subseteq F_1^\flat$, we have $q_i \in A'_*$. Overall $Q \subseteq A'_*$ and $A_* \subseteq A'_*$. □

Remark 4.2. In theorem 4.2, if F_1 has an attractor A_1, then $A_1 = A_\bullet$. This is because $A_1 \supseteq Q$, where Q is given in condition (H3).

Remark 4.3. The original definition of a lower transition attractor in [34] required that $A_\bullet$ is (F_1, Q')-invariant, where $Q' = \{q_{i_*}\}$, and i_* is a unique index $i \in \{1, \ldots, N\}$ with $\mathrm{Lip}(f_{(i,1)}) = 1$. Thanks to theorem 4.2, both definitions, the one in this paper and that in [34] coincide.

Under mild additional conditions on F_t, the lower transition attractor is compact. See corollary 4.1 and remark 4.4 below. These results require extending some concepts defined in section 1 to infinite IFSs, e.g., [22, 23]. Let F be a finite or infinite IFS on a complete metric space $\mathbb{X}$. The Hutchinson operator on $2^{\mathbb{X}}$ for an infinite IFS F is defined exactly as for a finite IFS in section 1. Further, as in the finite IFS case, we define

$$\mathrm{Lip}(F, d) := \sup_{f \in F} \mathrm{Lip}(f, d).$$

An IFS F on $\mathbb{X}$ will be called **compact** if $F(K)$ is compact for every compact set $K \subseteq \mathbb{X}$. Clearly, any finite IFS is compact.

Given an IFS F on $\mathbb{X}$, the **monoid** induced by F is

$$\mathbb{M}(F) := \{f_1 \circ \cdots \circ f_k : f_1, \ldots, f_k \in F, k \in \mathbb{N}\} \cup \{\mathrm{id}_{\mathbb{X}}\}.$$

A monoid can be treated as a new IFS. In particular, we may speak of a compact monoid.

Corollary 4.1. *Let F_t be as in theorem 4.2 and let $J = \{1 \leqslant i \leqslant N : \mathrm{Lip}(f_{(i,1)}, d) = 1\}$. If the monoid $\mathbb{M}(\{f_{(i,1)} : i \in J\})$ is compact, then the lower transition attractor $A_\bullet$ of F_t is compact.*

Proof. The statement follows from theorem 4.2(c) and from [32, theorem 4.1(E)]. □

Recall that a metric space is **proper** if its bounded and closed subsets are compact.

Remark 4.4. If either of the following two conditions hold, then the monoid $\mathbb{M}(\{f_{(i,1)} : i \in J\})$ is compact.

- $J = \{i_*\}$ for some $i_* \in \{1, \dots, N\}$, and $f_{(i_*,1)}$ is a periodic isometry, see [20];
- $\mathbb{X}$ is proper and all $f_{(i,1)}$, $i \in J$, have a common fixed point (not necessarily unique), see [32, theorem 4.2(ii), remark 2.2 item 3].

The compactness of the lower transition attractor $A_\bullet$ in corollary 4.1 cannot be inferred from (H1)–(H3) alone. Example 4.1 below is a counterexample.

Example 4.1 (A one-parameter family satisfying (H1)–(H3) whose lower transition attractor is not compact.)

On $\mathbb{R}$ let $F_t := \{g_t, f_t\}$, where $g_t(x) = -tx$ and $f_t(x) = -tx + t + 1$. For $t \in (0, 1)$ we have $A_t = [-t/(1-t), 1/(1-t)]$. In this case $A_\bullet = \mathbb{Z}$.

Example 4.2 below is a three-dimensional example illustrating the previous results in this section.

Example 4.2. In $\mathbb{R}^3$ let $F_t = \{f_{(i,t)}, 1 \leqslant i \leqslant 5\}$ be the one-parameter affine family where $f_{(i,t)}(v) = tL_i(v - q_i) + q_i$, and

$$L_1 = L_2 = L_3 = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The map L_4 is the rotation by $\pi/2$ about the y-axis, and L_5 is the reflection in the xz-plane. The fixed points are

$$q_i = \left(\cos\frac{2\pi(i-1)}{3}, \sin\frac{2\pi(i-1)}{3}, 0\right) \quad \text{for} \quad i = 1, 2, 3, \quad q_4 = (0, 1, 0), \quad q_5 = (0, 0, 1),$$

where q_1, q_2, q_3 are the third roots of unity in the xy-plane. Note that the attractor of the IFS $\{f_{(i,1)}, 1 \leqslant i \leqslant 3\}$ is the Sierpiński triangle in the xy-plane with vertices q_1, q_2, q_3.

For each $1 \leqslant i \leqslant 5$, the point q_i is a common fixed point of $f_{(i,t)}$ for $t \in [0, 1]$. However, q_i is not the only fixed point of $f_{(i,1)}$ for $i = 4, 5$. More precisely, $f_{(4,1)}$ has the whole y-axis as its set of fixed points; $f_{(5,1)}$ has the whole xz-plane as its set of fixed points; and $(0, 0, 0) \neq q_4, q_5$ is the only common fixed point of $f_{(4,1)}$ and $f_{(5,1)}$.

On the left in figure 6 is the attractor A_t of F_t for $t = 0.8$. By theorem 4.2 the lower transition attractor $A_\bullet$ for IFS family F_t of example 4.2 exists; it appears on the right in figure 6. By corollary 4.1 $A_\bullet$ is compact, since one can easily check that the monoid $\mathbb{M}(\{f_{(4,1)}, f_{(5,1)}\})$ is finite or one can apply remark 4.4. Figure 6 was generated using Mekhontsev's IFStile program [24]. To draw $A_\bullet$ using this program we have applied part (c) of theorem 4.2 which identifies $A_\bullet$ as a semiattractor of a suitable IFS $F_1^\flat := \{f_{(1,1)}, f_{(2,1)}, f_{(3,1)}, f_{(4,1)}, f_{(5,1)}, \check{q}_4, \check{q}_5\}$ related to F_t. Then the resulting IFS $F_1^\flat$ was replaced with a contractive IFS according to [32, theorem 4.1(B)].

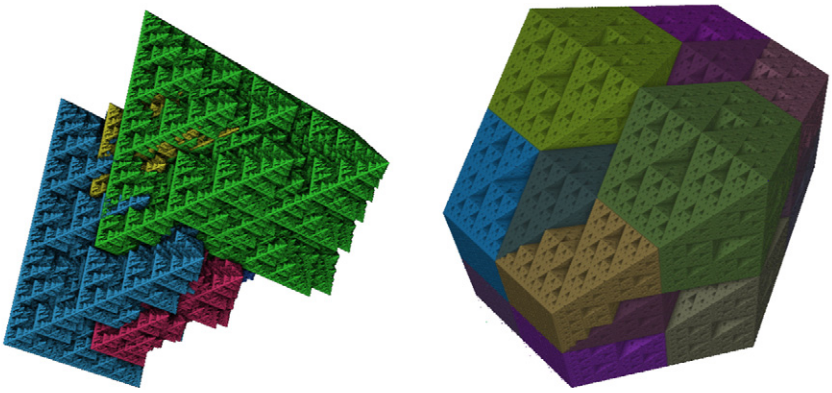

Figure 6. The attractor A_t for the one-parameter affine family F_t of example 4.2 for parameter value $t = 0.8$ and the lower transition attractor $A_\bullet$ of F_t.

4.2. The upper transition attractor

Definition 4.3. Call a compact set $A^\bullet$ an **upper transition attractor** of a one-parameter IFS family $F_t := \{f_{(1,t)}, f_{(2,t)}, \ldots, f_{(N,t)}\}, t \in [0, 1]$, if there is an increasing sequence $t_n \to 1$ such that

$$A^\bullet = \lim_{n\to\infty} A_{t_n}.$$

Theorems 4.3, 4.4, and proposition 4.1 below are strong versions of results on upper transition attractors and their relation to the lower transition attractor that were proved in [34] only for special cases of one-parameter similarity families.

