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A COMBINATORIAL APPROACH TO BINARY POSITIONAL NUMBER SYSTEMS

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Abstract. Although the representation of the real numbers in terms of a base and a set of digits has a long history, new questions arise even in the binary case – digits 0 and 1. A binary positional number system (binary radix system) with base equal to the golden ratio $(1+\sqrt{5})/2$ is fairly well known. The main result of this paper is a construction of infinitely many binary radix systems, each one constructed combinatorially from a single pair of binary strings. Every binary radix system that satisfies even a minimal set of conditions that would be expected of a positional number system, can be constructed in this way.

1. Introduction

The terms *positional number system, radix system*, and *β-expansion* that appear in the literature all refer to the representation of real numbers in terms of a given base or radix B and a given finite set D of digits. Historically the base is 10 and the digit set is $\{0,1,2,\dots,9\}$ or, in the binary case, the base is 2 and the digit set is $\{0,1\}$. Alternative choices for the set of digits goes back at least to Cauchy, who suggested the use of negative digits in base 10, for example $D=\{-4,-3,\dots,4,5\}$. The *balanced ternary system* is a base 3 system with digit set $D=\{-1,0,1\}$ discussed by Knuth in [10]. In the balanced ternary, every integer, positive or negative, has a representation of the form $\sum_{n=0}^{N}\omega_n 3^n$, where $\omega_n\in\{-1,0,1\}$ for all n. A well known system with digit set $\{0,1\}$ and base equal to the golden ratio $\left(1+\sqrt{5}\,\right)/2$ originated with [5]. Other early work on the representation of numbers using a non-integer base include those of Rényi [12] and Parry [11]. Positional number systems have also been extended to the representation of complex numbers [9] and, more generally, to the representation of points in $\mathbb{R}^d$ [13]. A vast array of additional references on positional number systems can be

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accessed by searching on the terms 'radix system' and 'β-expansion'. In particular, a number of papers, for example [1,6–8], have appeared on the binary representation of the number 1.

Our intention in this paper is to provide a combinatorial framework for the binary representation of the real numbers. By binary we mean with digit set $\{0,1\}$. A precise definition of a *binary radix system* appears in Section 2 (Definition 2.3). Starting from any pair of infinite strings of 0's and 1's that satisfy a few combinatorial conditions given in Section 3, a binary radix system is constructed (Theorem 4.1 and Theorem 5.2). Conversely, any binary radix system can be obtained by this construction (Theorem 4.3 and Theorem 5.4). This leads to infinitely many binary radix systems, some of whose properties are investigated in this paper.

The organization of this paper is as follows. The notion of an *admissible pair* (α,β) of strings is defined in Section 3. This concept was introduced in our previous paper [3], and we refer to results in that paper in some of the proofs in this paper. The construction of a binary radix system on the interval $[0,1]$ from an admissible pair of binary strings is the subject of Section 4. Moreover, every binary radix system on $[0,1]$ can be obtained by this construction from some admissible pair (α,β). Section 5 extends these two results of Section 4 to radix systems for the set $\mathbb{R}^+$ of non-negative real numbers. Section 7 contains proofs of results in the previous two sections. An algorithm is provided in Section 6 whose input is a particular binary radix system and a positive real number x and whose output is the decimal expansion of x in that binary radix system. In addition, it is shown that, not only are there infinitely many radix systems for $\mathbb{R}^+$, but there are infinitely many radix systems for any given base between 1 and 2. For the standard base 2 radix system there is an associated tiling of the real line, where a tile consists of all those points on the real line with the same integer part. This is the trivial tiling consisting of unit length intervals. There is an analogous tiling for any binary radix system but, in general, the tiles have various lengths, and the tiling is self-replicating but not periodic. This is the subject of Section 8.

2. Binary radix systems

Let $\Omega=\{0,1\}^\infty$ denote the set of infinite strings of the form $\omega:=\omega_0\omega_1\omega_2\cdots$, where $\omega_n\in\{0,1\}$ for all $n\geqq 0$. A line over a finite string denotes infinite repetition, for example $\overline{01}=010101\cdots$. The *lexicographic order* $\preceq$ on Ω is the total order defined by $\sigma\prec\omega$ if $\sigma\neq\omega$ and $\sigma_k<\omega_k$ where k is the least index such that $\sigma_k\neq\omega_k$.

DEFINITION 2.1. The *shift operator* $S:\Omega\to\Omega$ is defined by

$$S(\omega_0\omega_1\omega_2\cdots)=\omega_1\omega_2\cdots .$$

Let S^n denote the n^{th} iterate of S for all $n \geqq 0$. A subset $\Gamma \subset \Omega$ is *shift invariant* if $S^n(\Gamma) \subseteqq \Gamma$ for all $n \geqq 0$.

Extend Ω to a set $\Omega^\bullet$ of *decimals* by adding a "decimal point" as follows:

$$\Omega^\bullet := \{\omega_0\omega_1 \cdots \omega_N \,{}_\bullet\omega_{N+1}\omega_{N+2} \cdots : \, \omega_0\omega_1\omega_2 \cdots \in \Omega\}.$$

When no confusion arises, we omit an initial string of zeros (before the decimal point) and/or a terminal string of zeros (after the decimal point), for example $01{}_\bullet1000\cdots = 1{}_\bullet1$.

The lexicographic order on Ω can be extended to $\Omega^\bullet$ as follows. If $\omega \in \Omega^\bullet$, let $\widehat{\omega} \in \Omega$ denote the string ω with the decimal point removed. If

$$\sigma = \sigma_0\,\sigma_1 \cdots \sigma_N\,{}_\bullet\,\sigma_{N+1}\,\sigma_{N+2} \cdots \quad \text{and} \quad \omega = \omega_0\omega_1 \cdots \omega_N\,{}_\bullet\omega_{N+1}\omega_{N+2} \cdots$$

(where some leading entries may possibly be 0's), then define $\sigma \prec \omega$ if $\sigma \neq \omega$ and $\widehat{\sigma} \prec \widehat{\omega}$ in the lexicographic order on Ω. For example ${}_\bullet\overline{1} \prec 1{}_\bullet\overline{0}$ because $0{}_\bullet\overline{1} \prec 1{}_\bullet\overline{0}$ because $0\overline{1} \prec 1\overline{0}$.

Definition 2.2. A subset $\Gamma \subset \Omega^\bullet$ will be called *shift invariant* if, whenever $\omega \in \Gamma$, for any $n \geqq 0$, any decimal obtained by placing the decimal point at any position in $S^n(\widehat{\omega})$ (introducing zeros if needed), is also in Γ.

The conditions in the definition below are meant to be the minimum requirements that one would expect of a positional number system for the set $\mathbb{R}^+$ of non-negative real numbers.

Definition 2.3. A *binary radix system* for $\mathbb{R}^+$ is a pair (Γ, B), where $B > 1$ is a real number called the *base* and $\Gamma \subset \Omega^\bullet$ is called the *address space.* The following conditions must be satisfied:

(1) Γ is shift invariant, and

(2) the map $\overset{\bullet}{\pi} : \Gamma \to \mathbb{R}^+$ defined by

$$\overset{\bullet}{\pi}(\omega_0\omega_1 \cdots \omega_N\,{}_\bullet\omega_{N+1}\omega_{N+2} \cdots) = \sum_{n=-\infty}^{N} \omega_{N-n}\, B^n \tag{2.1}$$

it is bijective and strictly increasing. The map $\overset{\bullet}{\pi}$ is called the *radix map.*

Condition (1) is a consistency requirement. In the standard base two radix system, for example, if 1 is in the address space, then so must be

$$\ldots,\, {}_\bullet0001,\, {}_\bullet001,\, {}_\bullet01,\, {}_\bullet1,\, 1{}_\bullet,\, 10{}_\bullet,\, 100{}_\bullet,\, 1000{}_\bullet,\, \ldots,$$

and if ${}_\bullet111\cdots$ is in the address space, then so must be

$$\ldots,\, {}_\bullet000\overline{1},\, {}_\bullet00\overline{1},\, {}_\bullet0\overline{1},\, {}_\bullet\overline{1},\, 1{}_\bullet\overline{1},\, 11{}_\bullet\overline{1},\, 111{}_\bullet\overline{1},\, 1111{}_\bullet\overline{1},\, \ldots.$$

A binary radix system for the real numbers in the unit interval $[0,1]$, for technical reasons that should become clear subsequently, is slightly different.

DEFINITION 2.4. A *binary radix system* for $[0,1]$ is a pair (Γ, B), where $B>1$ is a real number called the *base* and $\Gamma \subset \Omega$ is called the *address space.* The following conditions must be satisfied, where $b = 1/B$:

(1) Γ is shift invariant, and

(2) the map $\pi : \Gamma \to [0,1]$ defined by

$$\pi(\omega_0\omega_1\omega_2\cdots) = (1-b)\sum_{n=0}^{\infty} \omega_n\, b^n \tag{2.2}$$

it is bijective and strictly increasing. The map π is called the *radix map.*

A remark on the factor $1-b$ in Equation (2.2) is in order. If $B=2$, the standard binary base, then $b = 1/2$ and

$$\pi(\omega_0\omega_1\omega_2\cdots) = \sum_{n=1}^{\infty} \omega_{n-1}\left(\frac{1}{2}\right)^n,$$

so that $.\omega_0\omega_1\omega_2\cdots$ is the usual decimal representation of a real number in the interval $[0,1]$. We hope that, by the end of the paper, the reader will be convinced that the factor $1-b$ is natural.

