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GRAPHIC MATROIDS, SHELLABILITY AND THE POINCARE CONJECTURE

ABSTRACT. In this paper we introduce a theory of edge shelling of graphs. Whereas the standard notion of shelling a simplicial complex involves a sequential removal of maximal simplexes, edge shelling involves a sequential removal of the edges of a graph. A necessary and sufficient condition for edge shellability is given in the case of 3-colored graphs, and it is conjectured that the result holds in general. Questions about shelling, and the dual notion of closure, are motivated by topological problems. The connection between graph theory and topology is by way of a complex ΔG associated with a graph G. In particular, every closed 2- or 3-manifold can be realized in this way. If ΔG is shellable, then G is edge shellable, but not conversely. Nevertheless, the condition that G is edge shellable is strong enough to imply that a manifold ΔG must be a sphere. This leads to completely graph-theoretic generalizations of the classical Poincaré Conjecture.

1. INTRODUCTION

Research involving the interplay between combinatorics and topology can enrich both fields. Some recent contributions are cited among the references. The purpose of this paper is to investigate certain questions concerning the edge set of a graph. These questions are motivated by problems about low-dimensional manifolds, in particular the classical Poincaré Conjecture.

The graphs G in this paper are endowed with a cover, i.e. a set of subgraphs covering the edges of G. The connection with topology is by way of a complex ΔG associated with the graph G. The complex ΔG is essentially the nerve of the cover. Examples of graphs with cover can be obtained naturally from edge-colored graphs, which are discussed in Section 5. These examples are important because any closed 2- or 3-manifold can be realized as ΔG for some colored graph G. A combinatorial fundamental group $\pi(G)$ is defined for the graph G in Section 2. This generalizes the edge-path construction of the fundamental group of a simplicial complex. It is shown that the combinatorial fundamental group of G is isomorphic to the topological fundamental group of ΔG.

The graph theoretic questions considered in this paper concern edge shelling of G. An edge shelling is a certain sequential removal of the edges of G. Shelling of simplicial complexes has been important in polyhedral theory, in manifold theory and, more recently, in connection with Cohen–Macaulay rings. Early calculations of the Euler characteristic of a convex polytope were based on a shelling argument. Shelling of complexes and edge shelling of graphs are related as follows: if ΔG is shellable, then G is edge shellable, but not conversely. Properties of edge shelling are treated in Section 6. For example, it is shown that if G is edge shellable, then $\pi(G)$ is trivial. It is con-

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jectured that the converse is true: if $\pi(G)$ is trivial then G is edge shellable. The conjecture is proved in the case of 3-colored graphs.

Dual to edge shelling is a concept of closure in the cycle matroid of G. In Sections 3 and 4 three closure operators (c_0, c_1 and c_2) are investigated. Basically c_0 is closure with respect to cycles in G; c_1 is closure with respect to null homotopic cycles; and c_2 is closure with respect to cycles in members of the cover. The closure operators are related by $c_2(A) \subseteq c_1(A) \subseteq c_0(A)$ for any subgraph A of G. For a spanning tree T in G we show that $c_1(T) = c_0(T)$ if and only if $\pi(G) = 0$. The conjecture that $c_2(T) = c_1(T)$ for some spanning tree T implies the conjecture mentioned above.

The topological significance of edge shelling and closure is examined in Section 7. The long-standing Poincaré Conjecture states that a simply connected closed 3-manifold is homeomorphic to a 3-sphere. It is well known that a triangulated 3-manifold that is shellable is homeomorphic to a 3-sphere. However, it has been impossible to use this fact to prove the Poincaré Conjecture, because there exist non-shellable 3-balls and 3-spheres. An alternative graph-theoretic approach leads to problems involving edge shellability and closure. The edge shellability of a 4-colored graph G is not as strong a condition as the shellability of the complex ΔG. Nevertheless we prove that if G is edge shellable, then a manifold ΔG is still a 3-sphere. In particular, the validity of either of the graph-theoretic conjectures stated above implies the validity of the Poincaré Conjecture.