Theorem 4.3. *Let F_t, $t \in [0, 1]$, be a one-parameter family of IFSs on a proper metric space $(\mathbb{X}, d)$. Assume that*

(a) *either F_t satisfies (H1′) and (H2);*

(b) *or F_t satisfies (H1) and (H2), and there is (possibly empty) $I \subseteq \{1, \ldots, N\}$ such that:*

1. *The sub-family $F_t^I := \{f_{(i,t)} : i \in I\}$ of F_t satisfies (H2′) and $\mathrm{Lip}(F_1^I, d) < 1$, i.e., $\sup\{\mathrm{Lip}(f_{(i,t)}, d) : t \in [0, 1], i \in I\} < 1$;*
2. *The maps $f_{(i,t)}$ have a common fixed point for all $t \in [0, 1]$, $i \notin I$.*

Then F_t admits at least one upper transition attractor.

Condition (2) in part (b) of the above theorem may look artificially strong. However, in view of remark 5.3, it is in line with condition (c) of theorem 5.2.

To prove theorem 4.3 we need the following lemmas.

Lemma 4.1. *Assume that (f_n) is a sequence of contractions on a complete metric space $(\mathbb{X}, d)$, uniformly convergent to some function f. Then the set of fixed points of maps f_n, $n \in \mathbb{N}$, is bounded.*

Proof. Let $d_{\sup}(f, g) := \sup_{x\in\mathbb{X}} d(f(x), g(x))$ for $f, g : \mathbb{X} \to \mathbb{X}$. For $n \in \mathbb{N}$, let x_n be the fixed point of f_n. Fix an $n_0 \in \mathbb{N}$ such that $d_{\sup}(f_n, f) < 1$ for all $n \geqslant n_0$. For every $n \geqslant n_0$, we have

$$\begin{aligned} d(x_n, x_{n_0}) &\leqslant d(f_n(x_n), f_{n_0}(x_n)) + d(f_{n_0}(x_n), f_{n_0}(x_{n_0})) \\ &\leqslant d_{\sup}(f_n, f_{n_0}) + \mathrm{Lip}(f_{n_0}) d(x_n, x_{n_0}) \\ &\leqslant d_{\sup}(f_n, f) + d_{\sup}(f, f_{n_0}) + \mathrm{Lip}(f_{n_0}) d(x_n, x_{n_0}). \end{aligned}$$

Hence

$$d(x_n, x_{n_0}) \leqslant \frac{d_{\sup}(f_n, f) + d_{\sup}(f, f_{n_0})}{1 - \mathrm{Lip}(f_{n_0})} \leqslant \frac{2}{1 - \mathrm{Lip}(f_{n_0})}.$$

Therefore

$$\mathrm{diam}\{x_n : n \in \mathbb{N}\} \leqslant 2 \max\left\{ d(x_1, x_{n_0}), \dots, d(x_{n_0-1}, x_{n_0}), \frac{2}{1 - \mathrm{Lip}(f_{n_0})} \right\} < \infty.$$

□

The lemma below will be used in section 5 as well.

Lemma 4.2. *Let $\mathbb{X}$ be a metric space and $f_t, t \in [0,1]$, be a family of nonexpansive selfmaps of $\mathbb{X}$ such that the map $[0,1] \ni t \mapsto f_t(x)$ is continuous for every $x \in \mathbb{X}$. Then*

(*a*) *For every nonempty and compact set $D \subseteq \mathbb{X}$, the map*

$$[0,1] \ni t \mapsto f_t(D) \in \mathcal{K}(\mathbb{X})$$

is uniformly continuous;

(*b*) *The IFS $F := \{f_t : t \in [0,1]\}$ is compact.*

Proof. Part (a). This result follows from remark 4.1(d). Indeed, observe that the one-parameter family $F_t := \{f_t\}$ satisfies (H1) and $\mathrm{Lip}(F_t, d) \leqslant 1$.

Part (b). Take any nonempty and compact set $D \subseteq \mathbb{X}$. By (a)

$$F(D) = \overline{\bigcup_{t\in[0,1]} f_t(D)} = \bigcup_{t\in[0,1]} f_t(D)$$

is compact thanks to [13, corollary 2.20, ch 2.1 p 42 and theorem 2.68, ch 2.2 p 62]. □

Proof of theorem 4.3. Fix a sequence $[0,1) \ni t_n \nearrow 1$. For every sequence of attractors A_{t_n} of F_{t_n}, we are going to find a bounded set B so that $A_{t_n} \subseteq B$. Since $\mathbb{X}$ is a proper metric space, (A_{t_n}) must have a convergent subsequence, whose limit is an upper transition attractor of F_t.

Part (a). Observe that

$$\begin{aligned} \sup_{K\in\mathcal{K}(\mathbb{X})} h(F_{t_n}(K), F_1(K)) &= \sup_{K\in\mathcal{K}(\mathbb{X})} h\left(\bigcup_{i=1}^{N} f_{(i,t_n)}(K), \bigcup_{i=1}^{N} f_{(i,1)}(K) \right) \\ &\leqslant \sup_{K\in\mathcal{K}(\mathbb{X})} \max_{i=1,\dots,N} h(f_{(i,t_n)}(K), f_{(i,1)}(K)) \\ &\leqslant \max_{i=1,\dots,N} d_{\sup}(f_{(i,t_n)}, f_{(i,1)}) \to 0. \end{aligned}$$

Therefore the sequence of Hutchinson operators (F_{t_n}) is uniformly convergent to F_1. Hence by lemma 4.1 the family of attractors $\{A_{t_n} : n \in \mathbb{N}\}$ is bounded in $\mathcal{K}(\mathbb{X})$. So $B := \bigcup_{n\in\mathbb{N}} A_{t_n} \subseteq \mathbb{X}$ is bounded.

Part (b). The case $I = \emptyset$ is immediate since $A_t = \{q_*\} =: B$, where q_* is a common fixed point of $f_{(i,t)}$'s guaranteed by condition (2.).

To find B in case $I \neq \emptyset$, consider the IFSs

$$F' := \bigcup_{i \in I} \{f_{(i,t)} : \ t \in [0,1]\}$$

$$F'' := \bigcup_{i \notin I} \{f_{(i,t)} : \ t \in [0,1]\}.$$

First observe that the IFS F' is compact since, by lemma 4.2(b), F' is a finite union of compact IFSs. Moreover, $\mathrm{Lip}(F', d) < 1$.

Second observe that by [32, remark 2.2, items (3) and (4)], the monoid $\mathbb{M}(F'')$ is compact. Moreover, $\mathrm{Lip}(F'', d) \leqslant 1$.

By the above observations, using [32, theorem 4.1, remark 2.2], we have that the IFS $F' \cup F''$ has a compact semiattractor B. In particular, $F_t(B) \subseteq F'(B) \cup F''(B) = B$. So, for each $t < 1$, $A_t \subseteq B$ for the attractor A_t of F_t. $\square$

The existence of an upper transition attractor in theorem 4.3 cannot be inferred from (H1)–(H3) alone; see example 4.1. Neither can it be inferred from (H1) and the assumption that all maps in F_t are contractions for all $t \in [0,1]$. This is in contrast to a lower transition attractor, which exists under conditions (H1)–(H3); see theorem 4.2. Nevertheless, if the attractor of F_1 exists and the upper/lower transition attractor of F_t exists, then they both coincide; see theorem 4.4(a) and remark 4.2. It is worth recalling that a one-parameter family F_t, $t \in [0,1]$, can satisfy (H1) and (H2) and have a unique upper transition attractor, while the attractor of F_1 does not exist, see [34, example 8.1]. Example 4.3 below shows that the opposite situation can occur. Namely, we construct a family F_t for which F_1 admits an attractor, but F_t does not have an upper transition attractor.