EXAMPLE 2.5 (standard binary radix system). The standard binary radix system on $[0,1]$ is actually two examples. Let $B=2$ and let Γ consist of all strings in Ω except those ending in $0111\cdots$. Then (Γ, B) is a binary radix system for $[0,1]$. Alternatively, with the same base $B=2$, let Γ consist of all strings in Ω except those ending in $1000\cdots$. Again (Γ, B) is a binary radix system for $[0,1]$. Note that an address space Γ with base $B=2$ cannot contain both an element that ends in $1\overline{0}$ and an element that ends in $0\overline{1}$. This is because, *by shift invariance*, if this were so, then both $1\overline{0}$ and $0\overline{1}$ must lie in Γ. But $\pi(1\overline{0}) = \pi(0\overline{1})$, contradicting condition (2), that π is bijective. It is precisely to avoid this type of inconsistency, i.e., that some numbers are represented by a decimal ending in $1\overline{0}$ and others are represented by a decimal ending in $0\overline{1}$, that we require shift invariance as a condition for a binary radix system.

EXAMPLE 2.6 (golden ratio radix system). In the introduction we referred to a binary radix system (Γ, B), where B is the golden ratio $\tau = \big(1+\sqrt{5}\big)/2$. For such a radix system on the interval $[0,1]$, the set Γ consists of all strings in Ω that do not contain 011 or $\overline{01}$ as a substring. The radix map $\pi : \Gamma \to [0,1]$ is

$$\pi(\omega) = (1-\overline{\tau})\sum_{n=0}^{\infty} \omega_n\, \overline{\tau}^n = \sum_{n=2}^{\infty} \omega_{n-2}\, \overline{\tau}^n,$$

where $\overline{\tau} := \frac{1}{\tau} = \left(\sqrt{5}-1\right)/2$. That π is bijective and strictly increasing is a special case of general results in Section 4.

3. Allowable and admissible pairs

In this section two related terms are defined, *allowable* and *admissible* pairs of strings. For $\alpha, \beta \in \Omega$, the notation

$$[\alpha, \beta] := \{\omega \in \Omega : \ \alpha \preceq \omega \preceq \beta\}$$

will be used for a closed interval in Ω; likewise for the open interval (α, β) and half open intervals $[\alpha, \beta)$ and $(\alpha, \beta]$.

DEFINITION 3.1. For $\omega \in \Omega$, let $\omega \mid n = \omega_0\omega_1 \cdots \omega_n$. For $\Gamma \subseteqq \Omega$, let

$$\Gamma_n = \{\omega \mid n : \ \omega \in \Gamma\}$$

and let $|\Gamma_n|$ denote the cardinality of Γ_n. Define the *exponential growth rate* $h(\Gamma)$ of $\Gamma \subseteqq \Omega$ by

$$h(\Gamma) = \limsup_{n\to\infty} \frac{1}{n} \log |\Gamma_n|.$$

DEFINITION 3.2. The *address spaces* associated with a pair (α, β) of strings in Ω are defined by

$$\Omega_{(\alpha,\beta,-)} := \left\{\omega \in \Omega : \ S^n\omega \notin (\alpha, \beta] \text{ for all } n \geqq 0\right\},$$

$$\Omega_{(\alpha,\beta,+)} := \left\{\omega \in \Omega : \ S^n\omega \notin [\alpha, \beta) \text{ for all } n \geqq 0\right\},$$

$$\Omega_{(\alpha,\beta)} := \Omega_{(\alpha,\beta,-)} \cup \Omega_{(\alpha,\beta,+)}.$$

DEFINITION 3.3. Call a pair (α, β) of elements of Ω *allowable* if α and β satisfy the following three conditions.

(1) $\alpha_0 = 0$, $\alpha_1 = 1$ and $\beta_0 = 1$, $\beta_1 = 0$,
(2) $S^n\alpha \notin (\alpha, \beta]$ and $S^n\beta \notin [\alpha, \beta)$ for all $n \geqq 0$,
(3) $h\big(\Omega_{(\alpha,\beta)}\big) > 0$.

There exist pairs that satisfy conditions (1) and (2), but not (3); see [3] for examples.

EXAMPLE 3.4. If $\alpha = 0\overline{1} = 0111\cdots$ and $\beta = 1\overline{0} = 1000\cdots$, then (α, β) is an admissible pair. The address space $\Omega_{(\alpha,\beta,-)}$ consists of all strings that do not end in $1000\cdots$, and $\Omega_{(\alpha,\beta,+)}$ consists of all strings that do not end in $0111\cdots$. Notice that $\Omega_{(\alpha,\beta,-)}$ and $\Omega_{(\alpha,\beta,+)}$ are exactly the two possible address spaces for the standard binary radix systems of Example 2.5.

Example 3.5. If $\alpha = \overline{01}$ and $\beta = 1\overline{0}$, then (α, β) is an allowable pair. Notice that $\Omega_{(\alpha,\beta,-)}$ is exactly the address space of the golden ratio based binary radix system of Example 2.6. The address space $\Omega_{(\alpha,\beta,+)}$ consists of all strings not containing 011 or $1\overline{0}$ as substrings, and $\Omega_{(\alpha,\beta)}$ consists of all strings not containing 011 as a substring.

Definition 3.6. A string $\omega \in \Omega$ will be called *periodic* if $\omega = \overline{s}$ for some finite string s. A string $\omega \in \Omega$ will be called *eventually periodic* if there is an $m \geqq 0$ such that $S^m \omega$ is periodic.

The address space $\Omega_{(\alpha,\beta)}$ in Example 3.5 can be characterized by a set of "forbidden" finite substrings (011 in that case). This is always true if α and β are periodic as explained in the next proposition.

Proposition 3.7. *If an allowable pair (α, β) is such that α and β are periodic, then there is a finite set T of finite strings such that $\Omega_{(\alpha,\beta)}$ is the set of $\omega \in \Omega$ such that s is not a substring of ω for any $s \in T$.*

Despite the fact that the conclusion of Proposition 3.7 holds in Example 3.5, the proposition is false in general if, instead of periodic, the hypothesis assumes only eventually periodic. For example, if $\alpha = 011\overline{01}$ and $\beta = 1\overline{0}$, then we would have to forbid an infinite set 0111, 011011, 01101011, 0110101011, ...

More instructive than giving a formal proof of the proposition is to give an example that makes clear how a proof would proceed. Referring to Definition 3.3 of allowable pairs, if $\alpha = \overline{01101}$ and $\beta = \overline{100}$, then the forbidden set is $T = \{0111, 011011, 1000\}$. The string 0111 is not allowed because $0111\gamma \succ \alpha$ for any string γ, and 011011 is not allowed because $011011\gamma \succ \alpha$ for any string γ. The string 000 is not allowed because $1000\gamma \prec \beta$ for any string γ. Because of the periodicity, no additional forbidden strings are required.

A proof of the following theorem appears in [3], a main result in that paper.

Theorem 3.8. *If (α, β) is allowable, then the equation*

$$\sum_{n=0}^{\infty} \alpha_n x^n = \sum_{n=0}^{\infty} \beta_n x^n \tag{3.1}$$

has a solution in the interval $\left[\frac{1}{2}, 1\right)$. There is no solution in the interval $\left(0, \frac{1}{2}\right)$.

Definition 3.9. For an allowable pair (α, β), denote the smallest real solution of equation (3.1) in the interval $(0, 1)$ by $r := r(\alpha, \beta)$.

For finite binary strings $a = \varepsilon_0\varepsilon_1 \cdots \varepsilon_{m-1}$ and $b = \delta_0\delta_1 \cdots \delta_{n-1}$, let

$$A(x) = \sum_{j=0}^{m-1} \varepsilon_j x^j \qquad \text{and} \qquad B(x) = \sum_{j=0}^{n-1} \delta_j x^j,$$

and define the polynomial

$$p_{(a,b)}(x) := (1 - x^m)\, B(x) - (1 - x^n)\, A(x).$$

For $r \in (0, 1)$, call finite binary strings a and b *r-equivalent* if r is a root of $p_{(a,b)}(x)$. It is easily checked that r-equivalence is indeed an equivalence relation on the set of finite binary strings.

EXAMPLE 3.10. The strings 01 and 100 are r-equivalent, where r is the real root of $x^3 + x^2 - 1$. In particular $p_{(01,100)}(x) = (1 - x)\big(1 - x^2 - x^3\big)$.