2. Graphs and covers

Graphs will be finite, without loops. Multiple edges are allowed. A collection $\mathscr{F}$ of connected subgraphs of a graph G is called a *cover* if $\cup \mathscr{F} = G$. In this section a complex $\Delta(G, \mathscr{F})$ will be associated with the pair $(G, \mathscr{F})$.

If the intersection of any set of subgraphs in $\mathscr{F}$ is connected, then $\Delta(G, \mathscr{F})$ could be defined as the nerve of the cover $\mathscr{F}$. This means that $\Delta(G, \mathscr{F})$ is a simplicial complex whose simplexes are nonempty subsets of $\mathscr{F}$ with nonempty intersection. In general, we proceed as follows: Consider any subset $\{F_1, F_2, \ldots, F_{k+1}\}$ of distinct elements of $\mathscr{F}$. If H is any connected component of $\cap_{i=1}^{k+1} F_i$, let σ_H denote a k-dimensional Euclidean simplex whose verti-

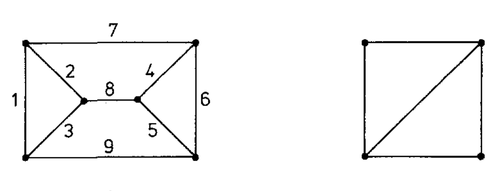

Fig. 1. Cover $\mathscr{F} = \{F_1, F_2, F_3, F_4\}$, where $F_1 = \langle 1, 2, 3 \rangle$, $F_2 = \langle 4, 5, 6 \rangle$, $F_3 = \langle 2, 4, 7, 8 \rangle$, $F_4 = \langle 3, 5, 8, 9 \rangle$.

ces are labeled $F_1, F_2, \ldots, F_{k+1}$. Each m-dimensional face $\langle F_{i_1}, F_{i_2}, \ldots, F_{i_{m+1}} \rangle$ of σ_H is labelled H', where H' is the connected component of $\cap_{j=1}^{m+1} F_{i_j}$ containing H. Let K be the disjoint union of the σ_H taken over all connected components H of intersections of all possible subsets of $\mathscr{F}$. In K identify two simplexes of the same dimension when all the faces are identically labeled. If $\sim$ denotes this equivalence, define $\Delta G := \Delta(G, \mathscr{F}) = K/\sim$. An example is shown in Figure 1. Here the subgraphs in $\mathscr{F}$ are denoted by the edges that induce them. In $\Delta(G, \mathscr{F})$ it is possible for a set of $m + 1$ vertices to be contained in two distinct m-dimensional simplexes. For this reason a complex whose cells are simplexes is referred to as a *pseudocomplex* in [7].

For simplicial complexes, the well-known [11] edge-path construction yields the fundamental group. We now generalize this procedure. A closely related notion of homotopy was used by Tits [13] in relation to chamber complexes. Consider a connected graph G with cover $\mathscr{F}$. A path in G is a sequence of directed edges $\{e_1, e_2, \ldots, e_m\}$ such that the initial vertex of e_{i+1} is the terminal vertex of e_i for $i = 1, 2, \ldots, m - 1$. If the terminal vertex of path α is the initial vertex of β, then the product $\alpha\beta$ is the concatenated path. The path α^{-1} is obtained by listing the edges of α in revers order. Two paths $\alpha = \beta\gamma\delta$ and $\alpha' = \beta\gamma'\delta$ are called elementary $\mathscr{F}$-homotopic if $\gamma\gamma'^{-1}$ is contained entirely in one of the subgraphs $F \in \mathscr{F}$. Two paths α and α' are *$\mathscr{F}$-homotopic* $\alpha \sim \alpha'$ if there is a sequence of elementary homotopies taking one to the other. If v is taken as a base point of G, the homotopy classes of paths with initial and terminal vertex v form a group in the usual way. This group, which is independent of base point, is denoted $\pi(G) := \pi(G, \mathscr{F})$.

EXAMPLE 1. Clearly $\pi(G, \{G\}) = 0$. At the other extreme let $E(G)$ denote the edge set of G. Then $\pi(G, E(G)) \cong \pi_1(G)$, where $\pi_1(G)$ is the topological fundamental group of G regarded as a 1-complex.