Example 4.3. Motivated by the construction in [28, example 1], we will construct a one-parameter family of IFSs F_t, $t \in [0,1]$, with the following properties:

(a) F_t satisfies (H1);

(b) For all $t \in [0,1]$ all maps in F_t are contractions, in particular F_t satisfies (H2);

(c) F_t has no upper transition attractor.

Let ℓ_1 be the Banach space of absolutely summable sequences of real numbers. We will construct functions $f_t : \ell_1 \to \ell_1$, $t \in [0,1]$, such that the one-parameter family $F_t = \{f_t\}$ will satisfy properties (a)–(c) above.

First, for every $n \in \mathbb{N}$, define $g_n : \ell_1 \to \ell_1$ by

$$g_n(x) := \left(\frac{1}{n} + \left(1 - \frac{1}{n}\right) x_n\right) \mathbf{e}_n, \quad x = (x_n) \in \ell_1,$$

where $\mathbf{e}_n = (0, \ldots, 0, 1, 0, \ldots)$ is a sequence with a 1 in the nth position and 0s elsewhere. Observe that

$$\|g_n(x)\| \leqslant \frac{1}{n} + |x_n|$$

and so

(a′) $g_n(x) \to \mathbf{0}$ for every $x \in \ell_1$,

where $\mathbf{0}$ is the sequence of zeros. Moreover, the sequence of functions g_n satisfies (b′) $\mathrm{Lip}(g_n) = 1 - \frac{1}{n} < 1$;

(c′) $\mathbf{e}_n$ is a unique fixed point of g_n but $\mathbf{e}_n \not\to \mathbf{0}$.

Next we construct a one-parameter family f_t, $t \in [0, 1]$, which interpolates the sequence g_n. It will be done in such a way that the properties (a′)–(c′) of g_n bootstrap to the properties (a)–(c) of $F_t = \{f_t\}$, $t \in [0, 1]$. In order to do that define

$$f_1(x) := \mathbf{0}, \quad x \in \ell_1.$$

Next choose any increasing sequence (a_n) of real numbers tending to 1 and such that $a_0 = 0$. We need three functions $\alpha, \beta : [0, 1) \to [0, 1]$, and $\nu : [0, 1) \to \mathbb{N} \cup \{0\}$. Dissect the unit interval into disjoint subintervals $[0, 1) = \bigcup_{n\in\mathbb{N}\cup\{0\}} [a_n, a_{n+1})$ and define on each subinterval

$$\alpha(t) := \begin{cases} 1, & \text{if } a_n \leqslant t \leqslant \dfrac{a_n + a_{n+1}}{2}, \\ \dfrac{2 \cdot (t - a_{n+1})}{a_n - a_{n+1}}, & \text{if } \dfrac{a_n + a_{n+1}}{2} \leqslant t < a_{n+1}, \end{cases}$$

$$\beta(t) := \begin{cases} \dfrac{2 \cdot (t - a_n)}{a_{n+1} - a_n}, & \text{if } a_n \leqslant t \leqslant \dfrac{a_n + a_{n+1}}{2}, \\ 1 & \text{if } \dfrac{a_n + a_{n+1}}{2} \leqslant t < a_{n+1}, \end{cases}$$

see figure 7. Finally, define $\nu(t) := \min\{n \in \mathbb{N} \cup \{0\} : t \geqslant a_n\}$ for $t \in [0, 1)$. (Equivalently, $\nu(t) = n$ when $t \in [a_n, a_{n+1})$.) The maps $f_t : \ell_1 \to \ell_1$, $t \in [0, 1)$, can now be defined as follows:

$$f_t := \alpha(t) g_{\nu(t)} + \beta(t) g_{\nu(t)+1}.$$

(In particular, $f_{a_n} = g_n$.)

The maps α, β, ν satisfy the following conditions:

(i) α, β are right continuous on $[0, 1)$, and are continuous on each interval (a_n, a_{n+1}), $n \in \mathbb{N} \cup \{0\}$;

(ii) For any $n \in \mathbb{N} \cup \{0\}$, we have that

  (iia) $\alpha(t) = 1$ for $t \in [a_n, (a_n + a_{n+1})/2]$, and $\lim_{t\to a_{n+1}^-} \alpha(t) = 0$;

  (iib) $\beta(a_n) = 0$, $\beta(t) = 1$ for $t \in [(a_n + a_{n+1})/2, a_{n+1})$ and $\lim_{t\to a_{n+1}^-} \beta(t) = 1$.

We will justify conditions (a)–(c) for the family $F_t = \{f_t\}$.

(a) The continuity of the map $t \mapsto f_t(x)$ at each $t \in [0, 1)$ follows from (a) and (b), whereas its continuity at $t = 1$ follows from (a′).

(b) From (b′) we easily see that for $t \in [0, 1)$,

$$\mathrm{Lip}(f_t) \leqslant \max\left\{ \left(1 - \frac{1}{\nu(t)}\right)\alpha(t), \left(1 - \frac{1}{\nu(t) + 1}\right)\beta(t) \right\} \leqslant 1 - \frac{1}{\nu(t) + 1} < 1.$$

(c) Straightforward calculations show that the unique fixed point of f_t, $t \in [0, 1)$, is

$$\mathbf{z}_t = \frac{\alpha(t)}{\nu(t)(1 - \alpha(t)) + \alpha(t)} \mathbf{e}_{\nu(t)} + \frac{\beta(t)}{(\nu(t) + 1)(1 - \beta(t)) + \beta(t)} \mathbf{e}_{\nu(t)+1}. \tag{4.3}$$

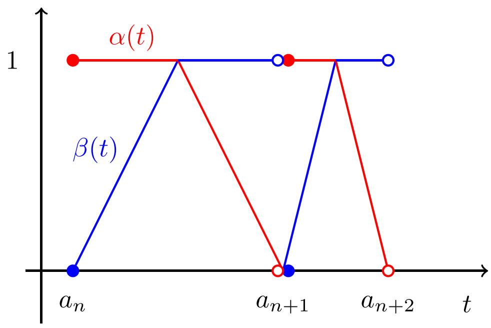

Figure 7. The graphs of α and β.

Thanks to (b), either the left or the right fraction in (4.3) equals 1. Hence $\|\mathbf{z}_t\| \geqslant 1$. In consequence, $\mathbf{z}_{t_n} \not\to \mathbf{0}$ for any $t_n \to 1$. Since $A_t = \{\mathbf{z}_t\}$ is an attractor of F_t, and the attractor $\{\mathbf{0}\}$ of F_1 is the only possible candidate for an upper transition attractor of a one-parameter family F_t (see discussion before this example), this yields (c).

Theorem 4.4. *Let F_t, $t \in [0, 1]$, satisfy (H1) and (H2). If $A^\bullet$ is any upper transition attractor of F_t, then*

(*a*) $F_1(A^\bullet) = A^\bullet$.

If, in addition, F_t satisfies (H3), then

(*b*) $A^\bullet \supseteq Q$*; equivalently* $A^\bullet \supseteq A_\bullet$,

where $A_\bullet$ is the lower transition attractor of F_t and Q is the set of limit fixed points from (H3).

