Periodicity, Quasiperiodicity, and Bieberbach's Theorem on Crystallographic Groups

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1. INTRODUCTION. This article contains an elementary proof of a fundamental geometric theorem of Bieberbach. Moreover, it affords the opportunity to digress onto subjects that motivated the proof—periodicity and quasiperiodicity. The proof is in Sections 5 and 6. Most of the article consists of observations on isometries of Euclidean space (Section 2), crystallographic groups (Section 3), and the role of Bieberbach's theorem in the theory of crystals and quasicrystals (Section 4).

A *crystallographic group* is a discrete, cocompact group of isometries of n-dimensional Euclidean space. All terms in this definition are explained in Section 3. For now, it suffices to say that the two-dimensional crystallographic groups, often called wallpaper groups, are familiar as symmetry groups of tilings of the plane, and the three-dimensional groups arise as symmetry groups of crystals. There are exactly 2 one-dimensional, 17 two-dimensional, and 230 three-dimensional crystallographic groups. In dimension four there are 4,783 crystallographic groups [2]; this enumeration relys heavily on the computer. The exact number in higher dimensions is unknown. The eighteenth of Hilbert's famous problems posed at the 1900 International Congress of Mathematicians asks, in part, whether the number of crystallographic groups is finite in all dimensions. An affirmative answer was provided by Bieberbach [1] in papers that appeared in 1911 and 1912. The two- and three-dimensional crystallographic groups were first classified in the 1890's by Fedorov [7] and, independently, by Schoenflies [14]. The classification of the three-dimensional crystallographic groups can be found in many texts on mathematical crystallography, but these texts usually assume the following result. This same result is the main step in Bieberbach's solution of Hilbert's eighteenth problem.

Theorem 1 (Bieberbach). *If G is an n-dimensional crystallographic group, then G contains translations in n linearly independent directions.*

Bieberbach's proof of Theorem 1 [1] depends on a nontrivial number theoretic result concerning the approximation of irrational numbers by rationals. More recent treatments by Wolf [17] and Charlap [5], also somewhat technical, are based on a proof of Frobenius [8] that appeared shortly after Bieberbach's proof. A shorter proof by P. Buser [4] was a result of his study of Gromov's work on almost flat manifolds. Gromov, in turn, has stated that his work on almost flat manifolds resulted from an attempt to understand the Bieberbach theorem [5]. Our proof is intended to be accessible to anyone with a basic undergraduate knowledge of abstract and linear algebra.

The concept that plays the central role in the proof is what we call the *axis* of an isometry g, the largest subspace of $\mathbb{R}^n$ on which g acts as a pure translation. A main step in the proof, a result also proved by Buser [4], is an analog in $\mathbb{R}^n$ of the well known Crystallographic Restriction in $\mathbb{R}^2$ and $\mathbb{R}^3$.

2. ISOMETRIES. An *isometry* is a mapping of $\mathbb{R}^n$ onto itself that preserves distance. The following representation of an isometry is well known and very easy to prove given the fact that an isometry with a fixed point is an orthogonal transformation. Given any point $p \in \mathbb{R}^n$, an isometry g can be expressed as the composition of an orthogonal transformation A, centered at p, and a translation:

$$g(x) = Ax + a. \tag{2.1}$$

The orthogonal map A will be referred to as the *rotational part* and translation by a the *translational part* of g. The rotational part is, up to conjugacy, independent of the point p. The main result in this section is a refinement of (2.1), obtained by making an appropriate choice of the origin p.

Lemma 1. *Let g be an isometry of $\mathbb{R}^n$. There exists a unique affine subspace F satisfying the following properties: (a) g is a translation when restricted to F, and (b) F is maximal with respect to property (a). Moreover, if the origin is chosen to lie in F, then*

$$g(x) = Qx + q,$$

where Q is orthogonal, F is the set of fixed points of Q, and $q \in F$.

The subspace F of Lemma 1 will be called the *axis* of g and denoted *axis*(g). As an application of Lemma 1, consider any isometry g of $\mathbb{R}^3$. If *axis*$(g) = \mathbb{R}^3$, then, according to Lemma 1, g is a translation. If *axis*(g) is a plane π, then g is the composition of a reflection through π and a translation in a direction along π. Such an isometry is called a *glide reflection* (or a *reflection* if the translation is the identity). If *axis*(g) is a line l, then g is the composition of a non-identity rotation and a translation along l. Such an isometry is called a *screw displacement* (or a *rotation* if the translation is the identity). Finally, if *axis*(g) is a point p, then g is an orthogonal transformation having only p as fixed point. Such an orthogonal transformation has canonical form

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which is the composition of a non-identity rotation about a line l with a reflection in a plane perpendicular to l. Such an isometry is called a *rotary reflection.* Thus Lemma 1 provides the following classification: every 3-dimensional isometry is a translation, rotation, reflection, glide, screw or rotary reflection.

