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# SCHEDULING PERIODIC EVENTS

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Graph theoretic methods are used to analyze a problem concerning periodically recurring events that has applications to transportation efficiency. Bounds on the optimum, results on complexity and an algorithm for the solution are obtained.

## 1. Introduction

This paper concerns the scheduling of periodically recurring events. The problem is to maximize the minimal distance between consecutive events. Consider, for example, the following problem concerning the efficient scheduling of trains.

**Problem 1.1.** Trains depart from a Central Station to $n$ destinations. For destination $i$, $1 \le i \le n$, consecutive trains depart every $m_i$ minutes where $m_i$ is integral. At what times should the departures of the trains be scheduled so that the minimum time interval between consecutive departing trains is maximized? (The departure times need not be integral.)

**Example 1.2.** Consider the case $n=3$ and suppose that trains leave every 6, 4 and 3 hours, respectively, for the three destinations. Hence $m_1=6$, $m_2=4$ and $m_3=3$. Schedule the departures of train 1 at times $0, 6, 12, \ldots$; train 2 at times $\frac{1}{2}, \frac{9}{2}, \frac{17}{2}, \ldots$; and train 3 at times $1, 4, 7, \ldots$. Then the minimum time interval between consecutive departing trains is $\frac{1}{2}$ hour. This cannot be improved.

Problem 1.1 is solved explicitly by Burkard [3] for the case $n=2$ and for arbitrary $n$, but with at most two distinct values $m_i$. Burkard attributes the problem to Guldan [5], who, in turn, attributes it to Černý [4]. In [4, 5] the problem is stated as follows: How should regular $p_i$-gons, $i=1,2,\ldots,n$, be inscribed in a circle so as to maximize the distance between the closest vertices on the circle. In [3] the max-min polygon problem is generalized to minimizing the objective function

$$\sum_{i=1}^{k} y_i^p,$$

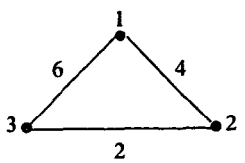
where $k$ is the total number of vertices on the circle; $y_i$ denotes the distance between

consecutive vertices on the circle; and $p$ is a fixed real parameter with $-\infty \leq p \leq 0$ or $1 \leq p \leq \infty$. The original problem, that of maximizing $\min_{1 \leq i \leq k} y_i$, is the special case $p \to -\infty$. In [1] this problem is further generalized to polygons that are not necessary regular. In that paper Brucker, Burkard and Hurink obtain an algorithm based on a decomposition of the set of all schedules into local regions in which the optimization problem is convex. This is analogous to Guldan's original algorithm [5] which characterizes local regions by an acyclic graph and solves the local problems by longest path computations. Our approach to Problem 1.1 is somewhat different.

Consider Problem 1.3 below which involves a notion similar to the cocycle of a graph. For a natural number $m$ and real $x$, let $|x|_m$ denote the distance from $x$ to the closest multiple of $m$. This acts like an absolute value modulo $m$. For example $|2|_5 = 2 = |3|_5$.

**Problem 1.3.** Let $(V, E)$ be the complete graph with an edge labeling $c: E \to \mathbb{N}$ by natural numbers. For any vertex labeling $g: V \to \mathbb{N}$, let $\delta g: E \to \mathbb{N}$ be defined as follows: if $e = \{i, j\}$, then $\delta g(e) = |g(i) - g(j)|_{c(e)}$. Find $\max_g \min_{e \in E} \delta g(e)$, where the maximum is taken over all vertex labelings $g$. Also find a function $g$ that realizes this maximum.

**Example 1.4.** For the graph




the maximum is 1, and the function $g(1) = 0$, $g(2) = 1$, $g(3) = 2$ realizes this maximum. It is a consequence of results in Section 2 that this solution to Problem 1.3 yields the solution to Problem 1.1 in Example 1.2.

For $n = 2$ or 3, explicit solutions to Problem 1.1 are easy to obtain (Lemma 2.6 and Theorem 3.2). In general, however, it is not even obvious that Problem 1.1 is discrete. On the other hand, Problem 1.3 is finite because the values of the function $g$ can be assumed, without loss of generality, less than the least common multiple of $\{c(e) \mid e \in E\}$. It is shown in Section 2 that Problem 1.1 is also finite (Theorem 2.5) and that it can be reformulated in the form of Problem 1.3 (Theorem 2.7). This yields an algorithm to solve Problem 1.1. As opposed to the algorithms of Guldan and of Brucker, Burkard and Hunrink mentioned above, the integrality of the optimum in Problem 1.3 allows for a straightforward search. We show in Section 4 (Theorem 4.1) that Problem 1.1 is NP-complete with respect to input $n$. Therefore the best that can be expected is a reasonable algorithm for small values of $n$. For a fixed $n$, our algorithm is polynomial in $m = \max m_i$. General upper and lower

bounds are given in Section 3 for the optimum in Problem 1.1, some in terms of graph chromatic number.