Proof. Let $[0, 1) \ni t_n \nearrow 1$ be such that $A_{t_n} \to A^\bullet$ with respect to h as $n \to \infty$. To establish (a) recall that each F_{t_n} and F_1 are nonexpansive with respect to h (part (b) of remark 4.1). Furthermore, according to lemma 4.2(a), we have

$$h(F_1(A^\bullet), F_{t_n}(A^\bullet)) \leqslant \max_{1\leqslant i\leqslant N} h(f_{(i,1)}(A^\bullet), f_{(i,t_n)}(A^\bullet)) \to 0.$$

Hence, by using $F_{t_n}(A_{t_n}) = A_{t_n}$ we get

$$\begin{aligned} h(F_1(A^\bullet), A^\bullet) &\leqslant h(F_1(A^\bullet), F_{t_n}(A^\bullet)) + h(F_{t_n}(A^\bullet), F_{t_n}(A_{t_n})) + h(F_{t_n}(A_{t_n}), A^\bullet) \\ &\leqslant h(F_1(A^\bullet), F_{t_n}(A^\bullet)) + 2h(A_{t_n}, A^\bullet) \underset{n\to\infty}{\to} 0. \end{aligned}$$

Now we establish (b). Observe that $q_{i,t_n} \in A_{t_n} \to A^\bullet$, and $q_{i,t_n} \to q_i \in Q$ as $n \to \infty$. Thus $Q \subseteq A^\bullet$. Hence $A^\bullet$ is (F_1, Q)-invariant, and therefore it contains $A_\bullet$. $\square$

Remark 4.5. Assuming (H1)–(H3), $A_\bullet$ is compact whenever $A^\bullet$ exists.

Proposition 4.1. *Assume that F_t, $t \in [0, 1]$, satisfies (H1) and (H2). Let $f_{(i_*,1)}$ be an isometry for some $i_* \in \{1, \ldots, N\}$.*

(*a*) *If there exists an upper transition attractor $A^\bullet$, then it is $f_{(i_*,1)}$-symmetric, i.e., $f_{(i_*,1)}(A^\bullet) = A^\bullet$.*

(*b*) *If there exists a lower transition attractor $A_\bullet$ that is compact, then it is $f_{(i_*,1)}$-symmetric.*

Proof. Observe that $f_{(i_*,1)}(A^\bullet) \subseteq A^\bullet$. Then the isometry $f_{(i_*,1)}$ is surjective on compactum $A^\bullet$. Analogously for $A_\bullet$. □

5. The existence of a unique upper transition attractor

This section addresses question 1.1 in the introduction. Theorem 5.2 below gives an affirmative answer for a large class of one-parameter IFS families.

We start with lemma 5.1 below for infinite IFSs, which is already known for finite IFSs. Here the Hutchinson operator $F : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ is as defined in section 4.

Definition 5.1. For a finite or infinite IFS F, a nonempty compact set A is a **Hutchinson attractor** on a complete metric space $\mathbb{X}$ if

- (*Invariance*) $F(A) = A$, and
- (*Attraction*) $A = \lim_{n\to\infty} F^{(n)}(S)$,

for every nonempty closed and bounded set $S \subseteq \mathbb{X}$, the limit with respect to the Hausdorff metric. Note that a Hutchinson attractor, if it exists, is unique.

A generalization of the Hutchinson theorem is the following (see [32] and the references therein):

Theorem 5.1. *If an IFS F on $(\mathbb{X}, d)$ satisfies $\sup_{f\in F} \mathrm{Lip}(f, d) < 1$ and is compact, then it admits a Hutchinson attractor.*

Roughly speaking, lemma 5.1 says that, if compact IFSs F, G are close to each other on a compact subinvariant set, in the sense that each map f from F has a close neighbour $g \in G$, and vice versa, then attractors of F and G are also close.

Lemma 5.1. *Let $G := \{g_i : i \in I\}$ and $H := \{h_j : j \in J\}$ be two compact IFSs on a complete metric space $(\mathbb{X}, d)$ such that $\mathrm{Lip}(G, d) < 1$ and $\mathrm{Lip}(H, d) < 1$. Let $B \subseteq \mathbb{X}$ be a compact set such that $G(B) \subseteq B$ and $H(B) \subseteq B$, and let $\delta > 0$ satisfy*

$$\forall_{i\in I}\, \exists_{j\in J}\, \forall_{x\in B} \quad d(g_i(x), h_j(x)) \leqslant \delta \quad \text{and} \quad \forall_{j\in J}\, \exists_{i\in I}\, \forall_{x\in B} \quad d(g_i(x), h_j(x)) \leqslant \delta. \tag{5.1}$$

Then

$$h(A_G, A_H) \leqslant \frac{\delta}{1 - \min\{\mathrm{Lip}(G, d), \mathrm{Lip}(H, d)\}},$$

where A_G and A_H are the Hutchinson attractors of G and H, respectively.

Remark 5.1. Given two compact IFSs G and H with attractors A_G and A_H, there always exists a nonempty compact $B \subseteq \mathbb{X}$ such that $G(B) \subseteq B$ and $H(B) \subseteq B$. Indeed, since G and H are compact, the IFS $G \cup H$ is also compact, hence admits the attractor $A_{G\cup H}$. Furthermore, for any nonempty compact set $D \subseteq \mathbb{X}$, the set

$$B := \mathrm{cl}\left(D \cup \bigcup_{n\in\mathbb{N}} (G \cup H)^{(n)}(D)\right) = A_{G\cup H} \cup D \cup \bigcup_{n\in\mathbb{N}} (G \cup H)^{(n)}(D)$$

is compact, and $G(B) \cup H(B) \subseteq B$.

Proof of lemma 5.1. By (5.1), we can easily see that for any compact $D \subseteq B$,

$$h(G(D), H(D)) \leqslant \delta. \tag{5.2}$$

Without loss of generality suppose $\alpha = \mathrm{Lip}(G, d) \leqslant \mathrm{Lip}(H, d)$. We will check inductively that for every $n \in \mathbb{N}$,

$$h(G^{(n)}(B), H^{(n)}(B)) \leqslant \delta \sum_{k=0}^{n-1} \alpha^k. \tag{5.3}$$

The case $n = 1$ of (5.3) is exactly (5.2) for $D := B$. Assume that the inequality (5.3) holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned} h(G^{(n+1)}(B), H^{(n+1)}(B)) &\leqslant h(G(G^{(n)}(B)), G(H^{(n)}(B))) + h(G(H^{(n)}(B)), H(H^{(n)}(B))) \\ &\leqslant \alpha h(G^{(n)}(B), H^{(n)}(B)) + \delta \\ &\leqslant \alpha\delta \sum_{k=0}^{n-1} \alpha^k + \delta = \delta \sum_{k=0}^{n} \alpha^k, \end{aligned}$$

where the penultimate inequality follows from (5.2) for $D := H^{(n)}(B)$, and the last inequality uses (5.3) for n. Thus (5.3) is true for $n + 1$. Now from (5.3) and the convergence of the Hutchinson iterates to the attractor, we get

$$h(A_G, A_H) \leqslant \delta \sum_{k=0}^{\infty} \alpha^k = \frac{\delta}{1-\alpha}.$$

This completes the proof. □

Recall that any surjective isometry $g : \mathbb{X} \to \mathbb{X}$ of a real normed space is of the following form:

$$g(x) = \hat{g}(x) + b = \hat{g}(x - x_*),$$

where $\hat{g} : \mathbb{X} \to \mathbb{X}$ is a linear isometry, $b = g(0) \in \mathbb{X}$ and $x_* = g^{-1}(0)$ (see [11] ch 1.3, Mazur–Ulam theorem).

Lemma 5.2. *Let $\mathbb{X}$ be a real Banach space; let $g : \mathbb{X} \to \mathbb{X}$ be a surjective isometry; let $x_* = g^{-1}(0)$; and let $\hat{g}$ be the linear part of g. For $t \in [0, 1]$, set*

$$g_t(x) := tg(x) + x_*, \quad x \in X.$$

The following statements hold:

(*a*) *For every $m \in \mathbb{N}$, $t_1, \ldots, t_m \in [0, 1]$ and for all $x \in \mathbb{X}$, we have*

$$g_{t_1} \circ \cdots \circ g_{t_m}(x) = t_1 \ldots t_m \, \hat{g}^{(m)}(x - x_*) + x_*.$$

(*b*) *g_1 is periodic if and only if $\hat{g}$ is periodic, and their periods are the same.*

(*c*) *If g_1 is periodic, then the monoid generated by the IFS $G := \{g_t : t \in [0, 1]\}$ is compact.*

Proof. By the preceding observations concerning surjective isometries, we have

$$g_{t_1}(x) = t_1 \hat{g}(x - x_*) + x_*$$

which gives us (a) for $m = 1$. Suppose that (a) is true for some $m \in \mathbb{N}$. Then we have

$$\begin{aligned} g_{t_1} \circ \cdots \circ g_{t_m} \circ g_{t_{m+1}}(x) &= t_1 \ldots t_m \hat{g}^{(m)}(t_{m+1}\hat{g}(x - x_*) + x_* - x_*) + x_* \\ &= t_1 \ldots t_{m+1} \hat{g}^{(m+1)}(x - x_*) + x_* \end{aligned}$$

so we obtain (a) for $m+1$. This ends the proof of (a).