EXAMPLE 2. Consider a finite simplicial complex Δ. Let Δ^1 be the 1-skeleton of Δ and $\mathscr{F}$ the set of 1-skeletons of 2-simplexes in Δ. Then $\pi(\Delta^1, \mathscr{F}) \cong \pi_1(\Delta)$. This is exactly the edge-path construction of the fundamental group of a simplicial complex.

The next result relates the combinatorial fundamental group of a graph G to the topological fundamental group of the complex ΔG.

THEOREM 2.1. *If G is a connected graph with cover, then $\pi(G) \cong \pi_1(\Delta G)$.*

Proof. Let $\mathscr{F}$ be the cover of G. By definition each $F \in \mathscr{F}$ is a vertex of ΔG. Let $\alpha = E_0 E_1 \ldots E_k$ be any edge path in ΔG based at $F_0 \in \mathscr{F}$. If E_i has endpoints $F_i, F_{i+1} \in \mathscr{F}$, where $F_{k+1} = F_0$, let H_i be the component of $F_i \cap F_{i+1}$ corresponding to E_i. If p_i is any vertex in H_i, let β_i be a path joining p_i and p_{i+1} and lying entirely in F_{i+1}. Then $\bar{\alpha} = \beta_1 \beta_2 \ldots \beta_k$ is a path in G with base point p_0 and it is easily checked that the $\mathscr{F}$-homotopy class $[\bar{\alpha}]$ of $\bar{\alpha}$ depends only on the edge-path homotopy class $[\alpha]$ of α. Thus the map $[\alpha] \mapsto [\bar{\alpha}]$ induces a homorphism $f : \pi_1(\Delta G) \to \pi(G)$. In the other direction let $\bar{\alpha} = e_0 e_1 \dots e_k$ be a path in G based at p_0. Then there are subgraphs $F_1, F_2, \dots, F_k$ in $\mathscr{F}$ such that $e_i \in F_i$. Let p_i be the common vertex of e_i and e_{i+1} and let α be the edge path $E_1 E_2 \dots E_k$ in ΔG, where E_i is the edge corresponding to the component of $F_{i-1} \cap F_i$ containing p_i. Then the edge-path homotopy class $[\alpha]$ of α depends only on the $\mathscr{F}$-homotopy class $[\bar{\alpha}]$ of $\bar{\alpha}$. Thus we have a homorphism $g : \pi(G) \to \pi_1(\Delta G)$ given by $[\bar{\alpha}] \mapsto [\alpha]$. It is easily checked that f and g are inverse to each other. □

THEOREM 2.2. *If T is a tree with cover, then $\pi(T) = 0$.*

Proof. Let $\alpha = e_1 e_2 \dots e_k$ be a loop in T. We claim that $\alpha \sim 0$. The proof is by induction on the length of α. It is obvious for $k = 0$. If $k \geqslant 1$, then $e_i = e_{i+1}^{-1}$ for some i. For any cover, $\alpha = e_1 \dots e_{i-1} e_i e_{i+1} e_{i+2} \dots e_k \sim e_1 \dots e_{i-1} e_{i+2} \dots e_k$. □

3. Closure

Throughout this section G is a connected graph with edge set E and cover $\mathscr{F}$. The topics in this section will be framed in the language of graphic matroids. For background on matroids see [1], [16]. If A is a set of edges of G, we often make no distinction between A and the subgraph of G induced by A. Recall that the cycles of G are the circuits of a matroid $M(G)$ on the edge set E. The matroid $M(G)$ is called the cycle matroid of G. If $A \subseteq E$, the following facts about the cycle matroid are elementary.

(1) An edge $e \in E - A$ belongs to the closure of $A \Leftrightarrow$ there is a cycle C in G with $e \in C \subseteq A \cup e$.
(2) A is a basis of $M(G) \Leftrightarrow A$ is a spanning tree in G.