## 2. Schedules and graphs

Assume that events $E_1, \dots, E_n$ occur periodically with periods $m_1, \dots, m_n$, respectively. Let $x_i$ denote any occurrence time of $E_i$. Since $E_i$ recurs at intervals $m_i$, it may be assumed that $0 \le x_i < m_i$ for all $i$. All the other occurrences of $E_i$ are uniquely determined by $x_i$ and are given by $x_i + km_i$, $k$ an integer. Therefore an $n$-tuple of times $x = (x_1, \dots, x_n)$, where $x_i$ is any occurrence time of $E_i$, determines all occurrences of all events. In Problem 1.1, $x_i$ is any departure time of a train to destination $i$, and $x$ determines the complete schedule of trains. Let $d_{ij}(x, y)$ denote the least distance between an occurrence of event $E_i$ and an occurrence of event $E_j$ for the pair $x = (x, y)$. Then Problem 1.1 asks for $\max_x \min_{1 \le i < j \le n} d_{ij}(x_i, y_j)$, where the maximum is over all possible $n$-tuples $x = (x_1, \dots, x_n)$. The notation gcd and lcm will be used for greatest common divisor and least common multiple.

**Lemma 2.1.** *With notation as above* $d_{ij}(x, y) = |x - y|_{m_{ij}}$, *where* $m_{ij} = \gcd(m_i, m_j)$.

**Proof.** Without loss of generality, take $i, j$ to be 1, 2, respectively. The possible intervals between times $x_1 + k_1 m_1$ of event $E_1$ and $x_2 + k_2 m_2$ of event $E_2$ are of the form $(x_1 - x_2) + (k_1 m_1 - k_2 m_2)$, $k_1, k_2$ integers, which is the same as $(x_1 - x_2) + \gcd(m_1, m_2)k$, $k$ integral. □

**Problem 2.2** (Problem 1.1 reformulated). Find $\max_x \min_{1 \le i < j \le n} d_{ij}(x_i, x_j)$ and a point $x = (x_1, \dots, x_n)$ that attains this maximum.

That this maximum is actually achieved can be seen as follows: Define a function $f(x_1, \dots, x_n) = \min_{1 \le i < j \le n} d_{ij}(x_i, x_j)$. Now $d_{ij}(x, y)$ is periodic of period $\gcd(m_i, m_j)$ in each variable; hence $f$ is periodic of period $m_i$ in the $i$th coordinate. So $f$ can be regarded as a function from $T_n = S^1 \times S^1 \times \cdots \times S^1$, the product of $n$ copies of the 1-sphere, to the reals $\mathbb{R}$. It is clear that $f: T_n \to \mathbb{R}$ is a continuous function on a compact set. Therefore there exists at least one point $x$ that maximizes $f$.

The value $\max_x \min_{1 \le i < j \le n} d_{ij}(x_i, x_j)$ will be denoted $M(m_1, \dots, m_n)$ and will be called the *optimum* for Problem 2.2. A point $x = x(m_1, \dots, m_n)$ that attains this optimum will be called an *optimum point*. An optimum point corresponds to an optimum schedule in the example of train scheduling. Usually the range of max and min will be clear and, in this case, they are omitted.

For any point $x = (x_1, \dots, x_n)$, define a directed graph $G(x) = (V, E)$ as follows: $V = \{1, 2, \dots, n\}$ and there is an arc $(i, j) \in E$, directed from vertex $i$ to vertex $j$, if

(1) $d_{ij}(x_i, x_j) = \min d_{ij}(x_i, x_j)$,
(2) for $x_i < y < x_i + d_{ij}(x_i, x_j)$ we have $d_{ij}(y, x_j) < d_{ij}(x_i, x_j)$.

In terms of scheduling trains, condition (1) says that the minimum interval between consecutive trains is realized between trains to destinations $i$ and $j$. Condition (2) says that slightly increasing the times of trains to destination $i$ causes the minimal interval between consecutive trains for destination $i$ and $j$ to decrease. Note that condition (1), by itself, implies the existence of an arc joining vertex $i$ and vertex $j$. Condition (2) then determines the direction of this arc.

**Lemma 2.3.** *For any point $x$ there is a point $y$ such that $G(y)$ is a connected graph and* $\min d_{ij}(y_i, y_j) = \min d_{ij}(x_i, x_j)$.