By (a), for every $m \in \mathbb{N}$ and $x \in \mathbb{X}$, we have

$$g_1^{(m)}(x) - x_* = \hat{g}^{(m)}(x - x_*).$$

Hence if $g_1^{(m)} = \mathrm{id}_{\mathbb{X}}$, then also $\hat{g}^{(m)} = \mathrm{id}_{\mathbb{X}}$, and vice versa. Thus (b) is true.

Now we prove (c). By (a), each element of the desired monoid $\mathbb{M}(G)$, distinct from the identity map, is of the form

$$g_{t_1} \circ \cdots \circ g_{t_m}(x) = t_1 \ldots t_m \hat{g}^{(m)}(x - x_*) + x_* = t^i \hat{g}^{(i)}(x - x_*) + x_* = g_t^{(i)}(x)$$

for some $i = 1, \ldots, p$ where p is the period of $\hat{g}$ and $t := \sqrt[i]{t_1 \ldots t_m}$. Hence

$$\mathbb{M}(G) = \{g_t^{(i)} : i = 1, \ldots, p,\ t \in [0,1]\}$$

(note that $g_1^p = \mathrm{id}_{\mathbb{X}}$). In particular, $\mathbb{M}(G)$ is the finite union of IFSs $\{g_t^{(i)} : t \in [0,1]\}$ over $i = 1, \ldots, p$, which are compact in view of lemma 4.2. Thus $\mathbb{M}(G)$ itself is compact. □

We now state the main result of this section, which shows that quite a wide class of IFS families possess a unique upper transition attractor. This gives an answer to question 1.1, but only a partial answer to conjecture 8.1 in [34]. Namely, we have to assume that the so-called special function in [34, definition 8.1], which is an isometry and is denoted by g_1 in theorem 5.2 below, has a periodic linear part. On the other hand, we are able to deal with [34, conjecture 8.1] in a more general setting. More precisely, we allow for the underlying space $\mathbb{X}$ to be not necessarily a Euclidean space and for the maps comprising the one-parameter IFS family to be not necessarily affine. Question 6.2 in the next section asks whether the periodicity assumption in theorem 5.2 can be dropped in a setting more restrictive than a real Banach space.

Theorem 5.2. *Let* $(\mathbb{X}, \|\cdot\|)$ *be a real Banach space and let* $g : \mathbb{X} \to \mathbb{X}$ *be a surjective isometry. Consider the one-parameter family*

$$F_t^g := F_t \cup \{g_t\}$$

on $\mathbb{X}$ *with* $t \in [0,1]$, *where*

$$F_t := \{f_{(i,t)} : 1 \leqslant i \leqslant N\}$$

and

$$g_t(x) := tg(x) + x_*.$$

Assume that F_t *satisfies:*

(*a*) *Condition* (*H1*), *and*

(*b*) *Condition* (*H2*$'$) *and* $\mathrm{Lip}(F_1, \|\cdot\|) < 1$.

Assume that g_t *satisfies:*

(*c*) $x_* = g^{-1}(0)$, *and*

(*d*) *The isometry* g *has a periodic linear part.*

Then F_t^g *has a unique upper transition attractor.*

Remark 5.2. Note that g has a periodic linear part if and only if g_1 has a periodic linear part. See also statement (b) in lemma 5.2.

Remark 5.3. Condition (c) is equivalent to any of the following:

(a) The maps g_t have a common fixed point for at least two different parameters $t \in [0, 1]$;
(b) The maps g_t have a common fixed point for all parameters $t \in [0, 1]$;
(c) x_* is a common fixed point of all g_t's.

Proof of theorem 5.2. Since all functions in F_t^g are contractions for $t < 1$, each IFS F_t^g has an attractor A_t for $t \in [0, 1)$ and we can consider upper transition attractors for the one-parameter family F_t^g, $t \in [0, 1]$. The existence of a unique upper transition attractor follows from the uniform continuity of the map

$$[0, 1) \ni t \mapsto A_t \in \mathcal{K}(\mathbb{X}),$$

see [34, proposition 8.1]. Hence we will prove that this map is uniformly continuous.

Step 1. Consider the IFSs

$$\begin{aligned} F &:= \bigcup_{t\in[0,1]} F_t = \{f_{(i,t)} : i = 1, \ldots, N,\ t \in [0, 1]\} = \bigcup_{i=1,\ldots,N} \{f_{(i,t)} :\ t \in [0, 1]\} \\ G &:= \{g_t : t \in [0, 1]\}. \end{aligned} \tag{5.4}$$

As in the proof of theorem 4.3(b), we can find a nonempty and compact set $B \subseteq \mathbb{X}$ so that

$$f_{(i,t)}(B) \subseteq B \quad \text{and} \quad g_t(B) \subseteq B \tag{5.5}$$

for all $t \in [0, 1]$ and $i = 1, \ldots, N$. One only needs to observe that the monoid $\mathbb{M}(G)$ is compact due to lemma 5.2(c).

Step 2. An alternative description of the attractor A_t of F_t^g.

Fix a real value $t \in [0, 1)$. Clearly,

$$\mathrm{Lip}(F_t^g, \|\cdot\|) \leqslant \max\{t, \mathrm{Lip}(F, \|\cdot\|)\} < 1.$$

Hence F_t^g generates a unique attractor A_t. Using [32, theorem 4.1] for IFSs F_t and $\{g_t\}$, we see that A_t can be viewed as the attractor of the IFS

$$M_t := \{g_t^{(m)} \circ f_{(i,t)} : i = 1, \ldots, N,\ m = 0, 1, 2, \ldots\} \tag{5.6}$$

where $g_t^{(0)} = \mathrm{id}_{\mathbb{X}}$. Note that the assumptions of [32, theorem 4.1] will be satisfied if we observe that the monoid

$$\mathbb{M}(\{g_t\}) = \{g_t^{(m)} : m = 0, 1, 2, \ldots\}$$

is compact. This is the case as it is a subset of a compact IFS $\mathbb{M}(G)$ considered in step 1. (Alternatively, we can observe that $\mathbb{M}(\{g_t\})$ is compact by using the fact $\mathrm{Lip}(g_t) \leqslant t < 1$.) Moreover, in view of (5.5), we see that $A_t \subseteq B$.

Step 3. Uniform continuity of the map $[0, t_0] \ni t \mapsto A_t$, where $t_0 \in [0, 1)$.

Fix any $t_0 \in [0, 1)$. Clearly,

$$\sup\{\mathrm{Lip}(F_t^g, \|\cdot\|) : t \in [0, t_0]\} \leqslant \max\{t_0, \mathrm{Lip}(F, \|\cdot\|)\} < 1.$$

Hence the assumptions of [15, theorem 2.6] are satisfied. This means that the map $[0, t_0] \ni t \mapsto A_t$ is continuous. As $[0, t_0]$ is compact, it is uniformly continuous.

Step 4. Uniform continuity of the map $[0, 1) \ni t \mapsto A_t$.