The three strings:

$$a = 011110, \quad b = 100111, \quad c = 101010$$

are pairwise r-equivalent, where r is the real root of $x^3 + x - 1$. In particular $p_{ab}(x) = \big(1 - x^6\big)\big(1 - x^2\big)\big(1 - x - x^3\big)$, $p_{ac}(c) = \big(1 - x^6\big)\big(1 - x - x^3\big)$, and $p_{bc}(x) = \big(1 - x^6\big)\, x^2\big(1 - x - x^3\big)$.

An r-equivalence class C is closed under concatenation and is therefore a submonoid of the free monoid $\{0, 1\}^*$ consisting of all finite strings with alphabet $\{0, 1\}$ and with concatenation as the operation. It is easy to check that if $a, ab \in C$ then $b \in C$, and if $b, ab \in C$ then $a \in C$, i.e., C is *left and right unitary.* In particular, C is itself a free monoid with a unique set G_C of free generators.

EXAMPLE 3.11. Some messy algebra suffices to show that $\{100, 011\}$ is the set of free generators of an r-equivalence class, where $r = \big(\sqrt{5} - 1\big)/2$ is the reciprocal of the golden ratio, i.e., the positive root of $1 - x - x^2$.

DEFINITION 3.12. Consider a pair (α, β) of binary strings

$$\alpha = a_0a_1a_2 \cdots \qquad \text{and} \qquad \beta = b_0b_1b_2 \cdots,$$

where $S := \{a_0, a_1, \ldots, b_0, b_1, \ldots\}$ is a finite subset (with repetition) of the set G_C of generators of an r-equivalence class C. Then (α, β) will be called *r-bad* unless $\alpha = aaa \cdots$, $\beta = bbb \cdots$. A pair of strings that is not r-bad will be called *r-good.* An $r(\alpha, \beta)$-good pair (α, β) of allowable strings will be called *admissible.*

REMARK 1. It follows immediately from the fact that $p(a, b)(x)$ is a polynomial that, if (α, β) is allowable and $r(\alpha, \beta)$ is not an algebraic number, then $r(\alpha, \beta)$ is admissible.

REMARK 2. It is proved in [3] that, if (α,β) is allowable and r-bad for some $r\in(0,1)$, then $\sum_{n=0}^{\infty}\alpha_n\,r^n=\sum_{n=0}^{\infty}\beta_n\,r^n$. This does not mean, however, that $r=r(\alpha,\beta)$, because r may not be the *smallest* solution of the equation $\sum_{n=0}^{\infty}\alpha_n\,x^n=\sum_{n=0}^{\infty}\beta_n\,x^n$ in the interval $(0,1)$. On the other hand, we have no example where it is not the smallest.

EXAMPLE 3.13 (allowable but not admissible pairs). The allowable pair

$$\alpha=011\overline{100},\qquad \beta=100\overline{011}$$

is bad. Solving equation (3.1), we find that the number $r:=r(\alpha,\beta)$ is the positive root of the polynomial x^2+x-1, which is approximately 0.6180. It is easy to check that 100 and 011 are r-equivalent. Therefore (α,β) is bad, hence not admissible.

The allowable pair $\alpha=01\overline{100}$, $\beta=100\overline{01}$ is also bad. Solving equation (3.1), we find that the number $r:=r(\alpha,\beta)$ is the real root of the polynomial x^3+x^2-1, which is approximately 0.52818. Again it is easy to check that 100 and 01 are r-equivalent. Therefore (α,β) is not admissible.

EXAMPLE 3.14 (another allowable but not admissible pair). Another bad allowable pair is

$$\alpha=accc\cdots\qquad\text{and}\qquad\beta=bababa\cdots,$$

where a, b and c are the finite strings given in Example 3.10. In this example a, b and c are pairwise r-equivalent, where r is the real root of x^3+x-1. It is straightforward to check that the (not roots of unity) solutions to equation (3.1) satisfy $(x^3+x-1)(x^{12}+x^6+x^2-1)=0$. Since the only positive real root of $x^{12}+x^6+x^2-1$ is approximately 0.8062, while the real root of x^3+x-1 is smaller, approximately 0.6823, we have $r(\alpha,\beta)=r$.

4. Radix systems on $[0,1]$ from admissible pairs

In the previous section, address spaces $\Omega_{(\alpha,\beta,-)}$ and $\Omega_{(\alpha,\beta,+)}$ were defined for every pair (α,β) of binary strings. This section explains how these two address spaces, in the case where (α,β) is admissible, become the address spaces of two corresponding binary radix systems for $[0,1]$. Moreover, the converse also holds. Every binary radix system for $[0,1]$ can be constructed in this way. The proofs of Theorem 4.1 and Theorem 4.3 below are omitted because they are essentially the same as the somewhat more difficult proofs of the analogous Theorems 5.2 and 5.4 for the reals in Section 5.

THEOREM 4.1. *If (α,β) is an admissible pair and $B_{(\alpha,\beta)}=1/r(\alpha,\beta)$, then $\big(\Omega_{(\alpha,\beta,-)},B_{(\alpha,\beta)}\big)$ and $\big(\Omega_{(\alpha,\beta,+)},B_{(\alpha,\beta)}\big)$ are binary radix systems for* $[0,1]$.

Definition 4.2. To simplify notation, let

$$R_{(\alpha,\beta,+)} := \left(\Omega_{(\alpha,\beta,+)}, B_{(\alpha,\beta)}\right) \quad \text{and} \quad R_{(\alpha,\beta,-)} := \left(\Omega_{(\alpha,\beta,-)}, B_{(\alpha,\beta)}\right)$$

denote the radix systems constructed from an admissible pair (α,β) as in Theorem 4.1. Call $R_{(\alpha,\beta,+)}$ and $R_{(\alpha,\beta,-)}$ the (α,β)-*radix systems.*

As stated in the next theorem, all binary radix systems are (α,β)-radix systems for some admissible (α,β).

Theorem 4.3. *For every binary radix system* (Γ, B) *for* $[0,1]$, *there is an admissible pair* (α,β) *such that either* $(\Gamma,B) = R_{(\alpha,\beta,-)}$ *or* $(\Gamma,B) = R_{(\alpha,\beta,+)}$.

Example 4.4 (standard binary radix system). Continuing from Example 2.5 and Example 3.4, consider the admissible pair $(0\overline{1}, 1\overline{0})$. The number $r := r(\alpha,\beta)$ in Theorem 4.1 is the smallest solution to the equation $\sum_{n=1}^{\infty} x^n = 1$, which, because the left hand side is a geometric series, reduces to $2x = 1$. So $r = 1/2$ and the base is $B_{(\alpha,\beta)} = 1/r = 2$. The radix systems $R_{(0\overline{1},1\overline{0},+)}$ and $R_{(0\overline{1},1\overline{0},-)}$ are the standard binary radix systems.

Example 4.5 (golden ratio radix system). Continuing from Example 2.6 and Example 3.5, consider the admissible pair $(\overline{01}, 1\overline{0})$. The number $r := r(\alpha,\beta)$ in Theorem 4.1 is the smallest solution to the equation $\sum_{n=0}^{\infty} x^{2n+1} = 1$, which reduces to $x^2 + x - 1 = 0$. So $r = \left(\sqrt{5}-1\right)/2$, and the base $B_{(\alpha,\beta)} = 1/r = \left(1+\sqrt{5}\right)/2$ is the golden ratio. The radix systems $R_{(\overline{01},1\overline{0},+)}$ and $R_{(\overline{01},1\overline{0},-)}$ are the golden ratio radix systems.

Example 4.6 (three radix systems with the same base). Consider the (α,β)-radix systems $R_{(\alpha_1,\beta_1,\pm)}$, $R_{(\alpha_2,\beta_2,\pm)}$, and $R_{(\alpha_3,\beta_3,\pm)}$, where the admissible pairs are:

$$\alpha_1 = \overline{01000}, \quad \alpha_2 = \overline{011}, \quad \alpha_3 = \overline{01}, \quad \beta_1 = 1\overline{0}, \quad \beta_2 = \overline{10}, \quad \beta_3 = \overline{100}.$$

In all three cases the number $r := r(\alpha,\beta)$ in Theorem 3.8 is the solution to the equation $x^3 + x^2 - 1 = 0$ in the interval $(0,1)$; approximately $r \approx 0.7549$. Therefore the base $B_{(\alpha,\beta)} \approx 1.3247$ is the same in all three cases. The three address spaces, however, are pairwise distinct. The address space $\Omega_{(\alpha_1,\beta_1)}$ consists of all strings that do not contain the substrings $11, 101, 1001$ or 10001. The address space $\Omega_{(\alpha_2,\beta_2)}$ consists of all strings that do not contain the substrings 100 or 111. The address space $\Omega_{(\alpha_3,\beta_3)}$ consists of all strings that do not contain the substrings 11 or 1000. See Corollary 1 in Section 6 for a general result concerning radix systems with the same base.

5. Binary radix systems for the non-negative reals

In this section, the method of the previous section for constructing radix systems for the interval $[0,1]$ is extended in order to construct radix systems for the set $\mathbb{R}^+$ of non-negative real numbers.

