$$\frac{1}{|C_n|}\mu\big([0,a]^n\big) = \frac{1}{n}a^n.$$

Combining these facts yields Cavalieri's formula.

3. COMMENTARY. It is well known that acting on the cube by S_n, the full symmetric group, yields the fundamental domain

$$S = \big\{(x_1, x_2, \dots, x_n) : 0 \le x_1 \le x_2 \le \cdots \le x_n \le a\big\}$$

(we can order the coordinates in increasing order). This is the standard simplex, whose volume in thus $a^n/|S_n| = a^n/n!$. From our perspective, this leads to the equation:

$$\int_{x_n=0}^{a} \cdots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} dx_1\, dx_2 \cdots dx_n = \frac{1}{n!}a^n,$$

which is an iterated form of Cavalieri's formula.

Our pyramid construction is precisely inverse to the standard geometric proof of the power rule for derivatives, $dx^n/dx = nx^{n-1}$, seen as the change in volume of an n-cube when the side-length is increased by dx: each of the n far faces of the cube increases the volume by $x^{n-1}\,dx$. We just integrate this and get n pyramids.

We can also interpret the observation in **[1]** that the graph $y = x^n$ is symmetric under nonhomogeneous dilation (it is unchanged if dilated in the x-direction by $a > 0$ and in the y-direction by a^n): this is because in one direction we are dilating in one dimension, and in the other we are dilating in n dimensions.

Computing the area under the curve x^n by comparison of the cross sections with a pyramid in $n + 1$ dimensions is an application of Cavalieri's principle (if cross-sectional areas are equal, then volumes are equal), but here changing dimension (which is legitimate as these areas and volumes are actually unitless *ratios* with respect to a standard square or cube: whether a curve is x^n or some multiple thereof depends on a choice of scale). In particular, the formulas for the quadrature of the parabola and the volume of a pyramid or cone are the same, though the ancient Greeks (Archimedes and Eudoxus) and Cavalieri computed them separately. By calculus we know these are both computed by $\int x^2$; here we have given a common geometric setting that illustrates why.

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A Property of Normal Tilings

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1. INTRODUCTION. Every polyhedron has a face with at most five edges. This is a consequence of Euler's formula for polyhedra. There is an analogous result for tilings of the plane.

Theorem 1. *Every normal tiling of the plane contains an infinite number of tiles each of which has at most six edges.*

This result appears in the classic text *Tilings and Patterns* [**4**], where a fairly involved proof is given. The goal of this note is to provide a short proof. Our proof requires only the Euler formula for planar graphs [**7**], which states that $v - e + t = 1$ for any finite, connected plane graph with v vertices, e edges, and t bounded regions.

Section 2 covers basic notions about tiling, in particular the definition and properties of a normal tiling. Theorem 1 is false without the assumption of normality. For example, the plane can be tiled with 7-gons if they are allowed to be arbitrarily long and thin. The proof of Theorem 1 appears in section 3, and remarks on a three-dimensional analog appear in section 4.

2. NORMAL TILINGS. Two sets in the plane are said to *overlap* if their interiors have nonempty intersection. A *plane tiling* T is a collection of compact sets that cover the plane without overlap. To eliminate some pathological cases we restrict our attention to normal tilings. A tiling T is *normal* if it satisfies the following three conditions:

(1) Every tile of T is a topological disk.

(2) The intersection of any two tiles of T is a connected set.

(3) The tiles of T are uniformly bounded.

It follows from conditions (1) and (2) that the intersection of two or more distinct tiles is either empty, a point (called a *vertex* of the tiling), or a Jordan arc (an *edge* of the tiling). *Uniformly bounded* in condition (3) means that there exist two positive constants U and u, the *parameters* of the tiling, such that every tile of T contains some circular disk of radius u and is contained in some circular disk of radius U. Condition (3) guarantees that the tiles in T do not get arbitrarily long or arbitrarily thin.

Let $D(r, P)$ denote the closed disk of radius r centered at point P in the plane. Given a tiling T, consider the set of all tiles whose intersection with $D(r, P)$ is nonempty. Add to this set any tiles needed to make their union simply-connected. Call the resulting set of tiles $T(r, P)$. Let $v(r, P)$, $e(r, P)$, and $t(r, P)$ denote the number of vertices, edges, and tiles in $T(r, P)$, respectively. The straightforward proofs of the following properties of normal tilings appear in [**4**]. These proofs are based on the simple fact that the area of each tile of a normal tiling T lies between πu^2 and πU^2. Statement (2) of Lemma 1, often referred to as the "Normality Lemma," implies that there are relatively few tiles near the boundary of $D(r, P)$.