The idea in the proof below is that if both $t, s < 1$ are appropriately less than 1, then we make use of uniform continuity proved in step 3, whereas if s, t are both sufficiently close to 1, then for each map of the form $g_t^m \circ f_{(i,t)}$ we will find sufficiently close neighbour $g_s^k \circ f_{(i,s)}$ (where k will be appropriately chosen), and vice versa. Then we will make use of lemma 5.1.

Let $\hat{g}$ be the linear part of g. Then by lemma 5.2, we see that for every $m \in \mathbb{N}$ and $x \in X$, we have:

$$g_t^{(m)}(x) = t^m \hat{g}^{(m)}(x - x_*) + x_*. \tag{5.7}$$

Let p be the period of $\hat{g}$. Take any $\varepsilon > 0$ and choose $r \in (0, 1)$ such that

$$(1 - r^p) \cdot (\mathrm{diam}(B \cup \{0\}) + \|x_*\|) < \frac{\varepsilon}{2}. \tag{5.8}$$

Then choose $\delta > 0$ such that:

(a) For $s, t \in [0, r]$, if $|t - s| < \delta$, then $h(A_t, A_s) < \varepsilon$;
(b) For $s, t \in [0, 1]$, if $|t - s| < \delta$, then

$$\sup\{\|f_{(i,t)}(x) - f_{(i,s)}(x)\| : i = 1, \ldots, N,\ x \in B\} < \frac{\varepsilon}{2};$$

(c) $(1 - (r - \delta)^p)\cdot$ $(\mathrm{diam}(B \cup \{0\}) + \|x_*\|) \leqslant \frac{\varepsilon}{2}$.

The choice of δ is possible by step 3 (for item (a)), by lemma 4.2(a) (for item (b)) and by condition (5.8) (for item (c)).

Now choose $s, t \in [0, 1)$ such that $|s - t| < \delta$. If $s, t \leqslant r$, then $h(A_t, A_s) \leqslant \varepsilon$ in view of (a). Hence assume that

$$\max\{s, t\} \geqslant r. \tag{5.9}$$

Take any $i = 1, \ldots, N$ and $m = 0, 1, 2, \ldots$, and let m', l' be such that $m = pm' + l'$, and $l' = 0, \ldots, p - 1$. Then let k' be the least nonnegative integer such that

$$s^{pk'+l'} \leqslant t^{pm'+l'}$$

and set $k := pk' + l'$. We will show that

$$|t^m - s^k| \leqslant 1 - (r - \delta)^p. \tag{5.10}$$

Using $s^k \leqslant t^m < s^{k-p}$, we have

$$\begin{aligned} |t^m - s^k| &= t^m - s^k \leqslant \min\{1, s^{k-p}\} - s^k = \min\{1 - s^k, s^{k-p}(1 - s^p)\} \\ &\leqslant \begin{cases} 1 - s^{l'} & \text{if } k' = 0 \\ s^{k-p}(1 - s^p) & \text{if } k' \geqslant 1 \end{cases} \\ &\leqslant 1 - s^p \leqslant 1 - (r - \delta)^p, \end{aligned}$$

where the last inequality follows from $r - \delta \leqslant s$ (thanks to (5.9)). Thus we have shown (5.10).

Now fix $i = 1, \ldots, N$ and choose any $x \in B$. Assume that $m \geqslant 1$ (which also implies $k \geqslant 1$). Set $z_t := f_{(i,t)}(x) - x_*$ and $z_s := f_{(i,s)}(x) - x_*$. Then by (b) and (c) and from the choice of δ, we have

$$\|z_t - z_s\| = \|f_{(i,t)}(x) - f_{(i,s)}(x)\| < \frac{\varepsilon}{2}$$

and

$$\|z_s\| \leqslant \|f_{(i,s)}(x) - 0\| + \|x_*\| \leqslant \mathrm{diam}(B \cup \{0\}) + \|x_*\| \leqslant \frac{\varepsilon}{2} \cdot (1 - (r - \delta)^p)^{-1}.$$

Hence by (5.7) and (5.10), and the fact that $\hat{g}^{(p)} = \mathrm{id}_{\mathbb{X}}$, we have

$$\begin{aligned}
&\|g_t^{(m)} \circ f_{(i,t)}(x) - g_s^{(k)} \circ f_{(i,s)}(x)\| \\
&\quad = \|t^m \hat{g}^{(m)}(f_{(i,t)}(x) - x_*) + x_* - s^k \hat{g}^{(k)}(f_{(i,s)}(x) - x_*) - x_*\| \\
&\quad = \|t^m \hat{g}^{(m)}(z_t) - s^k \hat{g}^{(k)}(z_s)\| \\
&\quad = \|t^m \hat{g}^{(pm'+l')}(z_t) - s^k \hat{g}^{(pk'+l')}(z_s)\| \\
&\quad = \|t^m \hat{g}^{(l')}(z_t) - s^k \hat{g}^{(l')}(z_s)\| \\
&\quad \leqslant \|t^m \hat{g}^{(l')}(z_t) - t^m \hat{g}^{(l')}(z_s)\| + \|t^m \hat{g}^{(l')}(z_s) - s^k \hat{g}^{(l')}(z_s)\| \\
&\quad = t^m \cdot \|\hat{g}^{(l')}(z_t - z_s)\| + |t^m - s^k| \cdot \|\hat{g}^{(l')}(z_s)\| \\
&\quad = t^m \cdot \|z_t - z_s\| + |t^m - s^k| \cdot \|z_s\| \\
&\quad \leqslant \|z_t - z_s\| + (1 - (r - \delta)^p) \cdot \|z_s\| \\
&\quad < \varepsilon.
\end{aligned}$$

When $m = 0$ (and consequently $k = 0$), we also have

$$\|g_t^{(m)} \circ f_{(i,t)}(x) - g_s^{(k)} \circ f_{(i,s)}(x)\| = \|f_{(i,t)}(x) - f_{(i,s)}(x)\| < \frac{\varepsilon}{2}.$$

Similar reasoning works when the roles of s and t are switched. Hence we see that condition (5.1) from lemma 5.1 is satisfied for IFSs M_t and M_s, whose attractors are A_t and A_s, respectively (for definitions of M_t and M_s, see (5.6)). Thus, using lemma 5.1, and the fact that

$$\mathrm{Lip}(M_s, \|\cdot\|), \mathrm{Lip}(M_t, \|\cdot\|) \leqslant \mathrm{Lip}(F, \|\cdot\|) < 1$$

(recall definition of F in (5.4) and notice that $\mathrm{Lip}(g^{(m)} \circ f) = \mathrm{Lip}(f)$ for $f \in F$), we get

$$h(A_t, A_s) \leqslant \frac{\varepsilon}{1 - \mathrm{Lip}(F, \|\cdot\|)}.$$

We conclude that the map $[0, 1) \ni t \mapsto A_t$ is uniformly continuous. □

The following corollary explains how theorem 5.2 answers question 1.1.

Corollary 5.1. *Let*

$$F_t := \{f_{(i,t)} : 1 \leqslant i \leqslant N\}$$

$t \in [0, \infty)$, *be a one-parameter family of IFSs on a real Banach space* $\mathbb{X}$. *Let* $g : \mathbb{X} \to \mathbb{X}$ *be a surjective isometry. Assume that there exist* $t_* \in (0, \infty)$ *and* $i_* \in \{1, \ldots, N\}$ *such that*

(*a*) *For any* $i = 1, \ldots, N$ *and* $x \in \mathbb{X}$, *the map* $[0, 1] \ni t \mapsto f_{(i,t)}(x)$ *is continuous;*
(*b*) $\sup\{\mathrm{Lip}(f_{(i,t)}, \|\cdot\|) : 1 \leqslant i \leqslant N, i \neq i_*, t \in [0, t_*]\} < 1$;
(*c*) $f_{(i_*,t)}(x) := \frac{t}{t_*} \cdot g(x) + x_*$ *for all* $t \in [0, \infty)$ *and* $x_* = g^{-1}(0)$;
(*d*) *The isometry* g *has a periodic linear part.*

Then $t_* = t_0 = \widehat{t_0}$ is a threshold for the existence of an attractor and for the contractivity of the one-parameter family F^t, and F_t has a unique upper transition attractor at t_*.