**Proof.** Denote $f(x) = \min d_{ij}(x_i, x_j)$. Proceed by induction on the number of components of $G(x)$. Assume $G(x)$ is disconnected and let $G_0$ be a connected component of $G$ with vertex set $V_0$. Then for any $j \in V_0$ and $k \in V - V_0$ it holds that $d_{jk}(x_j, x_k) > f(x)$. Uniformly increase all values of $x_i$, $i \in V_0$, until the first occurrence of $d_{jk}(x_j, x_k) = f(x)$ for some $j \in V_0$ and $k \in V - V_0$. This eventually occurs because the values of the $d_{ij}(x_i, x_j)$ remain unchanged for $i, j$ in $V_0$, and eventually $d_{jk}(x_j, x_k)$ decreases to $f(x)$ for some $j \in V_0$ and $k \in V - V_0$. This introduces an arc joining vertex $j$ to $k$, therefore decreasing the number of components of $G$. □

**Lemma 2.4.** *If $x$ is an optimum point, then $G(x)$ contains a directed cycle.*

**Proof.** Denote $f(x) = \min d_{ij}(x_i, x_j)$. Assume $G$ contains no directed cycle. Then there exists a vertex $i_1$ with outdegree$(i_1) = 0$. Removing $i_1$ and repeating this argument, gives an ordering of the vertices $(i_1, \ldots, i_n)$ such that, for all $j$, outdegree$(i_j) = 0$ in the subgraph induced by vertices $\{i_j, \ldots, i_n\}$. In other words, there is a set of rooted trees which span $G$ and with all edges directed away from the respective roots. Starting at the leaves and working toward the roots, slightly increase the value of the $x_i$. More precisely let $y_{i_j} = x_{i_j} + \varepsilon/j$. Then for sufficiently small $\varepsilon > 0$, we have $d_{ij}(y_i, y_j) > d_{ij}(x_i, x_j)$ for all arcs $(i, j)$ in $G$, and $d_{ij}(y_i, y_j) > f(x)$ for all $\{i, j\}$ in the complement of $G$. But this implies that $f(y_1, \ldots, y_n) > f(x_1, \ldots, x_n)$, contradicting the maximality of $x$. □

Theorem 2.5 below implies that Problem 2.2 can be solved by a finite search. In particular, it says that there exists an optimum point whose coordinates are rational numbers with bounded numerators and denominators.

**Theorem 2.5.** *There exists an integer $\alpha$, $1 \le \alpha \le n$, such that the optimum $M(m_1, \ldots, m_n)$ and an optimum point $x = (x_1, \ldots, x_n)$ are of the form*

$$M(m_1, \ldots, m_n) = b \gcd(m_1, \ldots, m_n)/\alpha,$$

$$x_i = a_i/\alpha,$$

*where the $a_i$ and $b$ are integers and $0 \le a_i < \alpha m_i$ for all $i$.*

**Proof.** Denote $f(x)=\min d_{ij}(x_i,x_j)$ and let $x$ be a maximum of $f$. Then by Lemma 2.4, $G(x)$ contains a directed cycle. Also by Lemma 2.3 we may assume, without loss of generality, that $G(x)$ is connected. Each arc $(i,j)$ in $G(x)$ implies, by Lemma 2.1, the existence of an integer $q$ such that

$$x_j=x_i+f(x)+q\gcd(m_i,m_j). \tag{1}$$

Let $i_1,\dots,i_\alpha$ be a directed cycle in $G$. Then apply (1) consecutively to each arc of this cycle and sum these equalities to obtain $\alpha f(x)=r\gcd(m_{i_1},\dots,m_{i_\alpha})$ or

$$f(x)=r\gcd(m_{i_1},\dots,m_{i_\alpha})/\alpha \tag{2}$$

for some integer $r$. This proves the first statement in the theorem.

There is no loss of generality in taking $x_1=0$. Let $j_1,j_2,\dots,j_p$ be a path (not necessarily directed) from vertex $1=j_1$ to vertex $k=j_p$. Then for $1\le s<p$,

$$x_{j_{s+1}}=x_{j_s}+q\gcd(m_{j_s},m_{j_{s+1}})+f(x) \quad \text{for some integer } q.$$

Use this formula successively for each vertex along the path and sum to obtain

$$x_k=a\gcd(m_{j_1},\dots,m_{j_p})+bf(x) \tag{3}$$

for some integers $a$ and $b$. From equations (2) and (3) it follows that each $x_k$ is of the form $a_i/\alpha$, where $a_i$ is an integer. The bound $0\le a_i<\alpha m_i$ follows because we may assume $0\le x_i<m_i$. □

Theorem 2.7 below shows that a solution to scheduling Problem 1.1 or 2.2 can be expressed in terms of solutions to the Graph Problem 1.3 in the introduction. A lemma, which solves Problem 2.2 in the case $n=2$, is needed.