Proof of Lemma. As in (2.1), write $g(x) = Ax + a$. Let V be the subspace of fixed points of A and $V^{\perp}$ the orthogonal complement of V. Note that both V and $V^{\perp}$ are invariant under A. Let q and $q^{\perp}$ be the components of a in the subspaces V and $V^{\perp}$, respectively. Since $I - A$ is nonsingular when restricted to $V^{\perp}$, the affine subspace $F = (I-A)^{-1}q^{\perp}$ is not empty. For $x \in F$ we have $Ax = x - q^{\perp}$, which implies that $g(x) = Ax + a = x + (a - q^{\perp}) = x + q \in F$. Therefore g is a translation when restricted to F. Define $Qx = Ax + q^{\perp}$. Then F is the set of fixed points of Q; Q is orthogonal because it has a fixed point; and $g(x) = Qx + q$. We leave to the reader the routine exercise of showing that F is unique, i.e., that there does not exist even a one dimensional subspace, not contained in F, upon which G acts as a translation. ∎

3. CRYSTALLOGRAPHIC GROUPS, DELAUNAY SETS AND VORONOI TILINGS. An n-dimensional *crystallographic group* G is a discrete, cocompact subgroup of isometries of $\mathbb{R}^n$. *Discrete* means that any ball contains at most finitely many points in the G-orbit of any point. *Cocompact* means that the quotient space $\mathbb{R}^n/G$ is compact, where the quotient is the set of orbits with the quotient topology. A less abstract, but equivalent, definition of crystallographic group is more appropriate for our purpose. A set X of points of $\mathbb{R}^n$ is called an (r, R)-*Delaunay set*, or simply Delaunay set, if

(1) X is *discrete*: there is a number r such that every ball of radius r centered at a point of X contains no other points of X.
(2) X is *uniform*: there is a number R such that every ball of radius R contains a point of X.

Let G be a group of isometries of $\mathbb{R}^n$ and p any point of $\mathbb{R}^n$. Then G is a crystallographic group if and only if the orbit of p is a Delaunay set. This can be restated in terms of Voronoi tilings as follows. Let P be the orbit of any point under the action of a group G of isometries of $\mathbb{R}^n$. For any $p \in P$, let D_p denote the *Voronoi region* of p. This is the set of points at least as close to p as to any other point of P:

$$D_p = \{x \in \mathbb{R}^n : |x - p| \le |x - y| \text{ for all } y \in P\}.$$

The Voronoi region D_p is the intersection of half space determined by the perpendicular bisectors of the line segments joining p to each of the other points of P. The group G is a crystallographic group if and only if each Voronoi region $\{D_p | p \in P\}$ is a bounded convex polytope. In particular, the Voronoi regions of any orbit of a crystallographic group tile $\mathbb{R}^n$; all the tiles are congruent; G acts transitively on these tiles; and the action of $g \in G$ on a single tile completely determines g.

This definition makes it easy to prove a first approximation to Bieberbach's theorem, a result Buser **[4]** calls Mini-Bieberbach. It states that an n-dimensional crystallographic group must contain n isometries that are nearly translations, in the sense that the translational parts are linearly independent and the rotational parts are close to the identity. As a measure of the proximity of the rotational part of an isometry g to the identity, define

$$rot(g) = \max_{x \in \mathbb{R}^n} \frac{|Ax - x|}{|x|},$$

where A is the rotational part of g.