DEFINITION 5.1. Extend the definition of address spaces $\Omega_{(\alpha,\beta,-)}$, $\Omega_{(\alpha,\beta,+)}$, $\Omega_{(\alpha,\beta)}$ to address spaces $\Omega^\bullet_{(\alpha,\beta,-)}$, $\Omega^\bullet_{(\alpha,\beta,+)}$, $\Omega^\bullet_{(\alpha,\beta)}$ of decimals as follows:

$$\Omega^0_{(\alpha,\beta,-)} := \big\{\omega \in \Omega_{(\alpha,\beta,-)} : 0\omega \preceq \alpha\big\}, \quad \Omega^\bullet_{(\alpha,\beta,-)} := \big[\Omega^0(\alpha,\beta,-)\big]^\bullet,$$

$$\Omega^0_{(\alpha,\beta,+)} := \big\{\omega \in \Omega_{(\alpha,\beta,+)} : 0\omega \prec \alpha\big\}, \quad \Omega^\bullet_{(\alpha,\beta,+)} := \big[\Omega^0(\alpha,\beta,+)\big]^\bullet,$$

$$\Omega^0_{(\alpha,\beta)} := \Omega^0_{(\alpha,\beta,-)} \cup \Omega^0_{(\alpha,\beta,+)}, \quad \Omega^\bullet_{(\alpha,\beta)} := \big[\Omega^0(\alpha,\beta)\big]^\bullet.$$

The reason for introducing the spaces Ω^0 with the added condition $0\omega \prec \alpha$ is to insure that the spaces $\Omega^\bullet$ are shift invariant; see Proposition 5.7 below.

THEOREM 5.2. *If (α,β) is an admissible pair, b the smallest solution to equation (3.1) in the interval $(0,1)$, and $B_{(\alpha,\beta)} = 1/b$, then $\big(\Omega^\bullet_{(\alpha,\beta,-)}, B_{(\alpha,\beta)}\big)$ and $\big(\Omega^\bullet_{(\alpha,\beta,+)}, B_{(\alpha,\beta)}\big)$ are binary radix systems for $\mathbb{R}^+$.*

DEFINITION 5.3. To simplify notation, let

$$R^\bullet_{(\alpha,\beta,+)} := \big(\Omega^\bullet_{(\alpha,\beta,+)}, B_{(\alpha,\beta)}\big) \quad \text{and} \quad R^\bullet_{(\alpha,\beta,-)} := \big(\Omega^\bullet_{(\alpha,\beta,-)}, B_{(\alpha,\beta)}\big)$$

denote the radix systems constructed from an admissible pair (α,β) as in Theorem 5.2. Call $R^\bullet_{(\alpha,\beta,+)}$ and $R^\bullet_{(\alpha,\beta,-)}$ the (α,β)-*radix systems*.

As stated in the next theorem, all binary radix systems are (α,β)-radix systems for some admissible (α,β).

THEOREM 5.4. *For every binary radix system (Γ, B) for $\mathbb{R}^+$, there is an admissible pair (α,β) such that either $(\Gamma,B) = R^\bullet_{(\alpha,\beta,-)}$ or $(\Gamma,B) = R^\bullet_{(\alpha,\beta,+)}$.*

EXAMPLE 5.5 (standard binary radix system). This is a continuation of Examples 2.5, 3.4, and 4.4 on the standard binary radix system. For the admissible pair $(0\overline{1}, 1\overline{0})$, in Definition 5.1, the space $\Omega^0_{(\alpha,\beta,\pm)} = \Omega_{(\alpha,\beta,\pm)}$. Therefore $\Omega^\bullet_{(\alpha,\beta,+)}$ consists of all decimals that do not end in $0\overline{1}$ and $\Omega^\bullet_{(\alpha,\beta,-)}$ consists of all decimals that do not end in $1\overline{0}$.

EXAMPLE 5.6 (golden ratio radix system). This is a continuation of Examples 2.6, 3.5, and 4.5 on the golden ratio radix system. The relevant admissible pair is $(\overline{01}, 1\overline{0})$. Note that $\Omega^0_{(\alpha,\beta)} \neq \Omega_{(\alpha,\beta)}$ since $11\overline{0} \in \Omega_{(\alpha,\beta)} \setminus \Omega^0_{(\alpha,\beta)}$. This is because $011\overline{0} \succ \alpha$. In particular, $11_\bullet \notin \Omega^\bullet_{(\alpha,\beta)}$. Indeed, it is an easy exercise to show that $\omega \in \Omega^\bullet_{(\alpha,\beta)}$ if and only if ω does not contain 11 as a substring.

The remainder of this section contains the proof of Theorem 5.2, beginning with the proof of shift invariance. The proof of Theorem 5.4 appears in Section 7.

PROPOSITION 5.7. *Given an admissible pair (α,β), the address spaces $\Omega^\bullet_{(\alpha,\beta,-)}$, $\Omega^\bullet_{(\alpha,\beta,+)}$, and $\Omega^\bullet_{(\alpha,\beta)}$ are shift invariant.*

PROOF. The proposition follows in a straightforward way from the definition of the address spaces, except for one detail. Consider the case of $\Omega^\bullet_{(\alpha,\beta,-)}$ (the case $\Omega^\bullet_{(\alpha,\beta,+)}$ is similar). Again, for $\omega \in \Omega^\bullet$, let $\widehat{\omega} \in \Omega$ denote the string ω with the decimal point removed. If $\omega := \omega_0 \cdots \omega_{N\bullet}\omega_{N+1}\omega_{N+2}\cdots \in \Omega^\bullet_{(\alpha,\beta,-)}$, then for $\Omega^\bullet_{(\alpha,\beta,-)}$ to be shift invariant it is necessary that ${}_\bullet 0\omega_0\omega_1\omega_2\cdots \in \Omega^\bullet_{(\alpha,\beta,-)}$. Therefore it is necessary, not only that $S^n\widehat{\omega} \notin (\alpha,\beta]$ for all $n \geqq 0$, but also that $0\widehat{\omega} \notin (\alpha,\beta]$, i.e., $0\widehat{\omega} \preceq \alpha$. But this is exactly the condition in the definition of $\Omega^0_{(\alpha,\beta,-)}$. $\square$

PROOF OF THEOREM 5.2. The shift invariance of $\Omega^\bullet_{(\alpha,\beta,\pm)}$, which is condition 1 in Definition 2.4, is Proposition 5.7.

We next show that the radix map $\pi : \Omega_{(\alpha,\beta,-)} \to [0,1]$ in (2.2) is strictly increasing and bijective. It is proved in [3, Lemma 3.10] that π is increasing, and in [3, Proposition 3.2] that π is continuous with respect to the following metric on Ω:

$$d(\omega,\sigma) = \begin{cases} 2^{-k} & \text{if } \ \omega \neq \sigma \\ 0 & \text{if } \ \omega = \sigma, \end{cases}$$

where k is the least index such that $\omega_k \neq \sigma_k$. To show that π is strictly increasing, let $\sigma,\omega \in \Omega_{(\alpha,\beta,-)}$ with $\sigma \prec \omega$. Without loss of generality (by taking a shift) it may be assumed that $\sigma_0 = 0$ and $\omega_0 = 1$. Hence $\sigma \preceq \alpha \prec \beta \prec \omega$. With notation as in Definition 3.1, let n be such that $\beta \mid (n-1) = \omega \mid (n-1)$, but $\beta_n = 0$, $\omega_n = 1$. Since $\alpha \prec S^n\omega$, we have $\pi(\alpha) \leqq \pi(S^n\omega)$. Since $S^n\beta \prec \alpha$, it is shown in [3, Lemma 4.6] that $\pi(S^n\beta) < \pi(\alpha)$. Therefore $\pi(S^n\beta) < \pi(\alpha) \leqq \pi(S^n\omega)$ and hence $\pi(\sigma) \leqq \pi(\beta) < \pi(\omega)$. That π is surjective follows from the continuity of π and the fact that $\pi(\overline{0}) = 0$, $\pi(\overline{1}) = 1$.