**Lemma 2.6.** *$M(m_1,m_2)=\frac{1}{2}\gcd(m_1,m_2)$ and $x(m_1,m_2)=(0,\frac{1}{2}\gcd(m_1,m_2))$.*

**Proof.** Lemma 2.1 implies that $M(m_1,m_2)\le\frac{1}{2}\gcd(m_1,m_2)$. We get equality by considering the particular point $x$ given in the statement of the lemma. □

Consider the complete graph $(V,E)$ on vertex set $V=\{1,2,\dots,n\}$ with edge labeling $c:E\to\mathbb{N}$. With notation as in Problem 1.3, let $M(c)=\max_g\min_{\{i,j\}}\delta g(i,j)$ denote a solution to Problem 1.3 for the edge labeling $c$. The relation between the solution $M(m_1,\dots,m_n)$ to the Scheduling Problem 2.2 and solutions $M(c)$ to the Graph Problem 1.3 is stated in Theorem 2.7.

**Theorem 2.7.** *Given integers $m_i$, $1\le i\le n$, let $c_\alpha(i,j)=\alpha\gcd(m_i,m_j)$. Then $M(m_1,\dots,m_n)=\max_{1\le\alpha\le n}(1/\alpha)M(c_\alpha)$.*

**Proof.** Let $m_{ij}=\gcd(m_i,m_j)$. For any integer $\alpha$,

$$M(c_\alpha)=\max_g \min_{\{i,j\}} |g(i)-g(j)|_{\alpha m_{ij}}$$
$$\le \max_x \min_{\{i,j\}} |x_i-x_j|_{\alpha m_{ij}}=M(\alpha m_1,\dots,\alpha m_n)=\alpha M(m_1,\dots,m_n).$$

The inequality above results because, given $g$, there is a point $x$ on $T_n$ defined by $x_i=g(i)$, $i=1,2,\dots,n$. Therefore $M(m_1,\dots,m_n)\ge\max(1/\alpha)M(c_\alpha)$. To prove equality, let $(x_1,\dots,x_n)=x(m_1,\dots,m_n)$ be an optimum point and let $\alpha$ be an appropriate integer as guaranteed by Theorem 2.5. Then $\alpha x(m_1,\dots,m_n)=x(\alpha m_1,\dots,\alpha m_n)$. Define a function $g: V\to\mathbb{N}$ by $g(i)=\alpha x_i$. Then

$$\alpha M(m_1,\dots,m_n)=M(\alpha m_1,\dots,\alpha m_n)$$
$$=\min|\alpha x_i-\alpha x_j|_{\alpha m_{ij}}=\min|g(i)-g(j)|_{\alpha m_{ij}}\le M(c_\alpha).$$

Thus $M(m_1,\dots,m_n)\le(1/\alpha)M(c_\alpha)$. □

**Remark 2.8.** In computing $M(c_\alpha)=\max_g\min_{\{i,j\}}\delta g(i,j)$ it may be assumed that $0\le g(i)<\alpha m_i$ for all $i$. This is because $g(i)$ is considered only modulo $\gcd(\alpha m_i,\alpha m_j)$ for values of $j\ne i$. Thus $g(i)$ can be considered modulo $\operatorname{lcm}_j \alpha\gcd(m_i,m_j)$, which divides $\alpha m_i$.

**Example 2.9.** Consider the case $n=5$ and $m_1=10$, $m_2=21$, $m_3=22$, $m_4=35$ and $m_5=33$. Using Theorem 2.7, the relevant graph for Problem 1.3 is in Fig. 1. It is

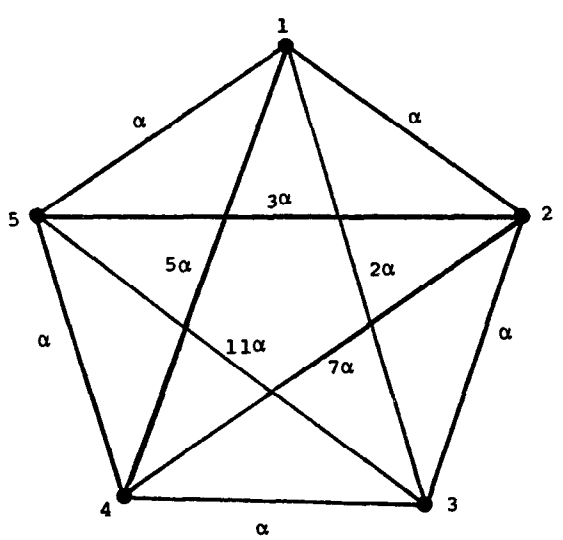



Fig. 1. Graph for optimization problem.

easy to check that $M(c_1)=0$, $M(c_2)=0$, $M(c_3)=1$, $M(c_4)=1$, $M(c_5)=2$. Hence $M(10,21,22,35,33)=\max(1/\alpha)M(c_\alpha)=M(c_5)=\frac{2}{5}$, and the optimum is attained at the point $(0,\frac{2}{5},\frac{4}{5},\frac{6}{5},\frac{8}{5})$.