We now introduce three closure operators on E. Let $\mathscr{C}_0$ be the set of cycles in G. Let $\mathscr{C}_1$ be the set of cycles in G that are $\mathscr{F}$-homotopic to 0. Let $\mathscr{C}_2$ be the set consisting of cycles C in G such that $C \subseteq F$ for some $F \in \mathscr{F}$. Clearly $\mathscr{C}_2 \subseteq \mathscr{C}_1 \subseteq \mathscr{C}_0$. Let A be a subset of E. An edge $e \in E - A$ is said to be *i-dependent* on A for $i \in \{0, 1, 2\}$ if there is a cycle $C \in \mathscr{C}_i$ such that $e \in C \subseteq A \cup e$. A subset $A \subseteq E$ is called *i-closed* if there does not exist an $e \in E - A$ that is i-dependent on A. The *i-closure* of A is the intersection of all i-closed sets containing A. The i-closure of A is denoted $c_i(A)$. Note that the closure c_0 is just the usual closure operator of the cycle matroid $M(G)$. It is easy to verify that all three satisfy the standard properties of a closure operator.

(1) $\quad A \subseteq c_i(A)$
(2) $\quad A \subseteq B \Rightarrow c_i(A) \subseteq c_i(B)$
(3) $\quad c_i c_i(A) = c_i(A)$.

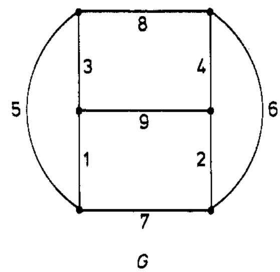

Fig. 2. Cover $\mathscr{F} = \{F_1, F_2, F_3\}$, where $F_1 = \langle 1,2,7,9 \rangle$, $F_2 = \langle 3,4,8,9 \rangle$ and $F_3 = \langle 5,6,7,8 \rangle$.

However, the closure operators c_1 and c_2 do not always determine a matroid on E. This is because, in general, the exchange property does not hold for c_1 or c_2. For example, consider the graph G in Figure 2. The set of edges $A = \{1,2,3,4,5\}$ is c_1- and c_2-closed. For $i = 1,2$ we have $6 \in c_i(A \cup \{7\})$, but $7 \notin c_i(A \cup \{6\})$.

The following properties of the closure operators are consequences of the definitions.

PROPOSITION 3.1. *Let G be a connected graph with cover $\mathscr{F}$. If A is a set of edges of G, then*

(1) *A 0-closed $\Rightarrow$ A 1-closed $\Rightarrow$ A 2-closed.*

(2) $c_2(A) \subseteq c_1(A) \subseteq c_0(A)$.

(3) *The intersection of i-closed sets is i-closed for $i \in \{0,1,2\}$.*

Let A be a subgraph of G. The set $\mathscr{F}/A = \{F \cap A \mid F \in \mathscr{F}\}$ will be referred to as the *restricted cover* of $\mathscr{F}$. By $\pi(A)$ we always mean $\pi(A, \mathscr{F}/A)$, the homotopy group with respect to the restricted cover. Similarly, $\Delta A = \Delta(A, \mathscr{F}/A)$.

THEOREM 3.2. *Let G be a connected graph with cover. If $A \subseteq G$, then there are subjective homorphisms $\pi(A) \to \pi(c_i(A))$ for $i = 1, 2$.*

Proof. The inclusion maps $A \hookrightarrow c_i(A)$ induce homomorphisms $f_i : \pi(A) \to \pi(c_i(A))$. We will show that f_1 is subjective. The proof for f_2 *is* almost identical. Let $A = A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = c_1(A)$ be a sequence of subgraphs of G such that $A_{i+1} = A_i \cup e_i$, where $e_i \in C_i \subseteq A_i \cup e_i$ and C_i is a cycle in A_{i+1} $\mathscr{F}$-homotopic to 0. By induction, it is sufficient to show that any based path $\alpha = E_1 E_2 \ldots E_m$ in A_{i+1} is $\mathscr{F}$-homotopic to a path in A_i. If e_i does not lie in α we are done. Otherwise $e_i = E_j$ for some j. If $C_i = E_j^{-1}\beta$, then $\alpha = E_1 \ldots E_{j-1} E_j E_{j+1} \ldots E_m \sim E_1 \ldots E_{j-1} \beta E_{j+1} \ldots E_m = \gamma$, where γ lies in A_i. $\square$

4. A CONJECTURE

Let G be a connected graph with cover. This section deals with the relationship between the closure operators c_0, c_1 and c_2. Specifically, let T be a spanning tree of G. In other words, T is a basis for the cycle matroid $M(G)$. By Proposition 3.1 we know that $c_2(T) \subseteq c_1(T) \subseteq c_0(T) = G$. Under certain conditions more can be said.