It now suffices to prove that the radix map $\overset{\bullet}{\pi} : \Omega^\bullet_{(\alpha,\beta,\pm)} \to \mathbb{R}^+$ in (2.1) is strictly increasing and bijective. We will prove it for $\Omega^\bullet_{(\alpha,\beta,+)}$; the proof for

$\Omega^\bullet_{(\alpha,\beta,-)}$ is the same. Abbreviate $B := B_{(\alpha,\beta)}$ and let $b = 1/B$. Let

$$p = (1-b)\sum_{n=0}^{\infty} \alpha_n\, b^n = (1-b)\sum_{n=0}^{\infty} \beta_n\, b^n.$$

Assume that $\omega \in \Omega_{(\alpha,\beta,+)}$. Since π is strictly increasing, we have $0\omega \prec \alpha$ if and only if $b\pi(\omega) = \pi(0\omega) < \pi(\alpha) = p$ if and only if $\pi(\omega) < Bp$. Therefore $\pi : \Omega^0_{(\alpha,\beta,+)} \to [0, Bp]$ is strictly increasing and bijective. Now let $\Omega^\bullet_N$ denote the set of elements of $\Omega^\bullet_{(\alpha,\beta,+)}$ of the form $\omega := \omega_0\omega_1\omega_2\cdots\omega_{N-1\bullet}\omega_N\omega_{N+1}\cdots$, where $\widehat{\omega} := \omega_0\omega_1\omega_2\cdots \in \Omega^0_{(\alpha,\beta,+)}$. The relationship between the radix maps π and $\overset{\bullet}{\pi}$ is given by $\overset{\bullet}{\pi}(\omega) = \frac{B^N}{1-b}\,\pi(\widehat{\omega})$, and therefore $\overset{\bullet}{\pi} : \Omega^\bullet_N \to \left[0, \frac{B^{N+1}p}{b-1}\right]$ is a bijection and strictly increasing. Since the sequence $\Omega^\bullet_0 \subset \Omega^\bullet_1 \subset \Omega^\bullet_2 \subset \cdots$ of sets is nested, and since $\bigcup_{N=0}^{\infty} \Omega^\bullet_N = \Omega^\bullet_{(\alpha,\beta,+)}$, the proof is complete. $\square$

6. Algorithm for determining the address

Given a binary radix system for $\mathbb{R}^+$, the radix map $\overset{\bullet}{\pi}$ assigns a non-negative real number to each decimal in the address space. In this section an algorithm is provided for converting in the opposite direction. Given a non-negative real number x, the algorithm determines its decimal representation in the binary radix system. More precisely, if $R^\bullet_{(\alpha,\beta,\pm)} = \left(\Omega^\bullet_{(\alpha,\beta,\pm)}, B_{(\alpha,\beta)}\right)$ has radix map $\overset{\bullet}{\pi} := \overset{\bullet}{\pi}_{(\alpha,\beta)}$ and $x \in \mathbb{R}^+$, then the algorithm finds $\sigma \in \Omega^\bullet_{(\alpha,\beta,-)}$ and $\omega \in \Omega^\bullet_{(\alpha,\beta,+)}$ such that $\overset{\bullet}{\pi}(\sigma) = \overset{\bullet}{\pi}(\omega) = x$.

We begin by defining a certain family of functions and introduce notation for the itineraries of points of this family of functions. Given B such that $1 < B \leqq 2$ and p such that $1-b \leqq p \leqq b$ where $b = 1/B$, consider the two functions $f_{(B,p,\pm)} : [0,1] \to [0,1]$ defined by

$$f_{(B,p,-)}(x) := \begin{cases} Bx & \text{if } \ 0 \leqq x \leqq p \\ Bx + (1-B) & \text{if } \ x > p, \end{cases} \tag{6.1}$$

and

$$f_{(B,p,+)}(x) := \begin{cases} Bx & \text{if } \ 0 \leqq x < p \\ Bx + (1-B) & \text{if } \ x \geqq p. \end{cases} \tag{6.2}$$

The facts that $1 < B \leqq 2$ and $1-b \leqq p \leqq b$ guarantee that $f_{(B,p,\pm)}$ has the form shown in Fig. 1. For a function f, the n^{th} iterate, i.e. f composed with itself n times, is denoted f^n.

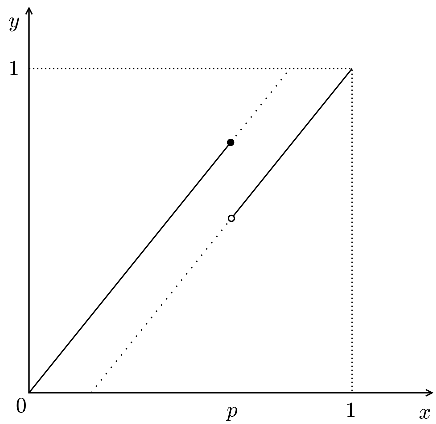

Fig. 1: The function $f_{(B,p,\pm)}$

Definition 6.1. Define the two *itinerary maps* $\tau_{(B,p,\pm)} : [0,1] \to \Omega$ by $\tau_{(B,p,-)} = \omega_0\omega_1\omega_2\cdots$ and $\tau_{(B,p,+)} = \sigma_0\sigma_1\sigma_2\cdots$, where

$$\omega_k = \begin{cases} 0 & \text{if } f^k_{(B,p,-)}(y) \leqq p \\ 1 & \text{if } f^k_{(B,p,-)}(y) > p, \end{cases} \qquad \text{and} \qquad \sigma_k = \begin{cases} 0 & \text{if } f^k_{(B,p,+)}(y) < p \\ 1 & \text{if } f^k_{(B,p,+)}(y) \geqq p. \end{cases}$$

In dynamical systems terminology, $\tau_{(B,p,-)}(y)$ and $\tau_{(B,p,+)}(y)$ are called the *itineraries* of the point y.

Algorithm:

Input: An admissible pair of strings (α,β) and an $x \in \mathbb{R}^+$.

Output: The decimals representations σ and ω of x in the binary radix systems $R^\bullet_{(\alpha,\beta,-)}$ and $R^\bullet_{(\alpha,\beta,+)}$, respectively.

1. Find the least $x \in (0,1)$ such that $\sum_{n=0}^\infty \alpha_n\, x^n = \sum_{n=0}^\infty \beta_n\, x^n$; call it b and let $B = 1/b$.
2. Compute $p := \pi(\alpha) = \pi(\beta)$, where $\pi : \Omega \to [0,1]$ is the radix map.
3. Find the minimum non-negative integer N such that $y := b^N(1-b)x < p$.
4. Compute $\tau_{(B,p,-)}(y) = \sigma_0\,\sigma_1\,\sigma_2\cdots$ and $\tau_{(B,p,+)}(y) = \omega_0\omega_1\omega_2\cdots$.
5. Return

$$\sigma = \sigma_0\,\sigma_1\,\sigma_2\cdots\sigma_{N\bullet}\,\sigma_{N+1}\,\sigma_{N+2}\cdots, \quad \text{and}$$

$$\omega = \omega_0\omega_1\omega_2\cdots\omega_{N\bullet}\omega_{N+1}\omega_{N+2}\cdots.$$

Definition 6.2. Denote the output σ and ω of the algorithm by $\tau_{(\alpha,\beta,-)}(x)$ and $\tau_{(\alpha,\beta,+)}(x)$, respectively, and call $\tau_{(\alpha,\beta,\pm)} : \mathbb{R}^+ \to \Omega^\bullet_{(\alpha,\beta,\pm)}$ the *section maps*. In other words, these give the decimal representations of x in $R_{(\alpha,\beta,\pm)}$. Note that $\tau_{(\alpha,\beta,\pm)}$ takes values in $\Omega^{\bullet}_{(\alpha,\beta,\pm)}$ while $\tau_{(B,p,\pm)}$ takes values in Ω.

EXAMPLE 6.3 (golden ratio radix system). This is a continuation of Examples 2.6, 3.5, 4.5, and 5.6 on the golden ratio radix system. The relevant admissible pair is $(\overline{01}, 1\overline{0})$. In this case $B = \frac{1+\sqrt{5}}{2}$ and $p = \frac{3-\sqrt{5}}{2}$. If, for example, $x = 2$, then $N = 2$ and

$$\tau_{(\alpha,\beta,-)}(2) = \tau_{(\alpha,\beta,+)}(2) = 10_{\bullet}0\overline{01}.$$

The following theorem suffices to prove the validity of the algorithm.

THEOREM 6.4. *The section map $\tau_{(\alpha,\beta,\pm)} : \mathbb{R}^+ \to \Omega^{\bullet}_{(\alpha,\beta,\pm)}$ is the inverse of the radix map $\overset{\bullet}{\pi} : \Omega^{\bullet}_{(\alpha,\beta,\pm)} \to \mathbb{R}^+$.*

PROOF. We will prove that $\tau_{(\alpha,\beta,-)}$ is the inverse of π; a similar proof holds for $\tau_{(\alpha,\beta,+)}$. We first show that the functions $f_{(B,p,\pm)}$ are well defined, i.e. $1 < B \leqq 2$ and $1-b \leqq p \leqq b$. Clearly $1 < B \leqq 2$ because $1 > b \geqq 1/2$ as stated in Theorem 3.8. Moreover $1-b \leqq p \leqq b$ because

$$1-b \leqq (1-b)\sum_{n=0}^{\infty} \beta_n b^n = p = (1-b)\sum_{n=0}^{\infty} \alpha_n b^n \leqq (1-b)\sum_{n=1}^{\infty} b^n = b.$$

To simplify notation, abbreviate $\tau := \tau_{(\alpha,\beta,-)}$ and $\widehat{\tau} := \tau_{(B,p,-)}$. We next show that $\pi \circ \widehat{\tau}$ is the identity on $[0,1]$. If $f_0(x) = Bx$, $f_1(x) = Bx + (1-B)$, and $g_0(x) = bx$, $g_1(x) = bx + (1-b)$, then f_0 and g_0 are inverses, as are f_1 and g_1. Expressing $g_i(x) = bx + i(1-b)$ for $i = 0,1$ and iterating

$$g_{\omega_0} \circ g_{\omega_1} \circ g_{\omega_2} \circ \dots \circ g_{\omega_k}(x) = b^k x + (b^{k-1}\omega_{k-1} + \dots + a\omega_1 + \omega_0)(1-b).$$