Proof. It is enough to check that $t_* = t_0 = \widehat{t_0}$. The rest is immediate from theorem 5.2. From conditions (b) and (c) we have that $t_0 \geqslant \widehat{t_0} \geqslant t_*$. Now, for $t > t_*$ the IFS F_t does not have an attractor, since $\mathrm{Lip}(f_{(i_*,t)}) = \frac{t}{t_*} > 1$. Therefore $t_0 \leqslant t_*$. □

Examples 5.1, 5.2, 5.3, and 5.4 below show that each of the assumptions (a)–(d) in theorem 5.2 is crucial. If any of these assumptions is removed, then not only may the upper transition attractor of the family F_t^g fail to be unique, but F_t^g may have no upper transition attractor at all. In particular, example 5.4 provides an infinite dimensional one-parameter family where the function g is not periodic and the one-parameter family has no upper transition attractor.

Example 5.1 (Theorem 5.2 may fail without assumption (a).)
Let $F_t^g := \{f_t, g_t\}$ be a one-parameter family on $\mathbb{R}$, where $f_t(x) = tx/2 + (2-t)/(1-t)$, $t \in [0, 1)$, f_1 is any continuous function, and $g_t(x) = tx$, $t \in [0, 1]$. Here F_t^g satisfies the assumptions of theorem 5.2 except that $[0, 1] \ni t \mapsto f_t(x)$ is not continuous at $t = 1$ for any x. The fixed point q_t of f_t is $q_t = 2/(1-t) \to \infty$ as $t \to 1$. Since $q_t \in A_t$, the limit $\lim_{t\to 1} A_t$ does not exist.

Example 5.2 (Theorem 5.2 may fail without assumption (b).)
On $\mathbb{R}$, let $F_t^g := \{f_t, g_t\}$, where $g_t(x) = -tx$, $f_t(x) = -tx + t + 1$. (This is example 4.1 from section 4.) Here F_t^g satisfies the assumptions of theorem 5.2 except that $\lim_{t\to 1} \mathrm{Lip}(f_t, \|\cdot\|) = 1$. For $t \in (0, 1)$ we have $A_t = [-t/(1-t), 1/(1-t)]$; therefore $\lim_{t\to 1} A_t$ does not exist.

Example 5.3 (Theorem 5.2 may fail without assumption (c).)
Let $F_t^g := \{f_t, g_t\}$, where $g(x) = x$, $g_t(x) = tx + 1$, and $f_t(x) = tx/2 + 1$. Here F_t^g satisfies the assumptions of theorem 5.2 except that $x^* = 1 \neq g^{-1}(0)$. In this case the fixed point q_t of g_t is $q_t = 1/(1-t) \to \infty$ as $t \to 1$. Since $q_t \in A_t$, the limit $\lim_{t\to 1} A_t$ does not exist.

Example 5.4 (Theorem 5.2 may fail without assumption (d).)
Let $\mathbb{X} := \ell^\infty(\mathbb{C})$ denote the real Banach space of all bounded complex sequences, endowed with the supremum norm. For $k \in \mathbb{N}$, set $\alpha_k := \frac{\pi}{2k}$, and define $g : \mathbb{X} \to \mathbb{X}$ by

$$g((x_k)) := \left(x_k\, \mathrm{e}^{\mathrm{i}\alpha_k}\right),$$

that is, each coordinate x_k is rotated around the origin by angle α_k. Next define $f : \mathbb{X} \to \mathbb{X}$ by

$$f((x_k)) := \left(\frac{1}{4}(x_k - 1)\right).$$

Observe that $f(\mathbf{1}) = \mathbf{0}$, where $\mathbf{1}$ and $\mathbf{0}$ are sequences of ones and zeroes, respectively. For $t \in (0, 1]$, define

$$g_t((x_k)) := tg((x_k)) = \left(tx_k\, \mathrm{e}^{\mathrm{i}\alpha_k}\right)$$

and

$$f_t((x_k)) := tf((x_k)) + \mathbf{1} = \left(\frac{t}{4}x_k + 1 - \frac{t}{4}\right).$$

Clearly, the map $[0, 1] \ni t \mapsto f_t(x)$ is continuous for every $x \in \mathbb{X}$, $\mathrm{Lip}(f_t) = \frac{t}{4}$ and $g^{-1}(\mathbf{0}) = \mathbf{0}$. Hence, setting $F_t := \{f_t\}$, all assumptions of theorem 5.2 are satisfied except that $\hat{g}$ (which here coincides with g) is not periodic. We will now show that F_t^g does not have any upper transition attractor.

Let

$$D := \{\mathbf{0}\} \cup \bigcup_{m=0}^{\infty} \overline{B}\left(\left(\frac{3}{4}t^m\,\mathrm{e}^{\mathrm{i}m\alpha_k}\right), \frac{1}{4}t^m\right), \tag{5.11}$$

where $\overline{B}(\cdot,\cdot)$ denotes the closed ball in $\mathbb{X}$, where the first coordinate is the center, and where the second coordinate is the radius. We first show that for every $t \in [0,1]$, the attractor A_t of $F_t^g := \{f_t, g_t\}$ is a subset of D. Clearly, the set $D \subseteq \overline{B}(\mathbf{0},1)$, and it is easy to see that

$$f_t(\overline{B}(\mathbf{0},1)) \subseteq \overline{B}\left(\left(\frac{3}{4}\right), \frac{1}{4}\right) \subseteq D,$$

(where $\left(\frac{3}{4}\right)$ is the constant sequence whose coordinates equal $\frac{3}{4}$). Hence

$$f_t(D) \subseteq D.$$

On the other hand, for every $m = 0, 1, 2, \ldots$, we have

$$g_t\left(\overline{B}\left(\left(\frac{3}{4}t^m\,\mathrm{e}^{\mathrm{i}m\alpha_k}\right), \frac{1}{4}t^m\right)\right) = \overline{B}\left(\left(\frac{3}{4}t^{m+1}\,\mathrm{e}^{\mathrm{i}(m+1)\alpha_k}\right), \frac{1}{4}t^{m+1}\right) \subseteq D$$

and $g_t(\mathbf{0}) = \mathbf{0}$; so we also have

$$g_t(D) \subseteq D.$$

Altogether we have $F_t^g(D) \subseteq D$. As D is closed, we get (5.11).

Now since the sequence $\mathbf{1}$ is the fixed point of f_t, it belongs to the attractor A_t, and hence also

$$\left(t^m\,\mathrm{e}^{\mathrm{i}m\alpha_k}\right) = g_t^{(m)}(\mathbf{1}) \in A_t \tag{5.12}$$

for every $m \in \mathbb{N}$.

We are ready to prove that (F_t^g) does not generate any upper transition attractor, that is, there is no sequence $t_n \in [0,1)$ with $t_n \nearrow 1$ so that (A_{t_n}) converges. First observe that it is enough to prove that

$$\forall_{s\in[\frac{1}{2},1)}\ \exists_{t_0<1}\ \forall_{t\in[t_0,1)}\ h(A_t, A_s) \geqslant \frac{1}{2}. \tag{5.13}$$

Indeed, suppose that (5.13) holds, and for some sequence $t_n \nearrow 1$ we have that (A_{t_n}) is convergent. Then (A_{t_n}) is a Cauchy sequence in $\mathcal{K}(\mathbb{X})$ and we can find $n_0 \in \mathbb{N}$ so that $h(A_{t_{n_0}}, A_{t_n}) < \frac{1}{2}$ for all $n \geqslant n_0$ and $t_{n_0} \geqslant \frac{1}{2}$. On the other hand, setting $s := t_{n_0}$ and using (5.13), we can find $n \geqslant n_0$ with $h(A_{t_n}, A_{t_{n_0}}) \geqslant \frac{1}{2}$, which gives a contradiction.