Lemma 2 (Mini-Bieberbach). *Let G be an n-dimensional crystallographic group. Given any point p and any $\varepsilon > 0$, there exist in G elements $g_i(x) = Ax + a_i$, $i = 1, 2, \dots, n$, centered at p, such that*

(1) $rot(g_i) < \varepsilon$ *for all i* *and* (2) $\{a_1, a_2, \dots, a_n\}$ *is linearly independent.*

Proof: Consider the Voronoi tiling with respect to the orbit of the point p. Further, let b be an arbitrary direction and consider the sequence $\{D_i\}$ of tiles that intersect the ray with endpoint p and direction b. Let $g_i(x) = A_i x + a_i$ be an element of G with rotational part A_i centered at p and such that g_i takes D_0 to D_i, where D_0 is the tile centered at p. Because the orthogonal group is compact, $\{g_i\}$ has a convergent subsequence. For us this means that there must exist two isometries g_j and g_k with the properties: (1) A_j and A_k are sufficiently close in the sense that $rot(g_k \circ g_j^{-1}) < \varepsilon$, and (2) $|a_j - a_k|$ is sufficiently large so that the angle between b and the vector from $g_j(p)$ to $g_k(p)$ is less than ε. Then the element $g = g_k \circ g_j^{-1}$ satisfies statement (1) in the lemma. The lemma follows by repeating this argument where, at the k^{th} stage, b is chosen orthogonal to the subspace spanned by $a_1, a_2, \dots, a_{k-1}$. ■

4. CRYSTALS AND QUASICRYSTALS. With the atoms and molecules of a real crystal in mind, define a *crystal* as the image of a finite number of points of $\mathbb{R}^n$ under a group generated by n linearly independent translations. The *symmetry group*, $\mathbf{sym}(X)$, of a crystal X is the group of isometries that leave the points of X as a whole invariant. A crystal clearly has the following properties: X is discrete; X is *periodic*, which means that $\mathbf{sym}(X)$ contains translations in n linearly independent directions; X is the union of finitely many lattices, a *lattice* being the image of a single point under a group generated by n linearly independent translations.

The notions, crystal and crystallographic group, are intimately related, as described in Theorem 2. Although it would be surprising if it were otherwise, this theorem is illustrative for a couple of reasons. First, it is another consequence of Bieberbach's theorem. Second, the proof uses two essential ingredients in crystallographic analysis, the translation group and the point group. Let g be an element of a crystallographic group G and let $p \in \mathbb{R}^n$. Consider the representation (2.1): $g(x) = Ax + a$, with respect to p. The mapping ϕ: $g \mapsto A$ induces a homomorphism of G into the orthogonal group. The kernel of ϕ is the *translation subgroup* T of G; the image of ϕ is the *point group* at p.

Theorem 2. *A set X of points in Euclidean space is a crystal if and only if X is discrete and* $\mathbf{sym}(X)$ *is a crystallographic group.*

Proof: Assume that X is a crystal, G its symmetry group, and T the subgroup of G generated by the n independent translations that define X. If $p \in X$ then $G(p)$, the orbit of p, is the union of finitely many lattices since $G(p)$ is invariant under T. Therefore, $G(p)$ is a Delaunay set, so G is a crystallographic group.

In the other direction, let G be the symmetry group of X, and let T be the translation subgroup of G, which, by Bieberbach's theorem, is generated by n independent translations. Let $p \in X$ and let L be the lattice that is the image of p under the action of T. From the fact that T is normal in G, it is easy to show that L is invariant under the action of the point group Φ at p and that, for any $A \in \Phi$, A is completely determined by its action on finitely many points of L. Then Φ, being a group of permutations of these points, is finite. Being isomorphic to Φ, the quotient G/T is also finite. Express $G = \bigcup_i Tg_i$ as the disjoint union of finitely many cosets of T. Denoting by D_p the Voronoi region at p, there are finitely many points in $X_p = X \cap D_p$ because X is discrete. Now we have $X = G(X_p) = (\bigcup_i Tg_i)(X_p) = T[\bigcup_i g_i(X_p)]$. Therefore X is the image of finitely many points under the action of the translation subgroup T. By Theorem 1, T contains translations in n linearly independent directions, so, by definition, X is a crystal. ■

It was actually quasicrystals, rather than crystals, that drew our attention to Bieberbach's result. A decade ago Shechtman, Blech, Gratias, and Cahn **[16]** discovered the first "quasicrystal," an alloy of aluminum and managanese whose electron diffraction pattern consisted of sharp spots exhibiting a 5-fold symmetry. This elicited great excitement in solid-state science for the following reason. A distinct diffraction pattern with sharp spots, called Bragg peaks, is evidence of "long range order," which, until that time, meant a crystal structure. On the other hand, the well known *Crystallographic Restriction* states that the only rotational symmetry possible for a crystal in two or three dimensions is 2, 3, 4, or 6-fold symmetry. In other words, a crystal structure and the observed 5-fold symmetry are incompatible. Since this original discovery, various similar materials (aluminum-lithium-copper, uranium-palladium-silicon, and other compositions) have been discovered and analyzed, and the consensus among solid-state scientists is that these materials cannot be explained within the framework of a periodic structure, that they are truly new. The "long range order" in quasicrystals, whatever is causing the Bragg peaks in the electron diffraction, is often referred to as "quasiperiodicity."