THEOREM 4.1. *Let G be a connected graph with cover and T a spanning tree of G. Then $c_1(T) = c_0(T)$ if and only if $\pi(G) = 0$.*

Proof. If $\pi(G) = 0$ then $\mathscr{C}_1 = \mathscr{C}_0$. This implies that $c_1(T) = c_0(T)$. Conversely, if $c_1(T) = c_0(T) = G$ then, by Theorem 3.2, there is a surjective homomorphism $\pi(T) \to \pi(G)$. But by Theorem 2.2 $\pi(T) = 0$. □

THEOREM 4.2. *If G is a connected graph with cover and T is a tree in G, then $\pi(c_2(T)) \cong \pi(c_1(T)) = 0$.*

Proof. By Theorem 3.2 there are surjective homomorphisms $\pi(T) \to \pi(c_i(T))$ for $i = 1, 2$, and by Theorem 2.2 $\pi(T) = 0$. Hence $\pi(c_i(T)) = 0$ for $i = 1, 2$. □

We would like to strengthen Theorem 4.2. A special case of the following conjecture is proved in Section 6, but we are unable to give a proof in general. This is not surprising in light of the topological implications discussed in Section 7.

CONJECTURE 1. *Let G be a connected graph with cover. There exists a spanning tree T such that $c_2(T) = c_1(T)$.*

There validity of Conjecture 1, together with Theorem 4.1, would imply the following statement.

CONJECTURE 2. *Let G be a connected graph with cover. If $\pi(G) = 0$ then there exists a spanning tree T such that $c_2(T) = c_1(T) = c_0(T) = G$.*

Conjecture 2 cannot be strengthened to state that if $\pi(G) = 0$ then $c_2(T) = G$ for *every* spanning tree T. As an example, consider the graph G in Figure 3. The simplicial complex ΔG is a topological 2-cell. Hence, by Theorem 2.1,

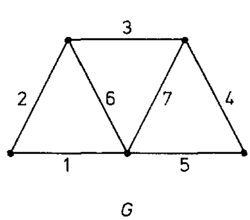

Fig. 3. Cover $\mathscr{F} = \{F_1, F_2, F_3\}$, where $F_1 = \langle 1,2,3,7 \rangle$, $F_2 = \langle 3,4,5,6 \rangle$ and $F_3 = \langle 1,2,3,4,5 \rangle$.

$\pi(G) = 0$. However $T = \langle 1, 5, 6, 7 \rangle$ is a spanning tree and $c_2(T) \neq G$. The graph G, however, is not a counterexample to Conjecture 2 because $T' = \langle 1, 2, 4, 5 \rangle$ is also a spanning tree and $c_2(T') = G$.

5. Colored graphs

Several authors have recently applied edge-colored graphs to topics in topology [4], [5], [6], [8], [15]. For us they furnish important examples of graphs with cover, because every closed 2- or 3-manifold can be realized as the associated complex ΔG of some colored graph G. In subsequent sections colored graphs are used in treating shellability of 2- and 3- manifolds and in an approach to the Poincaré Conjecture.

Let $I = \{1, 2, \ldots, n\}$ be fixed throughout this section. An *n-colored graph* is a connected graph, regular of degree n, whose edges are I-colored so that no two incident edges are the same color. The integer n is called the *rank* of G. For $J \subseteq I$ let $G(J)$ be the subgraph of G obtained by deleting all edges of color not in J. Each connected component of $G(J)$ is a colored graph of rank $|J|$, called a *J-residue* of G. The only residue of rank n is G itself. Each residue of rank 1 is a single edge, and each residue of rank 0 is a single vertex of G. Figure 4 shown a 3-colored graph and the six residues of rank 2.