## 3. Bounds

It is not difficult to solve Problem 2.2 explicitly for $n=2$ and 3. Lemma 2.6 does it for $n=2$; Theorem 3.2 below does it for $n=3$. Theorem 3.2 is also stated, but not proved, in [3].

**Lemma 3.1.** *If the $m_i$ are pairwise relatively prime, then $M(m_1,\dots,m_n)=1/n$ and this optimum is attained at the point $x=(0,1/n,2/n,\dots,(n-1)/n)$.*

**Proof.** With the notation as in Theorem 2.7, $c(i,j)=\alpha$ for all $i,j$ because the $m_i$ are relatively prime in pairs. If $\alpha<n$, then there exist $i,j$ such that $g(i)\equiv g(j)(\mathrm{mod}\,\alpha)$, which implies that $\delta g(i,j)=0$ and $M(c_\alpha)=0$. If $\alpha=n$, then again $\delta g(i,j)=0$ for some $i,j$ unless the $g(i)$ are the distinct integers modulo $n$, in which case $M(c_\alpha)=1$. By Theorem 2.7, $M(m_1,\dots,m_n)=\max(1/\alpha)M(c_\alpha)=1/n$. □

**Theorem 3.2.** *Denote $m_i'=m_i/\gcd(m_1,m_2,m_3)$, $i=1,2,3$. Then*

$$M(m_1,m_2,m_3)=\begin{cases}\frac{1}{3}\gcd(m_1,m_2,m_3), & \textit{if the } m_i' \textit{ are pairwise relatively prime,}\\ \frac{1}{2}\min\limits_{1\le i<j\le 3}\gcd(m_i,m_j), & \textit{otherwise.}\end{cases}$$

**Proof.** The first part follows, after scaling by $\gcd(m_1,m_2,m_3)$, from Lemma 3.1. Let $(m_i,m_j)=\gcd(m_i,m_j)$. For the second part, $\frac{1}{2}\min(m_i,m_j)$ is an upper bound for $M(m_1,m_2,m_3)$ by Lemma 2.6. To see that this upper bound can be achieved, assume, without loss of generality, that $\frac{1}{2}(m_1,m_2)$ is the minimum. Since $((m_1',m_3'),(m_2',m_3'))=(m_1',m_2',m_3')=1$, let $s$ and $t$ be integers such that $s(m_1',m_3')-t(m_2',m_3')=\lceil\frac{1}{2}(m_1',m_2')\rceil$. Let $r=2t(m_2',m_3')+2(m_1',m_2')+\varepsilon$ where $\varepsilon$ is 0 or 1, depending on whether $(m_1',m_2')$ is even or odd, respectively. Then use Lemma 2.1 to check that $x=(0,\frac{1}{2}(m_1,m_2),\frac{1}{2}r(m_1,m_2,m_3))$ realizes the upper bound except in the case $(m_1,m_2)=(m_2,m_3)$. If this occurs, then $(m_1,m_2)\mid(m_1,m_3)$, but $(m_1,m_3)\neq(m_1,m_2)$ because $(m_1',m_2',m_3')=1$. In this case $x=(0,\frac{1}{2}(m_1,m_2),(m_1,m_2))$ realizes the upper bound. □

Computing explicit formulas becomes complicated for $n\ge 4$. In the remainder of this section general bounds are obtained for the optimum solution to Problem 2.2. The first result follows directly from previous results. The second result is an improvement on the upper bound.

**Corollary 3.3.**

$$(1/n)\gcd(m_1,\dots,m_n)\le M(m_1,\dots,m_n)\le \tfrac{1}{2}\min_{1\le i<j\le n}\gcd(m_i,m_j).$$

**Proof.** The lower bound is immediate from Theorem 2.5. Also, for any point $x=(x_1,\dots,x_n)$ and any $1\le i<j\le n$, $M(m_1,\dots,m_n)\le M(m_i,m_j)$. The upper bound now follows from Lemma 2.6. □

For Example 2.9, Corollary 3.3 gives bounds $\frac{1}{5}\le M(10,21,22,35,33)\le\frac{1}{2}$. Recall that the actual optimum is $\frac{2}{5}$. The estimate in Corollary 3.3 is tight in that both bounds are attained. For example, it is easy to check that $m_1=m_2=\cdots=m_n=1$ gives the lower bound with $x=(1/n,2/n,3/n,\dots,1)$, and $m_1$, $m_2=m_3=\cdots=m_n$, $\gcd(m_1,m_2)=1$, $m_2\ge n-1$, gives the upper bound with $x=(\frac{1}{2},1,2,\dots,n-1)$. However the difference between the upper and lower bound may be large.