Therefore, for any x_0, we have

$$\lim_{k\to\infty} g_{\omega_0} \circ g_{\omega_1} \circ g_{\omega_2} \circ \dots \circ g_{\omega_k}(x_0) = (1-b)\sum_{k=0}^{\infty} \omega_k\, b^k = \pi(\omega).$$

Let $M_0 = [0,p]$ and $M_1 = (p,1]$. Let $x \in [0,1]$ and $\widehat{\tau}(x) = \omega$. It follows from the definition of $\widehat{\tau}$ that

$$x \in M_{\omega_0}, \quad f_{\omega_0}(x) \in M_{\omega_1}, \quad f_{\omega_1} \circ f_{\omega_0}(x) \in M_{\omega_2}, \quad f_{\omega_2} \circ f_{\omega_1} \circ f_{\omega_0} \in M_{\omega_3}, \dots$$

and therefore

$$x \in g_{\omega_0}(M_{\omega_1}), \quad x \in g_{\omega_0} \circ g_{\omega_1}(M_{\omega_2}), \quad x \in g_{\omega_0} \circ g_{\omega_1} \circ g_{\omega_2}(M_{\omega_3}), \dots .$$

Hence $(\pi \circ \widehat{\tau})(x) = \lim_{k\to\infty} g_{\omega_0} \circ g_{\omega_1} \circ g_{\omega_2} \circ \cdots \circ g_{\omega_k}(x_0) = x$.

To show that τ is the inverse of $\overset{\bullet}{\pi}$, with notation as in the algorithm and letting $\sigma := \tau(x)$, we have

$$(\overset{\bullet}{\pi} \circ \tau)(x) = \overset{\bullet}{\pi}(\sigma) = \frac{B^N}{1-b}\,\pi(\widehat{\sigma}) = \frac{B^N}{1-b}\,(\pi \circ \widehat{\tau})(y) = \frac{B^N}{1-b}\,b^N(1-b)x = x.$$

It remains to show that the image of any $x \in \mathbb{R}^+$ under the map τ lies in $\Omega^\bullet_{(\alpha,\beta,-)}$. With y as defined in the algorithm, it is sufficient to show that $\widehat{\tau}(y) \in \Omega^0_{(\alpha,\beta,-)}$. Let τ_- and τ_+ denote the itineraries of the point p of the functions $f_{(B,p,-)}$ and $f_{(B,p,+)}$, respectively. In [2, Theorem 5.1] it is proved that $\widehat{\tau}\big([0,1]\big) = \big\{\omega \in \Omega : S^n\omega \notin (\tau_-, \tau_+] \text{ for all } n \geqq 0\big\}$, and in [3, Theorem 1.1] it is proved that $\alpha = \tau_-$ and $\beta = \tau_+$. Therefore $\widehat{\tau}\big([0,1]\big) = \Omega_{(\alpha,\beta,-)}$. It only remains to show that if $\widehat{\omega} = \widehat{\tau}(y)$, then $0\widehat{\omega} \preceq \alpha$. However, since $y < p$ as in Step 1 of the algorithm, it follows immediately from the definition of the itinerary $\widehat{\tau}(y)$ of y that $\widehat{\omega}_0 = 0$. In particular $0\widehat{\omega} \prec \alpha$. $\square$

Corollary 1. *For each base B, $1 < B < 2$, there exist infinitely many binary radix systems with base B.*

Proof. Given B, let p be any real number such that $1 - 1/B \leqq p \leqq 1/B$, and consider the two functions f_- and f_+ defined in (6.1) and (6.2), respectively. If α_p is the itinerary of the point p for the function f_-, and β_p is the itinerary of the point p for the function f_+, then (α_p, β_p) is an admissible pair. The proof of this fact appears in [3, Theorem 4.7]. Therefore $R^\bullet_{(\alpha_p,\beta_p,-)}$ and $R^\bullet_{(\alpha_p,\beta_p,-)}$ are binary radix systems.

For each B, however, there are infinitely many choices for p. Each choice of p leads to a distinct admissible pair (α_p, β_p) because the maps $p \mapsto \alpha_p$ and $p \mapsto \beta_p$ are increasing as a function of $p \in [1 - 1/B, 1/B]$. To verify that $p \mapsto \alpha_p$ is increasing (the proof for $p \mapsto \beta_p$ is similar), let $f_p = f_{(B,p,-)}$ and $\tau_p = \tau_{(B,p,-)}$. Assume $p' > p$ and let $x_n = f_p^n(p)$, $x'_n = f_{p'}^n(p')$ and let $\alpha = \alpha_p$, $\alpha' = \alpha_{p'}$. Note that $\alpha_p \neq \alpha_{p'}$; otherwise $|x'_n - x_n| = B^n|p' - p|$ for all n, which is not possible because $B > 1$. Therefore assume that $\alpha_k = \alpha'_k$ for $0 \leqq k \leqq n-1$, but $\alpha_n \neq \alpha'_n$. By elementary analytic geometry, if $\alpha_k = \alpha'_k = 1$, then $x_k - x_{k+1} > x'_n - x'_{k+1} > 0$, and if $\alpha_k = \alpha'_k = 0$, then $x'_{k+1} - x'_k > x_{k+1} - x_k > 0$ for any k. From this it is easy to deduce that if $\alpha_{n-1} = \alpha'_{n-1} = 1$, then $\alpha_n = 0$ and $\alpha'_n = 1$ and hence $\alpha < \alpha'$, and if $\alpha_{n-1} = \alpha'_{n-1} = 0$, then $\alpha_n = 0$ and $\alpha'_n = 1$ and again $\alpha < \alpha'$. $\square$

7. Proof of Theorem 5.4

For $\Gamma \subseteqq \Omega^\bullet$, denote $\widehat{\Gamma} = \{\widehat{\omega} : \omega \in \Gamma\}$. Let

$$\alpha = \sup\{\gamma \in \widehat{\Gamma} : \gamma_0 = 0\}, \quad \beta = \inf\{\gamma \in \widehat{\Gamma} : \gamma_0 = 1\}.$$

We first show that the pair (α, β) satisfies conditions (1) and (2) in Definition 3.3 of an allowable pair. It follows readily from the definition of α and β and from the shift invariance of Γ that $S^n\alpha \notin (\alpha, \beta)$ and $S^n\beta \notin (\alpha, \beta)$ for all $n \geqq 0$, which is close to, but not quite, condition (2) in Definition 3.3 of an allowable pair. Condition (1) in Definition 3.3 holds because, by the shift invariance of Γ, there are strings in Γ_0 that begin with 01 and strings in Γ_1 that begin with 10.

To show that (α, β) satisfies condition (2) in Definition 3.3, it only remains to prove that there is no $n \geqq 0$ such that $S^n\alpha = \beta$ and no $n \geqq 0$ such that $S^n\beta = \alpha$. The fact that there is no element of $\widehat{\Gamma}$ between α and β in the lexicographic order and that $\overset{\bullet}{\pi}c$ is increasing and surjective forces $\overset{\bullet}{\pi}({}_\bullet\alpha) = \overset{\bullet}{\pi}({}_\bullet\beta)$. Moreover, either ${}_\bullet\alpha \in \Gamma$ or ${}_\bullet\beta \in \Gamma$, but not both. We will assume that ${}_\bullet\alpha \in \Gamma$ and ${}_\bullet\beta \notin \Gamma$; the proof in the case that ${}_\bullet\beta \in \Gamma$ is essentially the same. There is no $n \geqq 0$ such that $S^n\alpha = \beta$; otherwise the fact that $\alpha \in \widehat{\Gamma}$ and the shift invariance of Γ (and hence the shift invariance of $\widehat{\Gamma}$) would imply that $\beta \in \widehat{\Gamma}$, a contradiction. Finally assume, by way of contradiction, that there is an $n \geqq 0$ such that $S^n\beta = \alpha$. Then $\beta = t\alpha \prec t\beta$, where t is a finite string. If there exists a $\gamma \in \widehat{\Gamma}$ such that $t\alpha = \beta \prec \gamma \prec t\beta$, then $\alpha = S^n(t\alpha) \prec S^n\gamma \prec S^n(t\beta) = \beta$. But by the definition of α and β as sup and inf, there can be no such $\gamma \in \widehat{\Gamma}$. Therefore there is no such γ with $\beta \prec \gamma \prec t\beta$, which contradicts the definition of β as $\inf\{\gamma \in \widehat{\Gamma} : \gamma_0 = 1\}$ since $\beta \notin \widehat{\Gamma}$.