Let G be an n-colored graph. The set $\mathscr{R}$ of residues of rank $(n-1)$ is a cover of G. The associated complex $\Delta G := \Delta(G, \mathscr{R})$ is an $(n-1)$-dimensional pseudomanifold. The following result is proved in [4], [8].

THEOREM 5.1. *If M is a closed 3-manifold (2-manifold), then there is a 4-colored graph (3-colored graph) such that ΔG is homeomorphic to M.*

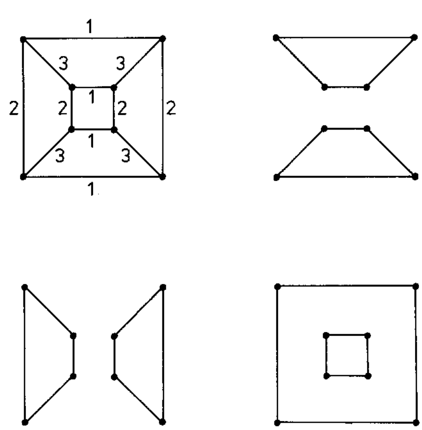

Fig. 4. The residues of rank 2 in a 3-colored graph.

6. Shellability

A simplicial complex is *pure* if its maximal simplexes all have the same dimension d. A pure finite simplicial complex is *shellable* if there is an ordering $(s_1, s_2, \ldots, s_N)$ of the maximal simplexes such that $s_k \cap \cup_{i=1}^{k-1} s_i$ is a nonempty union of maximal proper faces of s_k for $k = 2, 3, \ldots, N$. This implies, in particular, that this intersection is homeomorphic to either a $(d-1)$-ball or $(d-1)$-sphere. Therefore, a shellable d-dimensional pseudomanifold is either a d-ball or a d-sphere. The preceding definitions hold, exactly as stated, for pseudocomplexes.

Shellability has been a useful concept in polyhedral theory and, more recently, in connection with Cohen–Macaulay rings [12]. In this section a notion of edge shelling of graphs is formulated. Edge shelling is dual, in a sense, to closure. Also edge shelling for a graph is related to the standard shelling for simplicial complexes as defined above.

An edge e of a graph G is called a *bridge* if $G - e$ has more connected components than G. Let G be a graph with cover $\mathscr{F}$. In this section the terminology $\mathscr{C}$, c, dependent and closure refer, respectively, to $\mathscr{C}_2$, c_2, 2-dependent and 2-closure as defined in Section 4. Hence $\mathscr{C}$ is the set consisting of those cycles C such that $C \subseteq F$ for some $F \in \mathscr{F}$. Let A be a subgraph of G. An edge $e \in A$ is said to be shellable from A if either

(1) e is a bridge in A, or
(2) e is dependent on $A - e$.

The graph G is called *edge shellable* is there is an ordering $(e_1, e_2, \ldots, e_N)$ of the edges of G such that e_k is shellable from $\langle e_1, e_2, \ldots, e_k \rangle$ for $1 \leqslant k \leqslant N$. Edge shellability of a graph is related to closure.

THEOREM 6.1. *Let G be a connected graph with cover. The graph G is edge shellable if and only if G has a spanning tree T with $c(T) = G$.*

Proof. Assume $c(T) = G$, where $T = \{e_1, \ldots, e_k\}$. It is clear that for $1 \leqslant i \leqslant k$, e_i is shellable from $\langle e_1, e_2, \ldots, e_i \rangle$. Let $A = \{e_1, \ldots, e_k, e_{k+1}, \ldots, e_N\}$ be a set of edges of G, maximal with respect to the property that for $1 \leqslant i \leqslant N$, e_i is shellable from $\langle e_1, \ldots, e_i \rangle$. If A is the set of all edges of G, then we are done. If not, then A is not closed. There is an $e \notin A$ such that e is dependent on A. Hence e is shellable from $A \cup e$, contradicting the maximality of A.

Conversely, assume $(e_1, e_2, \ldots, e_N)$ is an edge shelling of G. Let $T = \{e'_1, \ldots, e'_k\}$ be a subset of $\{e_1, \ldots, e_N\}$ consisting of those e_i that are bridges in $\langle e_1, \ldots, e_i \rangle$. This implies that T contains no cycle. Also by the definition of edge shelling $c(T) = G$. Finally, T is a spanning tree; otherwise $G = c(T) \subseteq c_0(T) \neq G$. □

The next result follows immediately from Theorems 4.2 and 6.1.