Lemma 1. *If T is a normal tiling with parameters u and U, then:*

(1) *each vertex of T has at most $4(U/u)^2$ incident edges, and each tile has at most $9(U/u)^2 - 1$ edges;*

(2) *for any fixed positive number x and any point P in the plane*

$$\lim_{r\to\infty} \frac{t(r+x, P) - t(r, P)}{t(r, P)} = 0.$$

3. PROOF OF THEOREM 1. Throughout this section the point P in the plane is fixed. We thus abbreviate $D(r, P)$ to $D(r)$, and likewise for $T(r)$, $v(r)$, $e(r)$, and $t(r)$.

Lemma 2. *If T is a normal tiling, then for any fixed positive x*

$$\lim_{r\to\infty} \frac{e(r+x)}{e(r)} = 1.$$

Proof. Let $k = 9(U/u)^2 - 1$, where u and U are the parameters of T. Then by statement (1) of Lemma 1,

$$1 \le \frac{e(r+x)}{e(r)} = 1 + \frac{e(r+x) - e(r)}{e(r)} \le 1 + \frac{k\,[t(r+x) - t(r)]}{e(r)}. \tag{1}$$

Let $p(t)$ denote the number of edges in tile t. Because $\sum_{t\in T(r)} p(t)$ counts each edge in $T(r)$ at most twice, and each tile has at least three edges (by condition (2) in the definition of normality), we see that

$$3t(r) \le \sum_{t\in T(r)} p(t) \le 2e(r).$$

Combining this with inequality (1) yields

$$1 \le \frac{e(r+x)}{e(r)} \le 1 + \frac{2k}{3}\left[\frac{t(r+x) - t(r)}{t(r)}\right].$$

In view of statement (2) of Lemma 1, Lemma 2 follows from the last inequality. ■

Proof of Theorem 1. Assume, by way of contradiction, that all except finitely many tiles have at least seven edges. Then, for any $\epsilon > 0$, the average number of edges per tile in $T(r)$ is greater than $7 - \epsilon$ if r is sufficiently large. With notation as in Lemma 2, we conclude that

$$\frac{2e(r)}{t(r)} \ge \frac{\sum_{t\in T(r)} p(t)}{t(r)} > 7 - \epsilon.$$

Therefore, for any $\epsilon > 0$,

$$t(r) < \frac{2e(r)}{(7-\epsilon)} \tag{2}$$

if r is sufficiently large.

Note that every vertex in $T(r)$ is contained in the disk $D(r + 2U)$, where U is the upper parameter of the tiling. If $q(v)$ denotes the number of edges of T incident with vertex v, then $\sum_{v\in T(r)} q(v)$ counts each edge of $T(r + 2U)$ at most twice, yielding

$$3v(r) \le \sum_{v\in T(r)} q(v) \le 2e(r + 2U). \tag{3}$$

According to Euler's formula for planar graphs $1 = v(r) - e(r) + t(r)$ for any r. Combining this with inequalities (2) and (3) yields

$$1 < \frac{2}{3}e(r+2U) - e(r) + \frac{2e(r)}{(7-\epsilon)} = e(r)\left[\frac{2}{3}\frac{e(r+2U)}{e(r)} - 1 + \frac{2}{(7-\epsilon)}\right] \tag{4}$$

for any $\epsilon > 0$ and r sufficiently large. But by Lemma 2 the right-hand side of inequality (4) is less than 0 for r sufficiently large, clearly a contradiction. ■

4. TILING EUCLIDEAN 3-SPACE. There is no analog of Theorem 1 for tilings of Euclidean three-space. Danzer, Grünbaum, and Shephard [**1**] gave examples of normal, face-to-face tilings of $\mathbb{E}^3$ by (combinatorially equivalent) convex polyhedra with n faces, where n can be arbitrarily large. Interesting constructions of this kind of tiling were also described by Schulte [**5**]. (A tiling is said to be *face-to-face* if neighboring tiles intersect in a face of each.) In fact, there is no known upper bound for the number of faces of a convex polyhedron with the property that congruent copies of it will tile $\mathbb{E}^3$. Engel [**3**] gave an example of such a face-to-face tiling by copies of a polyhedron with thirty-eight faces, but it is unknown whether the number thirty-eight is maximum in this regard. In 1961 Delone [**2**] showed that any polyhedron that admits an isohedral face-to-face tiling of $\mathbb{E}^3$ can have at most 390 faces. Delone's bound was slightly improved to 378 by Tarasov [**6**] in 1997. (A tiling is *isohedral* if its symmetry group acts transitively on the set of tiles.)

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