An improvement in the upper bound in Corollary 3.3 can be obtained by using the chromatic number. Let $K_n$ be the complete graph on $n$ vertices where edge $\{i,j\}$ is labeled $\gcd(m_i,m_j)$. For an integer $k$, let $G_k$ be the subgraph of $K_n$ consisting of only those edges of $K_n$ labeled $k$. Note that if $m=\max m_i$, then $1\le k\le m$. Further, let $\chi$ denote the chromatic number of a graph, i.e. the minimum number of colors needed to properly color the vertices.

**Theorem 3.4.** *Given natural numbers $m_1,\dots,m_n$ and $m=\max m_i$*

$$M(m_1,\dots,m_n)<\min_{1\le k\le m}\frac{k}{\chi(G_k)-1}.$$

**Proof.** Given a subgraph $H$ of $K_n$, consider the generalization of Problems 2.2 and 1.3 where the minima are restricted to only those edges in $H$. Thus $M_H(m_1,\dots,m_n)$ denotes the solution to Problem 2.2 where only the intervals between those pairs of trains corresponding to edges in $H$ are considered. Similarly $M_H(c)=\max_g\min_{\{i,j\}\in H}\delta g(i,j)$ denotes the solution to Problem 1.3 for $H$. Then the arguments of Lemmas 2.3 and 2.4 and Theorems 2.5 and 2.7 all carry through without change to show that $M_H(m_1,\dots,m_n)=(1/\alpha)\max M_H(c_\alpha)$. Hence for all $k$,

$$\begin{aligned}M(m_1,\dots,m_n)&\le M_{G_k}(m_1,\dots,m_n)=\max(1/\alpha)M_{G_k}(c_\alpha)\\&=k\max(1/\alpha)M_{G_k}(c'_\alpha),\end{aligned}$$

where $c_\alpha(i,j)=k\alpha$ and $c'_\alpha(i,j)=\alpha$, respectively. Now it is sufficient to show that

$$(1/\alpha)M_{G_k}(c'_\alpha)<1/(\chi(G_k)-1)\quad\text{for all }\alpha.$$

By way of contradiction, assume that $M_{G_k}(c'_\alpha)\ge\alpha/(\chi(G_k)-1)$ for some $\alpha$. Then there exists a function $g:V\to\mathbb{N}$ such that $|g(i)-g(j)|_\alpha\ge\alpha/(\chi(G_k)-1)$ for all edges $\{i,j\}$ in $G_k$. By reducing the values of the function $g$ modulo $\alpha$, there is no loss

of generality in assuming that $1 \le g(i) \le \alpha$ for all $i$. Now define $g': V \to \mathbb{N}$ by $g'(i) = \lceil g(i)[(\chi(G_k)-1)/\alpha] \rceil$. Then $1 \le g'(i) \le \chi(G_k)-1$. Also if $q = (\chi(G_i)-1)/\alpha$, then

$$|g'(i)-g'(j)| = |\lceil qg(i) \rceil - \lceil qg(j) \rceil| > |qg(i)-qg(j)| - 1$$
$$\ge q|g(i)-g(j)|_\alpha - 1 \ge 0.$$

So $g'(i) \ne g'(j)$ for all eges $\{i,j\}$ in $G_k$. Therefore $g'$ is a $\chi(G_k)-1$ coloring of $G_k$, a contradiction. □

**Example 3.5.** Consider Example 2.9. In this case $G_1 = C_5$, $G_2 = G_3 = G_5 = G_7 = G_{11} = K_2$, where $C_5$ is the 5-cycle and $K_2$ is the complete graph on two vertices. Hence $M(10,21,22,33,35) < \min_k k/(\chi(G_k)-1) = 1/(3-1) = \frac{1}{2}$. It is known by Theorem 2.5 that $M(10,21,22,33,35)$ is rational with denominator $\le 5$. The largest such rational elss than $\frac{1}{2}$ is $\frac{2}{5}$. Therefore $M(10,21,22,33,35) \le \frac{2}{5}$. Recall that the actual optimum is $\frac{2}{5}$. The upper bound $\frac{1}{2}$ of Corollary 3.3 is not as accurate in this example.

## 4. Complexity

In this section it is proved that Problem 2.2 is NP-complete with respect to input $n$ (the number of trains). Then an algorithm, based on Theorem 2.7, is given that is fairly efficient for small values of $n$. In proving NP-completeness, the two relevant problems are stated in their decision form, MAX-MIN and GRAPH COLORING. The proof exhibits a polynomial transformation from MAX-MIN to GRAPH COLORING.

**MAX-MIN.** Given natural numbers $m_1, \ldots, m_n$ and a real number $\beta$, is it true that $M(m_1, \ldots, m_n) \ge \beta$?

**GRAPH COLORING.** Given a graph on $n$ vertices and an integer $k$, does there exist a proper vertex coloring with $\le k$ colors?