THEOREM 6.2. *Let G be a connected graph with cover. If G is edge shellable, then $\pi(G) = 0$.* □

Via Theorem 6.1 the Conjecture 2 can be restated in terms of edge shellability. The equivalent restatement is exactly the converse of Theorem 6.2.

CONJECTURE 2′. *Let G be a connected graph with cover. If $\pi(G) = 0$, then G is edge shellable.*

Conjecture 2′ can be settled in the case of 3-colored graphs. Recall that the cover in this case is the set of residues of rank 2.

THEOREM 6.3. *A 3-colored graph G is edge shellable if and only if $\pi(G) = 0$.*

Proof. The implication in one direction is a direct consequence of Theorem 6.2. Conversely, assume that $\pi(G) = 0$. Note that the rank 2 residues of G are cycles in G. Let T be any spanning tree of G and assume, by way of contradiction, that $c(T) \neq G$. Then there must exist a set $S = \{C_i\}, 0 \leqslant i \leqslant m$, of rank 2 residues and a set $A = \{e_i\}, 0 \leqslant i \leqslant m$, of edges not in T, such that $e_i \in C_i \cap C_{i+1}$. Here the indices are labeled modulo m. In particular, each cycle C_i contains exactly two elements of A. Let α_0 be the unique cycle contained in $T \cup e_0$. Note that α_0 contains exactly one edge in A. If α is any based path containing an odd number of edges of A and α' is elementary homotopic to α, then α' must also contain an odd number of edges of A. Hence it is not possible that $\alpha_0 \sim 0$. Therefore, $\pi(G) \neq 0$. □

To understand how edge shellability relates to the usual notion of shellability of pseudo complexes, consider the case where G is an n-colored graph. The set $\mathscr{R}_m$ of residues of rank $m \geqslant 1$ is a cover of G. In this case the elements of $\mathscr{C}$ are cycles in G that are m-colored. The pseudocomplex ΔG, as defined in the previous section, is a pure pseudocomplex.

THEOREM 6.4. *Let G be an n-colored graph. If ΔG is shellable, then G is edge shellable with respect to the cover $\mathscr{R}_m$ for any $m \geqslant 2$.*

Proof. Assume that ΔG is shellable. It is sufficient to prove the result for the cover $\mathscr{R}_2$. It was proved in [15] that ΔG shellable is equivalent to the existence of an ordering $(v_1, v_2, \ldots, v_N)$ of the vertices of G such that for $1 \leqslant k \leqslant N$, v_k belongs to a regular residue of rank d in the induced graph $H = \langle v_1, \ldots, v_k \rangle$, where d is the degree of v_k. If $\{e_1, e_2, \ldots, e_d\}$ is the set of edges incident with v_k, then e_j is shellable from $H - \{e_1, \ldots, e_{j-1}\}$. for $1 \leqslant j \leqslant d$. This process can be repeated for $k = N, N-1, \ldots, 1$ and the result follows by induction. □

The converse of Theorem 6.4 is false. Consider two identical copies of the graph in Figure 5. Let G be the 4-colored graph formed by joining correspond-

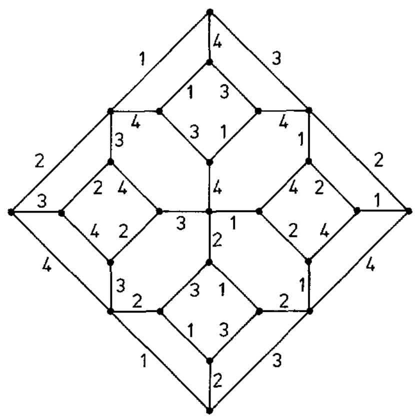

Fig. 5

ing vertices of the two copies by appropriately colored edges. It can be checked that G is edge shellable. However, it is shown in [15] that ΔG is a non-shellable sphere.