In any study of quasiperiodicity, a minimum that should be required of a set X of points is that X be a Delaunay set. However, this alone implies little about global order, an example being the molecules of a gas in a closed container. Senechal and Taylor **[15]** inquire about the consequences of requiring the following additional local congruence property. For $x \in X$ and real number ρ, let $N_\rho(x)$ denote the intersection of X with the ball of radius ρ centered at x.

Property $\mathbf{N}_\rho$: For any two points $x, y \in X$, the neighborhoods $N_\rho(x)$ and $N_\rho(y)$ are congruent by a congruence taking x to y.

Unfortunately, as Senechal and Taylor point out, a theory based on the local regularity Property $\mathbf{N}_\rho$ will not be interesting because it already implies that X is a crystal.

Theorem 3. *Let X be an (r, R)-Delaunay set in $\mathbb{R}^n$. There exists a number ρ, depending only on r, R, and n, such that if property $\mathbf{N}_\rho$ holds, then X is a crystal.*

Theorem 3 is again a consequence of Bieberbach's theorem. The proof of Theorem 3 is in two parts. First, in 1976 Delaunay and his colleagues **[6]** gave an elegant proof that, under the conditions of Theorem 3, the symmetry group G of X acts transitively. Since the orbit X of G is a Delaunay set, G is a crystallographic group. Theorem 3 now follows directly from Theorem 2, which, in turn, was a consequence of Bieberbach's theorem.

Theorem 3 implies that any investigation into quasiperiodicity requires ideas more subtle than the local homogeneity given by property $\mathbf{N}_\rho$. Advances in this direction have been made by Penrose **[12]**, de Bruijn **[3]**, Kramer and Neri **[10]**, Katz and Duneau **[9]**, Mozes **[11]**, Radin **[13]**, and many others, but these results lie outside the scope of this note.

5. CONJUGACY IN A CRYSTALLOGRAPHIC GROUP. Very informally, to say that two isometries of Euclidean space are *conjugate* means that they do the same thing, but in different places. In $\mathbb{R}^3$, for example, the conjugate kgk^{-1} of a $(\pi/2)$-rotation g about a line l is a $(\pi/2)$-rotation about the image line $k(l)$. Lemma 3 is a more formal statement. The notation is as follows. Let g be an isometry; use Lemma 1 to express it in the form $g(x) = Qx + q$, where Q is orthogonal and $q \in axis(g)$. Define $trans(g) = q$, where $trans(g)$ is considered as a free vector so, in statement (2) of Lemma 3, k maps both the initial and terminal point of the vector.

Lemma 3. *If g and k are isometries and $h = kgk^{-1}$ then*

$$(1)\ axis(h) = k(axis(g)) \qquad (3)\ rot(h) = rot(g)$$
$$(2)\ trans(h) = k(trans(g)) \qquad (4)\ rot(hg^{-1}) \le 2rot(k)rot(g).$$

Proof: The first three statements are routine to verify. The following proof of statement (4) is due to Buser **[4]**. Let A and B be the orthogonal parts of g and k, respectively, centered at the same point. Then $BAB^{-1}A^{1} - I = ((B - I)(A - I) - (A - I)(B - I))B^{-1}A^{-1}$ and, since $|B^{-1}A^{-1}x| = |x|$, it follows that

$$rot(kgk^{-1}g^{-1}) = rot(BAB^{-1}A^{-1}) \le 2rot(B)rot(A) = 2rot(k)rot(g). \qquad \blacksquare$$

Although somewhat technical, the next lemma is essential to the proof of Bieberbach's Theorem. A rough sketch of how it comes into play is as follows. Let G be a crystallographic group. Mini-Bieberbach (Lemma 2) implies the existence of n isometries with translational parts in independent directions and with rotational parts that are close to the identity. To prove Bieberbach's theorem it remains to show only that each such isometry g must necessarily be a translation. Assume the contrary, that g is not a translation. Under this assumption, a certain set C of conjugates of g, each distinct from g, is not empty. Lemma 4 is used to prove that $axis(\bar{g})$ and $axis(g)$ are not too close to each other if $\bar{g} \in C$. So among the isometries in C, let h have axis closest to the axis of g. Then it can be shown that $axis(hgh^{-1})$ is even closer to $axis(g)$ than is $axis(h)$, a contradiction if $hgh^{-1} \in C$. Lemma 4 is required again to show that $hgh^{-1} \in C$. The complete proof appears in Section 6.