We will now prove (5.13). Choose any $s \in [\frac{1}{2}, 1)$, and find the least $k_0 \in \mathbb{N}$ such that $s^{k_0} < \frac{1}{2}$. As $s \geqslant \frac{1}{2}$, we see that $s^{k_0} \geqslant \frac{1}{4}$. Since $1 - s^{k_0} > \frac{1}{2}$, we can find $t_0 < 1$ such that for $t \in [t_0, 1)$ we have

$$t^{2k_0} - s^{k_0} > \frac{1}{2}. \tag{5.14}$$

Choose any $(x_k) \in A_s$. By the definition of D (see (5.11)) and the fact that $A_t \subset D$, we can consider three cases.

Case 1. $(x_k) \in \overline{B}\left(\left(\frac{3}{4}s^m\,\mathrm{e}^{\mathrm{i}m\alpha_k}\right), \left(\frac{1}{4}s^m\right)\right)$ for some $m \leqslant k_0$.
Since $m\alpha_{k_0} \leqslant k_0\frac{\pi}{2k_0} = \frac{\pi}{2}$, we have

$$t^{2k_0} \leqslant \left|t^{2k_0} + \frac{3}{4}s^m\,\mathrm{e}^{\mathrm{i}m\alpha_{k_0}}\right| \leqslant \left|t^{2k_0} + x_{k_0}\right| + \left|-x_{k_0} + \frac{3}{4}s^m\,\mathrm{e}^{\mathrm{i}m\alpha_{k_0}}\right| \leqslant \left|t^{2k_0} + x_{k_0}\right| + \frac{1}{4},$$

so by (5.14) we get

$$\left|t^{2k_0} + x_{k_0}\right| \geqslant t^{2k_0} - \frac{1}{4} \geqslant t^{2k_0} - s^{k_0} > \frac{1}{2}.$$

Case 2. $(x_k) \in \overline{B}\left(\left(\frac{3}{4}s^m\,\mathrm{e}^{\mathrm{i}m\alpha_k}\right), \left(\frac{1}{4}s^m\right)\right)$ for some $m \geqslant k_0$.
Since $t^{2k_0} > s^{k_0} \geqslant s^m$, we have

$$\begin{aligned} t^{2k_0} - \frac{3}{4}s^{k_0} \leqslant t^{2k_0} - \frac{3}{4}s^m &\leqslant \left|t^{2k_0} + \frac{3}{4}s^m\,\mathrm{e}^{\mathrm{i}m\alpha_{k_0}}\right| \leqslant \left|t^{2k_0} + x_{k_0}\right| + \left|-x_{k_0} + \frac{3}{4}s^m\,\mathrm{e}^{\mathrm{i}m\alpha_{k_0}}\right| \\ &\leqslant \left|t^{2k_0} + x_{k_0}\right| + \frac{1}{4}s^m \leqslant \left|t^{2k_0} + x_{k_0}\right| + \frac{1}{4}s^{k_0}. \end{aligned}$$

Thus by (5.14),

$$\left|t^{2k_0} + x_{k_0}\right| \geqslant t^{2k_0} - \frac{3}{4}s^{k_0} - \frac{1}{4}s^{k_0} > \frac{1}{2}.$$

Case 3. $(x_k) = \mathbf{0}$.
In this case

$$\left|t^{2k_0} + x_{k_0}\right| = t^{2k_0} > \frac{1}{2}.$$

Summing up, we have that

$$\left\|\left(t^{2k_0}\,\mathrm{e}^{\mathrm{i}\frac{2k_0}{2k}\pi}\right) - (x_k)\right\| \geqslant \left|t^{2k_0}\,\mathrm{e}^{\mathrm{i}\frac{2k_0}{2k_0}\pi} - x_{k_0}\right| = |-t^{2k_0} - x_{k_0}| = |t^{2k_0} + x_{k_0}| > \frac{1}{2}.$$

By (5.12) we see that $\left(t^{2k_0}\,\mathrm{e}^{\mathrm{i}\frac{2k_0}{2k}\pi}\right) \in A_t$, so the above shows that

$$h(A_t, A_s) \geqslant \inf_{(x_k)\in A_s} \left\|\left(t^{2k_0}\,\mathrm{e}^{\mathrm{i}\frac{2k_0}{2k}\pi}\right) - (x_k)\right\| \geqslant \frac{1}{2}$$

and the proof of (5.13) is complete.

6. Open problems

Examples 3.1–3.3 show that an IFS with an attractor need not be contractive. In example 3.2 no function in the IFS F is a contraction. In fact, with respect to any equivalent metric d on the circle, $\mathrm{Lip}(f, d) > 1$ for all $f \in F$. This is not the case in example 3.3. It can be asked whether such a strong counterexample exists for $\mathbb{R}^n$.

Question 6.1. Is there an example of an IFS F on $\mathbb{R}^n$ that has an attractor A with basin $\mathbb{R}^n$ but with respect to any metric d equivalent to the Euclidean metric we have $\mathrm{Lip}(f, d) > 1$ for all $f \in F$.

For a large class of one-parameter IFS families, theorem 5.2 guarantees the existence of a unique upper transition attractor $A^\bullet$ such that $A^\bullet = \lim_{t\to t_0} A_t$ at a threshold t_0. The theorem, however, assumes that the linear part of the special function g is periodic. Example 5.4 shows that, in general, the assumption of periodicity of the linear part cannot be dropped. But the underlying space in that example is a non-separable infinite dimensional space.

Question 6.2. Can the assumption of periodicity of the linear part of the function g in theorem 5.2 be dropped assuming a less exotic space? In particular, can the assumption be dropped for a one-parameter similarity family with threshold t_0 satisfying the following properties:

- All $f_t \in F_t$ are contractions for $t \in [0, t_0]$, g_t is a contraction for $t \in [0, t_0)$ and $\mathrm{Lip}(g_{t_0}) = 1$, and
- The unique fixed point of each $f_t \in F_t$ and g_t is independent of $t \in [0, t_0)$.

Theorem 4.3 guarantees the existence, but not necessarily the uniqueness, of an upper transition attractor in this context.

In [34, theorem 8.2] relationships between the upper and lower transition attractors are given for a special type of one-parameter family. It can be asked whether the same relationships hold in a more general setting. In particular:

Conjecture 6.1. If F_t satisfies properties (H1)–(H3) of section 4 and if $A_\bullet = A^\bullet$ for some upper transition attractor of F_t, then $A^\bullet$ is the unique upper transition attractor of F_t and $A^\bullet$ is an attractor of F_1.

Recall that in a metric space $(\mathbb{X}, d)$, a *segment* with ends $x, y \in \mathbb{X}$ is defined by $[x, y] := \{z \in \mathbb{X} : d(x, z) + d(z, y) = d(x, y)\}$. A set $S \subseteq \mathbb{X}$ is *metrically convex* if $[x, y] \subseteq S$ for all $x, y \in S$. The *metrically convex hull* of $S \subseteq \mathbb{X}$ is $\mathrm{conv}_d\, S := \bigcup_{x,y\in S}[x, y]$.

Conjecture 6.2. If the functions in F_t map metrically convex sets onto metrically convex sets, then the metrically convex hulls of $A_\bullet$ and $A^\bullet$ in $(\mathbb{X}, d)$ coincide: $\mathrm{conv}_d\, A_\bullet = \mathrm{conv}_d\, A^\bullet$.

Acknowledgments

We would like to thank the referee and the editor for their valuable suggestions which substantially improved the presentation of our paper. In particular, these ideas allowed us to simplify example 4.3 and to extend theorem 4.3 and simplify its proof. The contribution of the second author to the research in this paper was done while she was at University College Dublin.

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