7. Poincaré Conjecture

Let G be a 4-colored graph. We know that if ΔG is shellable, then ΔG is a 3-sphere. By Theorem 6.4 and the remarks following it, the edge shellability of G is weaker than the shellability of ΔG. Nevertheless, it is proved in this section that if G is edge shellable, then a manifold ΔG is a 3-sphere. Topological implications of this result, particularly for the Poincaré Conjecture, are discussed.

Recall that for an n-colored graph, the set of residues of rank $(n-1)$ is the cover.

THEOREM 7.1. *Let G be an n-colored graph, $n \leqslant 4$, such that ΔG is a manifold. If G is edge shellable, then ΔG is an $(n-1)$-sphere.*

Proof. Only the case $n = 4$ is proved. The cases $n < 4$ follow from easier versions of the same argument. Assume that G is edge shellable. Then there is a spanning tree T such that $c(T) = G$. Because ΔT is shellable, ΔT is a 3-ball. We will rather regard ΔT as the complement of an open 3-ball in S^3. In general let $\mathscr{B}$ be the class of 3-dimensional pseudomanifolds that are the complements of a finite number of disjoint open 3-balls in S^3. Thus $\Delta T \in \mathscr{B}$. To prove Theorem 7.1, it is sufficient to show that $\Delta G \in \mathscr{B}$ because ΔG is without boundary. This will be done by induction. Assume that $A \subseteq G$ and $\Delta A \in \mathscr{B}$. Let $e \notin A$ be an edge that is dependent on A. We have only to show that $\Delta(A \cup e) \in \mathscr{B}$. By the construction described above, $\Delta(A \cup e)$ is obtained

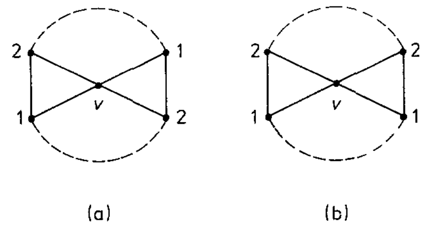

Fig. 6

from ΔA by identifying two 2-simplexes, s and s'. The fact that e is dependent on A means that s and s' share a vertex v in ΔA. If s and s' belong to the boundaries of distinct 3-balls in $S^3 - \Delta A$, then link v cannot be a 2-sphere in ΔG, contradicting the assumption that ΔG is a manifold. Therefore s and s' belong to the boundary of the same 3-ball in $S^3 - \Delta A$. Figure 6 shows simplexes s and s' situated on the boundary of this 3-ball. There are two ways that s and s' can be identified in $\Delta(A \cup e)$. If they are identified along like-numbered vertices as in Figure 6a, then link v cannot be orientable in ΔG, contradicting the assumption that ΔG is a manifold. Hence the identification is as in Figure 6b. It is possible that some of the vertices labeled 1 and 2 and some of the edges labeled (12), $(1v)$, $(2, v)$ are already identified in ΔA. In any case $S^3 - \Delta(A \cup e)$ remains the disjoint union of 3-balls. □

Remark 1. Theorems 2.1, 5.1, 6.3 and 7.1 imply that a simply connected closed 2-manifold must be a 2-sphere. Of course, this is well known by the classification of 2-dimensional surfaces. Our proof may be of interest because it is completely combinatorial.

Remark 2. If a triangulated closed 3-manifold is shellable, then it is a 3-sphere. As pointed out by Bing [2], this fact provides a natural approach to the classical Poincaré Conjecture.

POINCARÉ CONJECTURE. *A simply connected closed 3-manifold is homeomorphic to a 3-sphere.*

To verify the conjecture it is sufficient to show that every simply connected closed 3-manifold is shellable. Unfortunately, there exist triangulated 3-balls [10] and 3-spheres [15] that are not shellable. An alternative is to use the concept of edge shellability. By Theorems 2.1, 5.1 and 7.1, a weaker graph theoretic statement still implies the Poincaré Conjecture:

CONJECTURE 3. *Let G be a 4-colored graph. If $\pi(G) = 0$, then G is edge shellable.*

Conjecture 3 is a special case of Conjecture 2 or 2′, which in turn is a special case of Conjecture 1. Each of the three may therefore be considered a generalized Poincaré Conjecture.

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