As apparent from this outline, the minimum distance between the axes of two isometries g and h plays a crucial role. We use the notation

$$d(g,h) = \min\{|x - y| : x \in axis(g),\, y \in axis(h)\}.$$

Lemma 4. *If g is an element of a crystallographic group, then there exist positive numbers δ and c with the following property. Let $h = kgk^{-1}$ be a conjugate of g. If $rot(k) < \delta$ and either*

$$(1)\quad d(g,h) \le c \qquad \text{or} \qquad (2)\quad hgh^{-1} = g,$$

then $h = g$.

Proof: Let p and $\bar{p}$ be closest points on $axis(g)$ and $axis(h)$, respectively, and consider the Voronoi tiling with respect to the orbit of p. Choose c small enough so that, if $d(g, h) \le c$, then $\bar{p}$ lies in the interior of tile D_p. Choose $\delta < \sqrt{2}$ and, in addition, small enough so that both of the following conditions are satisfied.

(1) If $\bar{p}$ lies in the interior of tile D_p and $rot(k) < \delta$, then $g(D_p) \cap h(D_p) \neq \emptyset$. This is possible due to statement (2) of Lemma 3.
(2) If $f(D_p) = D_p$ and $rot(f) < 4\delta$, then f must act as the identity on D_p. This is possible because D_p is a bounded polytope with finite symmetry group.

Now assume that $rot(k) < \delta$ and $d(g, h) \le c$. By statement (1) we have $g(D_p) \cap h(D_p) \neq \emptyset$, which implies that $g(D_p) = h(D_p)$ and $g^{-1}h(D_p) = D_p$. By parts (2) and (4) of Lemma 3, $rot(g^{-1}h) = rot(hg^{-1}) \leq 2rot(k)rot(g) \leq 4rot(k) \leq 4\delta$. So by condition (2), with $f = g^{-1}h$, the isometry $g^{-1}h$ acts as the identity on D_p. Since an element of G is determined by its action on D_p, we have $h = g$.

Next assume that $g_0 := hgh^{-1} = g$. We claim that $d(g,h) = 0$, in which case $h = g$ follows from what has already been proved. To prove the claim, express $h(x) = Qx + q$ as in Lemma 1, where Q is orthogonal and $q \in V := axis(h)$. Taking the center of Q as the origin, let $W + a$ be the axis of g, where W is a linear subspace of $\mathbb{R}^n$. Using statement (1) of Lemma 3, $Q(W) + Qa + q = h(W + a) = h(axis(g)) = axis(g_0) = axis(g) = W + a$. This implies both (a) $Q(W) = W$ and (b) $(I - Q)a \in q + W$. But $V = axis(h) = k(axis(g)) = k(a + W)$, which, by the same reasoning as above, implies (c) $V = A(W)$, where A is the rotational part of k centered at the origin. We next prove, by contradiction, that $W = V$. Since subspaces V and W have the same dimension, assume that there exists a $w \in W \setminus V$. Let $w = v + v^{\perp}$, where $v \in V$ and $v^{\perp} \in V^{\perp}$ and let $x = w - Qw$. Then $x \in W$ because of statement (a), and $x \in V^{\perp}$ because $x = (v + v^{\perp}) - (Qv + Qv^{\perp}) = v^{\perp} - Qv^{\perp} \in V^{\perp}$. Hence, by statement (c), we know that A takes the element x of $V^{\perp}$ to an element of V. This contradicts $rot(k) < \sqrt{2}$. Now $W = V$ and, from statement (b), we have $(I - Q)a \in V$, which implies that $a \in V$ because $I - Q$ leaves both V and $V^{\perp}$ invariant and is non-singular when restricted to $V^{\perp}$. Hence $axis(g) = W + a = V + a = V = axis(h)$. ∎

6. A CRYSTALLOGRAPHIC RESTRICTION AND THE PROOF OF BIEBERBACH'S THEOREM. Theorem 4 is an analog in $\mathbb{R}^n$ of the Crystallographic Restriction discussed in Section 4. In particular, if X is a crystal, then Theorems 2 and 4 eliminate the possibility of X possessing a k-fold rotation about a codimension two axis if $k \geq 13$. Also notice that Bieberbach's Theorem is an immediate corollary of Theorem 4 because the existence of translations in n linearly independent directions is guaranteed by Lemma 2.