**Theorem 4.1.** *MAX-MIN is NP-complete.*

**Proof.** Let $G$ be an arbitrary graph on $n$ vertices. An instance of MAX-MIN is constructed as follows. Choose natural numbers $m_1, \ldots, m_n$ such that $\gcd(m_i, m_j) = 1$ if vertices $i$ and $j$ of $G$ are adjacent and $\gcd(m_i, m_j) \ge 2n$ otherwise. This can be done by considering the maximal cliques $C_1, \ldots, C_s$ in the complement of $G$. Let $q_i$, $1 \le i \le s$, be integers $\ge 2n$ and relatively prime in pairs. For vertex $i$ let $A_i = \{j \mid i \in C_j\}$ and let $m_i = \prod_{j \in A_i} q_j$.

It is now sufficient to show that $M(m_1, \ldots, m_n) \ge 1/k$ if and only if $\chi(G) \le k$. Assume that $G$ has at least one edge. (The case where $G$ has $n$ isolated vertices is

an easy exercise left to the reader.) First it is claimed that $\max_x \min_{\{i,j\}} d_{ij}(x_i, x_j) = \max_x \min_{\{i,j\}\in G} d_{ij}(x_i, x_j)$. Clearly the left-hand side is less than or equal to the right. To prove equality consider a point $(x_1, \ldots, x_n)$ realizing the maximum on the right-hand side. Since $\gcd(m_i, m_j) = 1$ for $(i, j) \in G$, it may be assumed, by reducing modulo 1, that $0 \le x_i < 1$ for all $i$. Now let $y_i = x_i + 2(i-1)$. The point $(y_1, \ldots, y_n)$ has the following properties:

(1) $1 < |y_i - y_j| < 2n - 1$;
(2) $d_{ij}(y_i, y_j) = d_{ij}(x_i, x_j)$ for all $i$ and $j$ adjacent in $G$;
(3) $d_{ij}(x_i, x_j) < 1 \le d_{ij}(y_i, y_j)$ for all $i$ and $j$ not adjacent in $G$.

The last inequality in (3) results from (1) and the fact that $\gcd(m_i, m_j) \ge 2n$ on edges in the complement of $G$. This proves the claim. Now assume that $G$ has a proper $k$ coloring $c: V \to \{0, 1, 2, \ldots, k-1\}$ and consider the point $(y_1, \ldots, y_n)$ with $y_i = c(i)/k$. We have

$$M(m_1, \ldots, m_n) = \max_x \min_{\{i,j\}} d_{ij}(x_i, x_j) = \max_x \min_{\{i,j\}\in G} d_{ij}(x_i, x_j)$$
$$\ge \min_{\{i,j\}\in G} d_{ij}(y_i, y_j) = \min_{\{i,j\}\in G} |(c(i) - c(j))/k|_1 \ge 1/k.$$

Conversely assume there is a point $x = (x_1, \ldots, x_n)$ such that $\min d_{ij}(x_i, x_j) \ge 1/k$. Let $y_i \equiv x_i (\mathrm{mod}\ 1)$ with $0 \le y_i < 1$ and let $c(i) = \lfloor k y_i \rfloor$. Note that $|x_i - x_j| \ge 1/k$ implies $|y_i - y_j| \ge 1/k$ for all $i, j$ adjacent in $G$, because $\gcd(m_i, m_j) = 1$. Therefore $|k y_i - k y_j| \ge 1$, which in turn implies that $c(i) \ne c(j)$ if $i$ and $j$ are adjacent in $G$. Since $0 \le c(i) < k$ for all $i$, the $c(i)$ constitute a proper coloring of $G$ with $\le k$ colors. □

**Remark 4.2.** Theorem 4.1 shows that Problem 2.2 is NP-complete, but not that it is strongly NP-complete. In other words, MAX-MIN was not limited to those instances where the largest integer appearing in the problem is bounded by a polynomial in $n$. For the instance of MAX-MIN constructed in the proof of Theorem 4.1, the size of the largest $m_i$ is, in general, exponential in $n$. This leaves open the possibility of a pseudo-polynomial algorithm [6], which solves Problem 2.2 in time bounded by a polynomial in $n$ and $m$. We believe that Problem 2.2 is strongly NP-complete, and a proof would be of interest.