The next claim is that $\Gamma = \Omega^\bullet_{(\alpha,\beta,-)}$. The definition of α and β implies that $\widehat{\Gamma} \subseteqq \Omega_{(\alpha,\beta,-)}$. That Γ is shift invariant further implies that $\widehat{\Gamma} \subseteqq \Omega^0_{(\alpha,\beta,-)}$, and therefore that $\Gamma \subseteqq \Omega^\bullet_{(\alpha,\beta,-)}$. To prove the equality we take a somewhat circuitous route. Let $b = 1/B$ and define $\pi_b : \Omega \to [0,1]$ by

$$\pi_b(\omega) := (1-b)\sum_{n=0}^{\infty} \omega_n\, b^n,$$

which is just the radix map in (2.2) except defined on all of Ω. Define

$$\Gamma' := \{t\gamma : \gamma \in \widehat{\Gamma},\ t \text{ a finite string of 1's including the empty string}\}.$$

We claim that $\pi_b : \Gamma' \to [0,1]$ is strictly increasing and surjective. To see that it is strictly increasing, note that, since $\overset{\bullet}{\pi}$ is strictly increasing on Γ, the map $\pi_b : \widehat{\Gamma} \to [0, q]$ is stringly increasing and surjective for some $0 < q \leqq 1$. Moreover, if $\omega \in \Gamma' \setminus \widehat{\Gamma}$ and $\gamma \in \widehat{\Gamma}$, then, by Definition 3.2 of the address spaces (recall that $\Gamma \subseteqq \Omega^{\bullet}_{(\alpha,\beta,-)}$), we have $0\gamma \preceq \alpha \prec 0\omega$. Therefore $\gamma \prec \omega$, i.e., every element of $\widehat{\Gamma}$ is less than every element of $\Gamma' \setminus \widehat{\Gamma}$. In addition, since $0\gamma \preceq \alpha$, we have $\pi_b(\gamma) \leqq (1/b)\, \pi_b(\alpha)$ by the fact that π is stringly increasing on Γ. And similarly, since $\alpha \prec 0\omega$, we have $(1/b)\, \pi_b(\alpha) < \omega$. Therefore $\pi_b(\gamma) < \pi_b(\omega)$. Finally, if σ and ω both lie in $\Gamma' \setminus \widehat{\Gamma}$ and $\sigma \prec \omega$, we will show that $\pi_b(\sigma) < \pi_b(\omega)$. Let $\sigma = t_1\gamma_1 \in \Gamma' \setminus \widehat{\Gamma}$, where t_1 is a string of m ones and $\gamma_1 \in \widehat{\Gamma}$, and $\omega = t_2\gamma_2 \in \Gamma' \setminus \widehat{\Gamma}$, where t_2 is a string of $n \geqq m$ ones and $\gamma_2 \in \widehat{\Gamma}$. Then $t\gamma_2 \succ \gamma_1$, where t is a string of $n - m$ ones. Therefore $\pi_b(t\gamma_2) \succ \pi_b(\gamma_1)$ and hence $\pi_b(\omega) \succ \pi_b(\sigma)$. Thus π_b is strictly increasing on Γ'.

To show that $\pi_b : \Gamma' \to [0, 1]$ is surjective, we express Γ' as the union of non-overlapping intervals and show that the images of these intervals under π_b leave no gaps. First note that the greatest element of $\widehat{\Gamma}$ is $S\alpha$; the smallest element of $1\widehat{\Gamma} \setminus \widehat{\Gamma}$ is $1S^2\alpha$ and the largest is $1S\alpha$; the smallest element of $11\widehat{\Gamma} \setminus (\widehat{\Gamma} \cup 1\widehat{\Gamma})$ is $11S^2\alpha$ and the largest is $11S\alpha$; etc. However, $\pi_b(1S^2\alpha) = \pi_b(S\alpha)$, and therefore $\pi_b(11S^2\alpha) = \pi_b(1S\alpha)$, etc. Hence $\pi_b : \Gamma' \to [0, 1]$ is surjective.

To conclude the proof that $\Gamma = \Omega^{\bullet}_{(\alpha,\beta,-)}$, call a map $\tau : [0, 1] \to \Omega$ such that $\pi_b \circ \tau$ is the identify a *section* of $\pi_b : \Omega \to [0, 1]$. If $\tau\big([0, 1]\big)$ is shift invariant, then τ is called a *shift invariant section*. In our case, the inverse of π_b restricted to Γ' is a shift invariant section. Call this section τ_b. A *mask* (in our case) is a partition of the interval $[0, 1]$ into two parts M_0 and M_1. Given a mask $M = \{M_0, M_1\}$, define a function

$$f_{(B,M)} = \begin{cases} Bx & \text{if } x \in M_0 \\ BX + (1 - B) & \text{if } x \in M_1. \end{cases}$$

The *itinerary map* $\tau_{(B,M)} : [0, 1] \to \Omega$ is defined by

$$\big[\tau_{(B,M)}(x)\big]_k = \begin{cases} 0 & \text{if } f^k_{(B,M)}(x) \in M_0 \\ 1 & \text{if } f^k_{(B,M)}(x) \in M_1. \end{cases}$$

To continue the proof, we use [4, Theorem 4] which states the following: the itinerary map $\tau_{(B,M)}$ is a section of π_b, and conversely, every shift invariant section of π_b is of the above form. Hence $\tau_b = \tau_{(B,M)}$ for some mask M. Because τ_b is increasing and because $0\sigma \prec 1\omega$ for any $\sigma, \omega \in \Omega$, the mask must be of the form $M_0 = [0, p]$, $M_1 = (p, 1]$ for some $p \in (0, 1)$. In particular, $\tau_b = \widehat{\tau}_{(B,p,-)}$ as in Definition 6.1 for some $p \in (0, 1)$. Let τ_- and τ_+ denote the itineraries $\widehat{\tau}_{(B,p,-)}(p)$ and $\widehat{\tau}_{(B,p,+)}(p)$, respectively, of the point p.

Then $\Omega_{(\tau_-,\tau_+,-)} = \tau_b\big([0,1]\big) = \Gamma' \subseteqq \Omega_{(\alpha,\beta,-)}$, the first equality by [2, Theorem 5.1]. If $\Gamma' \neq \Omega_{(\alpha,\beta,-)}$, then either $\tau_- \prec \alpha$ or $\tau_+ \succ \beta$. In either case there is a contradiction to the definition of α as a sup or β as an inf. Therefore $\alpha = \tau_-$, $\beta = \tau_+$ and $\Gamma' = \Omega_{(\alpha,\beta,-)}$, which implies that $\widehat{\Gamma} = \Omega^0_{(\alpha,\beta,-)}$, which in turn implies that $\Gamma = \Omega^\bullet_{(\alpha,\beta,-)}$.

To conclude the proof that (α,β) is allowable, we must prove condition (3) in Definition 3.3. A result of Parry [11, p. 373] on the topological entropy of a dynamical system on the unit interval, where the function is of the form $f_{(B,p,\pm)}$ shown in Fig. 1, provides the first equality in

$$0 < \log{(B)} = h\Big(\tau_b\big([0,1]\big)\Big) = h\big(\Omega_{(\tau_-,\tau_+,-)}\big) = h\big(\Omega_{(\alpha,\beta,-)}\big).$$

The first inequality is because $B > 1$. Since (α,β) is the pair of critical itineraries (τ_-,τ_+) of functions of the form in (6.1) and (6.2), and since the critical itineraries of such a function are an admissible pair [3, Theorem 4.7], we now know that (α,β) is an admissible pair.

To prove that $(\Gamma,B) = R^\bullet_{(\alpha,\beta,-)}$ it must be shown that

(1) $\Gamma = \Omega^\bullet_{(\alpha,\beta,-)}$, and

(2) $B = B_{(\alpha,\beta)} := 1/r_{(\alpha,\beta)}$.

We have already proved (1). Concerning (2), it was part of the proof of Theorem 6.4 that $\Omega_{(\alpha,\beta,-)} = \tau_{(B_{(\alpha,\beta)},\pi(\alpha),-)}\big([0,1]\big)$, and it was shown above that $\Omega_{(\tau_-,\tau_+,-)} = \widehat{\tau}_{(B,p,-)}$. Again using the result [11, p. 373] we have

$$\log{(B)} = h\Big(\widehat{\tau}_{(B,p,-)}\big([0,1]\big)\Big) = h\big(\Omega_{(\tau_-,\tau_+,-)}\big) = h\big(\Omega_{(\alpha,\beta,-)}\big)$$

$$= h\Big(\tau_{(B_{(\alpha,\beta)},\pi(\alpha),-)}\big([0,1]\big)\Big) = \log{\big(B_{(\alpha,\beta)}\big)}. \quad \square$$

8. Radix tilings of the real line

For an element $\omega \in \Omega^\bullet$, let $\omega_\bullet$ denote the finite substring of ω to the left of the decimal point.