Theorem 4. *If g is any non-identity element of a crystallographic group such that $rot(g) \leq 1/2$, then g must be a translation.*

Proof: By way of contradiction, assume that g is not a translation. Let δ and c be as in Lemma 4, and let $\varepsilon = \min(\delta, c/(4|trans(g)|))$. Consider the set C consisting of all conjugates $\bar{g} = kgk^{-1}$ of g in G such that

$$(1)\quad \bar{g} \neq g \qquad \text{and} \qquad (2)\quad rot(k) < \varepsilon.$$

The set C is not empty for the following reason. Since g is not a translation, $axis(g) \neq \mathbb{R}^n$. Lemma 2, with point p on $axis(g)$, then guarantees the existence of an isometry $k \in G$ such that $rot(k) < \varepsilon$ and the translational part of k does not lie in $axis(g)$. The latter condition implies that $axis(\bar{g})$ and $axis(g)$ are distinct because, by Lemma 3, $axis(\bar{g}) = k(axis(g))$. Since $axis(\bar{g}) \neq axis(g)$, also $\bar{g} \neq g$.

Let

$$d = \inf_{\bar{g} \in C} d(g, \bar{g}) > c > 0,$$

the inequalities resulting directly from Lemma 4. The contradiction that will finish the proof is the existence of a $g_0 \in C$ such that $d(g, g_0) < d$. Let $h \in C$ be such that

$$d(g,h) \leq \tfrac{5}{4}d.$$

Then $g_0 = hgh^{-1}$ is such an element. It remains to show only that $g_0 \in C$ and that $d(g, g_0) < d$.

We first show that $g_0 \in C$. Because $h \in C$, we have $h \neq g$ and $h = kgk^{-1}$, where $rot(k) < \varepsilon \leq \delta$. Therefore $g_0 \neq g$ by Lemma 4. To verify the second condition in the definition of C, we show that there exists a $\bar{k} \in G$ such that $g_0 = \bar{k}g\bar{k}^{-1}$, where $rot(\bar{k}) < \varepsilon$. Since $g_0 = hgh^{-1} = (hg^{-1})g(hg^{-1})^{-1}$, statement (4) of Lemma 3 implies that $rot(hg^{-1}) \leq 2rot(g)rot(k) \leq rot(k) < \varepsilon$. Hence take $\bar{k} = hg^{-1}$.

To show that $d(g, g_0) < d$, let $V = axis(g)$ and $V' = axis(h)$. Let $p \in V$ and $p' \in V'$ be closest points on V and V', respectively. Further, let $\bar{V}$ denote the image of V under the translational part of h, and let $\bar{p} \in \bar{V}$ be a closest point to p on $\bar{V}$. If $trans(h) = 0$ then $\bar{p} = p$. Otherwise, since $h \in C$ express $h = kgk^{-1}$, and let α be the angle between $trans(g)$ and $trans(h)$. By elementary trigonometry we have $\sin(\alpha) \leq rot(k)$. Condition (2) in the definition of C and statement (2) of Lemma 3 yield

$$|p - \bar{p}| \leq |trans(h)|\sin(\alpha) \leq |trans(g)|rot(k) < \tfrac{1}{4}c \leq \tfrac{1}{4}d$$

$$|\bar{p} - p'| \leq |\bar{p} - p| + |p - p'| < \tfrac{1}{4}d + d(g,h) \leq \tfrac{3}{2}d.$$

If p'' is the image of $\bar{p}$ under the rotational part of h then, using statement (3) of Lemma 3,

$$|\bar{p} - p''| \leq |\bar{p} - p'|rot(h) < \tfrac{3}{2}d\, rot(g) \leq \tfrac{3}{4}d.$$

But $p \in axis(g)$ and p'', being in the image of $axis(g)$ under h, lies in $axis(g_0)$. Therefore

$$d(g, g_0) \leq |p - p''| \leq |p - \bar{p}| + |\bar{p} - p''| < \tfrac{1}{4}d + \tfrac{3}{4}d = d. \qquad \blacksquare$$