Theorem 2.7 naturally suggests an algorithm for solving Problem 2.2. According to Theorem 2.7 and Remark 2.8:

$$M(m_1, \ldots, m_n) = \max_{1 \le \alpha \le n} (1/\alpha) M(c_\alpha), \tag{4}$$

where

$$M(c_\alpha) = \max_x f_\alpha(x). \tag{5}$$

Here the maximum is over all points $x = (x_1, \ldots, x_n)$ with $0 \le x_i < \alpha m_i$ and

$$f_\alpha(x) = \min_{1 \le i < j \le n} |x_i - x_j|_{\alpha m_{ij}}, \tag{6}$$

where $m_{ij}=\gcd(m_i, m_j)$. Let $m=\max\{m_i\}$. In seeking a solution, it can be assumed, without loss of generality, that $x_0=0$. To compute $M(m_1,\dots,m_n)$ by (4), there are $n$ values of $\alpha$ for which to compute $M(c_\alpha)$. To compute $M(c_\alpha)$ for a particular $\alpha$ by (5), there are at most $(\alpha m)^{n-1}$ points $x$ for which to compute $f_\alpha(x)$. To calculate $f_\alpha(x)$ for a particular $\alpha$ and $x$ by (6), there are at most $\binom{n}{2}$ pairs of integers, $i, j$, to try. Therefore at most

$$\begin{aligned}\binom{n}{2}\sum_\alpha (\alpha m)^{n-1} &< n^2 m^{n-1}\sum_\alpha \alpha^{n-1} = n^{n+1}m^{n-1}\sum_\alpha (\alpha/n)^{n-1}\\ &\le n^{n+1}m^{n-1}\int_0^1 n\left(x+\frac{1}{n}\right)^{n-1}\mathrm{d}x \le n^{n+1}m^{n-1}\left(1+\frac{1}{n}\right)^n\\ &= O(m^{n-1}n^{n+1})\end{aligned}$$

time is required to solve Problem 2.2 by a complete search. If $n$ is considered constant, then the complexity of the scheduling Problem 2.2 is polynomial in $m$. Of course, the constant and the degree of the polynomial may be large. On the other hand, it was shown in Theorem 4.1 that Problem 2.2, as a function of input $n$, is NP-complete.

**Remark 4.3.** (1) The Euclidean algorithm computes all greatest common divisors $m_{ij}$ in $O(n^2 \log m)$ time, where $m=\max m_i$. This is subsumed in the complexity estimate $O(n^{n+1}m^{n-1})$.

(2) In formula (4) it is not necessary to check every value of $\alpha$ between 1 and $n$. If the optimum occurs for a value of $\alpha$ between 1 and $\lfloor \frac{1}{2}n \rfloor$, then it also occurs, by scaling, for a value of $\alpha$ between $\lceil \frac{1}{2}n \rceil$ and $n$. Therefore, only values of $\alpha$ between $\lceil \frac{1}{2}n \rceil$ and $n$ need be tried. This, however, improves only the constant in the complexity estimate $O(n^{n+1}m^{n-1})$.

(3) The optimum can be estimated to within $1/n$ by checking only the value $\alpha=n$ in formula (4). More precisely, it is always the case that

$$\frac{M(c_n)}{n} \le M(m_1,\dots,m_n) < \frac{M(c_n)+1}{n}.$$

The first inequality is clear and the second is a consequence of Theorem 4.4 below. In Example 2.9, $n=5$ and $\frac{1}{5}M(c_5)=\frac{2}{5}$. Therefore $\frac{2}{5}\le M<\frac{3}{5}$. No fraction with denominator 1, 3 or 4 lies within these bounds, and the only fraction with denominator 2 that does is $\frac{1}{2}$. Therefore $M=\frac{1}{2}$ or $\frac{2}{5}$. (Recall from Section 2 that the optimum is $\frac{2}{5}$.)

**Theorem 4.4.** *For any integer $\alpha$, $M(m_1,\dots,m_n)<(M(c_\alpha)+1)/\alpha$.*

**Proof.** Let $x=(x_1,\dots,x_n)$ be an optimum point. Define $g: V\to \mathbb{N}$ by $g(i)=\lfloor \alpha x_i \rfloor$, $1\le i\le n$. Let $m_{ij}=\gcd(m_i,m_j)$. If $c(i,j)=\alpha m_{ij}$, then $\delta g(i,j)=|g(i)-g(j)|_{\alpha m_{ij}}$. We

claim that $\delta g(i,j) > \alpha d_{ij}(x_i, x_j) - 1$ for all $i, j$. By Lemma 2.1 it is sufficient to show that

$$\alpha|x_i - x_j|_{m_{ij}} - |\lfloor \alpha x_i \rfloor - \lfloor \alpha x_j \rfloor|_{\alpha m_{ij}} = |\alpha x_i - \alpha x_j|_{\alpha m_{ij}} - |\lfloor \alpha x_i \rfloor - \lfloor \alpha x_j \rfloor|_{\alpha m_{ij}} < 1.$$

But the stronger statement $|a-b| - |\lfloor a \rfloor - \lfloor b \rfloor| < 1$ actually holds for the ordinary absolute value and any real numbers $a$ and $b$. The claim implies that $M(c_\alpha) > \alpha M(m_1, \ldots, m_n) - 1$. □

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