Definition 8.1. Given a binary radix system (Γ,B) with radix map $\overset{\bullet}{\pi}$ and a finite string s, let

$$T'_s := \big\{ \overset{\bullet}{\pi}(\omega) : \omega \in \Gamma \text{ and } \omega_\bullet = s \big\}.$$

Note that, for many values of s, the set T'_s may be empty. For instance, in Example 2.6 the set $T'_{011} = \emptyset$. If $T'_s \neq \emptyset$, then the closure T_s of T'_s is a closed interval which we call a *tile*. If Ω_F denotes the set of all finite binary strings, let

$$\mathcal{T} := \{T_s : s \in \Omega_F\}.$$

Then $\mathcal{T}$ is a collection of non-overlapping intervals whose union is $\mathbb{R}^+$. The set $\mathcal{T}$ will be referred to as the *tiling* of $\mathbb{R}^+$ for the binary radix system (Γ, B). For an (α,β)-radix system, the corresponding tiling is denoted by $\mathcal{T}_{(\alpha,\beta)}$. The tiling for $R^{\bullet}_{(\alpha,\beta,-)}$ is the same as for $R^{\bullet}_{(\alpha,\beta,+)}$.

EXAMPLE 8.2 (standard radix system). For the standard binary radix system (Examples 2.5, 3.4, and 4.4), the tiling is the set $\mathcal{T} = \big\{ [n, n+1] : n \geqq 0 \big\}$ of unit length intervals.

EXAMPLE 8.3 (golden ratio radix system). For the golden ratio based radix system (Examples 2.6, 3.5, and 4.5), there are tiles of two lengths in the ratio $1 : \frac{1+\sqrt{5}}{2}$. There are tiles of length $1/\tau$ whose "fractional part" (the part to the right of the decimal point) ranges from ${\bullet}000\cdots$ to ${\bullet}010101\ldots$ and tiles of length 1 whose "fractional part" ranges from ${\bullet}000\cdots$ to ${\bullet}101010\ldots$, The tiling $\mathcal{T}$ is a well known non-periodic tiling of the line. If the tiles are denoted 1 and B (for their relative lengths), then the sequence of tiles in the tiling of $\mathbb{R}^+$, from left to right, is

$$B\,1\,B\,B\,1\,B\,1\,B\,B\,1\,B\,B\,1\,B\,1\,B\,B\,1\,B\,1\,B\,B\,1\cdots. \tag{8.1}$$

This tiling is self-replicating in the following sense. If $f_B : \mathbb{R}^+ \to \mathbb{R}^+$ is the function $f_B(x) = Bx$, then $B(\mathcal{T}) := \big\{ f_B(T) : T \in \mathcal{T} \big\}$ is also a tiling of $\mathbb{R}^+$, and each tile in $B(\mathcal{T})$ is the union of tiles in $\mathcal{T}$. More specifically, each tile of type B is an interval of the form $T_B := [s0{\bullet}\overline{0}, s0{\bullet}\overline{10}]$ for some finite string s, and $B(T_b) = [s00{\bullet}\overline{0}, s01{\bullet}\overline{01}] = [s00{\bullet}\overline{0}, s00{\bullet}\overline{10}] \cup [s01{\bullet}\overline{0}, s01{\bullet}\overline{01}]$, which is the union of a tile of type B and a tile of type 1. Likewise each tile of type 1 is an interval of the form $T_B := [s{\bullet}\overline{0}, s{\bullet}\overline{01}]$ for some finite string s, and $B(T_1) = [s0{\bullet}\overline{0}, s0{\bullet}\overline{10}]$ which is a tile of type B. Therefore, the sequence (8.1) above can be recursively generated, starting from B and using the substitution rules $B \leftarrow B1$, $1 \leftarrow B$. In other words, the tiling is recursively generated as follows:

$$B \to B1 \to B1B \to B1BB1 \to B1BB1B1B \to \cdots.$$

DEFINITION 8.4. With notation as in the example above, call a tiling $\mathcal{T}$ *self-replicating* if each tile in $B(\mathcal{T})$ is the union of tiles in $\mathcal{T}$.

THEOREM 8.5. *If (α,β) is an admissible pair, then*
(1) *the tiling $\mathcal{T}_{(\alpha,\beta)}$ is self-replicating, and*
(2) *if α and β are eventually periodic, then there are at most finitely many lengths of tiles in the tiling $\mathcal{T}_{(\alpha,\beta)}$.*

PROOF. Let (Γ, B) be the binary radix system with radix map $\overset{\bullet}{\pi}$. Concerning statement (1), consider the set $X = \{x_1, x_2, \ldots\}$ of right endpoints of the tiles in $\mathcal{T}_{(\alpha,\beta)}$. It suffices to prove that $Bx_n \in X$ for all $n \geqq 1$. For any $n \geqq 1$, the decimal representation of x_n must be of the form

$$\tau_{(\alpha,\beta,-)}(y) = v\alpha_0\alpha_1 \cdots \alpha_{N\bullet}\alpha_{N+1}\alpha_{N+2} \cdots$$

for some finite string v and some $N \geqq 0$. To see this, suppose that x_n is the right endpoint of the tile $T = \{ \overset{\bullet}{\pi}(\omega) : \omega_\bullet = s \}$ and note that s has the form $s = s'\alpha_0\alpha_1 \cdots \alpha_N$ for some finite string s' and some N. Therefore any point in T has a decimal representation of the form

$$s'\alpha_0\alpha_1 \cdots \alpha_{N\bullet}\omega_1\omega_2 \cdots \preceq s'\alpha_0\alpha_1 \cdots \alpha_{N\bullet}\alpha_{N+1}\alpha_{N+2} \cdots \in \Omega^\bullet_{(\alpha,\beta)}$$

for some string $\omega_1\omega_2 \cdots$. Conversely, any point with decimal representation $v\alpha_0\alpha_1 \cdots \alpha_{N\bullet}\alpha_{N+1}\alpha_{N+2} \cdots \in \Omega^\bullet_{(\alpha,\beta)}$ is a right endpoint of some tile in $\mathcal{T}_{(\alpha,\beta)}$ because any decimal $v\alpha_0\alpha_1 \cdots \alpha_{N\bullet}\omega_1\omega_2 \cdots \in \Omega^\bullet_{(\alpha,\beta)}$ is less than $v\alpha_0\alpha_1 \cdots \alpha_{N\bullet}\alpha_{N+1}\alpha_{N+2} \cdots$ in the lexicographic order. Therefore $Bx_n \in X$ since the decimal representation of Bx_n is obtained by shifting the decimal point one space: $v\alpha_0\alpha_1 \cdots \alpha_N\alpha_{N+1\bullet}\alpha_{N+2} \cdots$.

Concerning statement (2), from the form of the decimal representation of the endpoints of the intervals of the tiling (as discussed above), the lengths of the tiles have values $\overset{\bullet}{\pi}({}_\bullet S^m\alpha) - \overset{\bullet}{\pi}({}_\bullet S^n\beta)$ for some $n, m \geqq 0$. Since α and β are eventually periodic, there are at most finitely many shifted strings $S^m\alpha$ and $S^n\beta$. $\square$

Example 8.6. This is a continuation of Example 4.6 in which three distinct (α,β)-radix systems have the same base B such that $B^3 - B - 1 = 0$. The three associated self-replicating tilings, however, are distinct. From the self-replicating property, in a similar way as was done in Example 8.3, substitution rules for the three tilings can be determined.

In the notation of Example 4.6, for the tiling $\mathcal{T}_{(\alpha_1,\beta_1)}$ there are five tile types T_0, T_1, T_2, T_3, T_4 of relative lengths 1, B, B^2, B^3, B^4, respectively. To simplify notation we refer to tile T_i as simply i. Then the substitution rules are:

$$0 \leftarrow 1, \qquad 1 \leftarrow 2, \qquad 2 \leftarrow 3, \qquad 3 \leftarrow 4, \qquad 4 \leftarrow 40,$$

and the tiling begins, from left to right:

$$4\,0\,1\,2\,3\,4\,4\,0\,4\,0\,1\,4\,0\,1\,2\,4\,0\,1\,2\,3\,4\;0\,1\,2\,3\,4\,4\,0\,1\,2\,3\,4\,4\,0 \ldots.$$

For the tiling $\mathcal{T}_{(\alpha_2,\beta_2)}$ there are 4 tile types T_0, T_1, T_2, T_3 of relative lengths 1, B, B^2, B^6, respectively. The substitution rules are:

$$0 \leftarrow 1, \qquad 1 \leftarrow 2, \qquad 2 \leftarrow 10, \qquad 3 \leftarrow 32,$$

and the tiling begins, from left to right:

$$3\,2\,1\,0\,2\,1\,1\,0\,2\,2\,1\,1\,0\,1\,0\,2\,2\,1\,2\,1\,1\;0\,1\,0\,2\ldots.$$

For the tiling $\mathcal{T}_{(\alpha_3,\beta_3)}$ there are 4 tile types T_0, T_1, T_2, T_3 of relative lengths 1, B, B^2, B^5, respectively. The substitution rules are:

$$0 \leftarrow 1, \qquad 1 \leftarrow 2, \qquad 2 \leftarrow 01, \qquad 3 \leftarrow 31,$$

and the tiling begins, from left to right:

$$3\,1\,2\,0\,1\,1\,2\,2\,0\,1\,0\,1\,1\,2\,1\,2\,2\,0\,1\,2\,0\;1\,0\,1\,1\,2\ldots.$$

All three of the above tilings are self-replicating under expansion by the factor B.

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