REFERENCES

1. L. Bieberbach, Über die Bewegungsgruppen der Euklidischen Raume, I, *Math. Ann.* 70 (1911), 297–336; II, Math. Ann. 72 (1912), 400–412.
2. H. Brown, R. Bülow, J. Neubüser, H. Wondratschek, and H. Zassenhaus, *Crystallographic Groups of Four-Dimensional Space*, Wiley, New York, 1978.
3. N. G. de Bruijn, Algebraic theory of Penrose's non-periodic tilings of the plane I, II, *Proc. Konink. Ned. Akad. Wetensch* A84 (1981), 39–66.
4. P. Buser, A geometric proof of Bieberbach's theorems on crystallographic groups, *L'Ensignment Mathématique* 31 (1985), 137–145.
5. L. S. Charlap, *Bieberbach Groups, and Flat Manifolds*, Springer-Verlag, New York, 1986.
6. B. Delone, N. Dolbilin, M. Shtogrin, and R. Galiulin, A local criterion for the regularity of a system of points, *Soviet Math. Dokl.* 17 (1976), 319–322.
7. E. S. Fedorov, Symmetry in the plane, *Zapiski Rus. Mineralog. Obščestva, Ser. 2* 28 (1891), 345–390.
8. C. Frobenius, Über die unzerlegbaren diskreten Bewegengsgruppen, *Sitzungsber. Akad. Wiss. Berlin* 29 (1911), 654–665.
9. A. Katz and M. Duneau, Quasiperiodic patterns and icosahedral symmetry, *J. Physique* 47 (1986), 181–196.
10. P. Kramer and R. Neri, On periodic and non-periodic space fillers of E^m obtained by projection, *Acta Cryst.* A40, 580–587.
11. S. Mozes, Tilings, substitution systems and dynamical systems generated by them, *J. Analyse Math.* 53 (1989), 139–186.
12. R. Penrose, The role of aesthetics in pure and applied mathematics, *Bull. Inst. Math. Applications* 10 (1974), 266–271.

13. C. Radin, Symmetry of tilings of the plane, *Bull. Amer. Math. Soc.* 29 (1993), 213–217.
14. A. Schoenflies, *Kristallsysteme und Kristallstruktur*, Teubner, Leipzig, 1981.
15. M. Senechal and J. Taylor, Quasicrystals: the view from Les Houches, *Math. Intelligencer* 12 (1990), 54–64.
16. D. Shechtman, I. Blech, D. Gratias, and J. Cahn, Metallic phase with long-range orientational order and no translational symmetry, *Phys. Rev. Lett.* 53 (1984), 1951–1954.
17. J. Wolf, *Spaces of Constant Curvature*, McGraw-Hill, New York, 1967.

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Note on Gauss-Bonnet

Several geometers have pointed out that I have not given due credit to Allendoefer and Weil in my recent article on the Gauss-Bonnet Theorem, All the Way with Gauss-Bonnet and the Sociology of Mathematics, *Monthly* 103 (1996), 457–469

As I described on p. 464, Hopf asked for an intrinsic proof and generalization of the even dimensional case of his Satz VI, the topological Gauss-Bonnet theorem. In 1940, Allendoerfer and Fenchel independently found the correct formula, which today is called the Gauss-Bonnet-Chern theorem, but only for Riemannian submanifolds in Euclidean space. However, in 1943, Allendoerfer and Andre Weil proved that in fact it held for all Riemannian manifolds. This is clearly stated in Theorem I on p. 101 of C. B. Allendoerfer and Andre Weil, *Trans. Amer. Math. Soc.* 53 (1943), 101–129. This is is the complete Gauss-Bonnet-Chern theorem. But their proof involved local embeddings, and hence was not intrinsic.

Later, Weil told Chern about the lack of an intrinsic proof for the Gauss-Bonnet-Chern Theorem. Chern found his famous proof in short order, and published it in the *Annals* in 1944 with a title that accurately reflected his contribution: A Simple Intrinsic Proof of the Generalized Gauss-Bonnet Theorem. The Nash embedding theorem (every Riemannian manifold can be found as a submanifold of Euclidean space) was proved in the 1950's. When combined with the Allendoerfer-Fenchel result of 1940, Nash's theorem renders Allendoerfer-Weil's step superfluous. Chen's proof and Nash's theorem probably helped create the current widespread misconception about the credit due to Allendoerfer and Weil.

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