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Reconstruction of the Set of Branches of a Graph

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Abstract. It is proved that the set of branches of a graph G is reconstructible except in a very special case. More precisely the set of branches of a graph G is reconstructible unless all the following hold: (1) the pruned center of G is a vertex or an edge, (2) G has exactly two branches, (3) one branch contains all the vertices of degree one of G and the other branch contains exactly one end-block. This is the best possible result in the sense that in the special excluded case, the reconstruction of the set of branches is equivalent to the reconstruction of the graph itself.

1. Introduction

All graphs in this paper are finite, simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *degree sequence of* G is the sequence obtained by listing the degrees of its vertices in nondecreasing order. A vertex of degree k is called a *k-vertex*. For all $v \in V(G)$, the vertex-deleted subgraph $G - v$ is the subgraph of G obtained by deleting v and all its incident edges. Let $\{G - v\} = \{G - v | v \in V(G)\}$ be the collection of all vertex-deleted subgraphs of G. A *reconstruction* of G is a graph H such that $\{G - v\} = \{H - u\}$, that is, there is a bijection f between $V(G)$ and $V(H)$ such that $G - v \cong H - f(v)$, for all $v \in V(G)$. A property or parameter P is *reconstructible* if, for any graph G with the property or parameter P, all its reconstructions also have the property or parameter P. For example, being disconnected is a reconstructible property and the degree sequence is a reconstructible parameter [1]. A graph G is *reconstructible* if, for any reconstruction H of G, we have $H \cong G$.

A graph G is called *separable* if G is connected and there is a vertex $v \in V(G)$ such that $G - v$ is disconnected. For a separable graph G which is not a tree, define the *pruned graph* of G, *pruned*(G), to be the maximal subgraph of G having no 1-vertex. A *2-block* in a graph is a maximal 2-connected subgraph. Notice that a 2-block could be an edge, which will be called a *trivial 2-block*. The *block-cutpoint tree* of a graph G, denoted *block*(G), has the set of all cutpoints and all 2-blocks of G as its vertex set. Two vertices of *block*(G) are joined by an edge if and only if one of them is a 2-block of G and the other is a cutpoint of G

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on that 2-block. The 2-block or cutpoint P of G which is the center of the tree *block*(*pruned*(G)) is called the *pruned center* of G. A *branch* B of G is a maximal subgraph of G that contains exactly one vertex u of the pruned center P, called the *root* of B, such that $B - u$ is connected. A branch B is called a *rooted branch* if the root of B is labeled. (For example the root can be labeled by coloring it blue.) Hereafter, branch will always mean rooted branch. The set of all branches of G is denoted by $\mathbf{B}_G$, or simply $\mathbf{B}$, if G is clear from the context.

In 1969, Greenwell and Hemminger [2] proved that, for a separable graph G, the set $\mathbf{B}_G$ is reconstructible if there is a 1-vertex v such that $G - v$ has at least two branches with 1-vertices. This result is extended in Theorem 2 in Section 2 of this paper. Basically this result states that, except in very special circumstances, the set of branches of a graph is reconstructible.

2. The Reconstruction of the Set $\mathbf{B}_G$

Let $d_G(v)$ denote the degree of vertex v in the graph G. *Throughout this section, a 2-block is always nontrivial unless otherwise stated.*

Lemma 1. *Let G be a 2-connected graph, not a cycle, and let u, $v \in V(G)$. Then either* (1) *G is obtained from a cycle containing v by adding an edge between the two neighbors of v; or* (2) *there is a vertex $w_1 \neq u$, v such that $G - w_1$ contains at most two 1-vertices and exactly one 2-block and u is in this 2-block, and in case $G - w_1$ contains exactly two 1-vertices, v is one of them; and there is a vertex $w_2 \neq u$ such that $G - w_2$ contains at most one 1-vertex and exactly one 2-block and u is in this 2-block.*

Proof. It is not hard to show that any 2-connected graph G can be constructed in stages: $G_0 \subset G_1 \subset \cdots \subset G_m = G$, where G_0 is a cycle containing u and v, and $G_{i+1} = G_i \cup p_{i+1}$ for a path p_{i+1} where $G_i \cap p_{i+1}$ is exactly the endvertices of p_{i+1}. Moreover G_i is 2-connected for each i. Consider two cases.

Case 1. All paths $p_1, p_1, \ldots p_m$ are single edges. Then G consists of a single cycle G_0 with chords. A chord divides the cycle G_0 into two arcs A and B; say $u \in A$. (If u lies on the chord, then choose A as the arc clockwise from u.) Call a chord *minimal* if there exists no other chord with both endvertices on B. Since G is not a cycle there exists a minimal chord. Choose a minimal chord and choose w_1 on the corresponding arc B adjacent to an endvertex of B. This can always be done so that $w_1 \neq v$ and condition (2) of the lemma is satisfied except in either of the following cases. (i) There is exactly one chord, its corresponding arc B has length 2 and vertex v is the interior vertex of B, in which case the condition (1) of the lemma is satisfied. (ii) There is more than one chord, but only one of them is minimal and its corresponding arc B has only one interior vertex, which is v. In this case, if w_2 is chosen to be v and w_1 is chosen to be an endvertex of B that is not incident with all the chords, then condition (2) of the lemma is satisfied. Note that in $G - w_1$, there are at most two 1-vertices, which have to be the two vertices adjacent to w_1 on the cycle G_0 and, of these two vertices, one of them is v.

Case 2. One of the paths $p_1, p_1, \dots p_m$ is not a single edge. Let k be the largest integer such that p_k is a path of length greater than 1. Then $p_{k+1}, p_{k+2}, \dots, p_m$ are all single edges. Some of these edges may be chords of path p_k. A chord subtends an arc B on path p_k. As before call such a chord *minimal* if there exists no other chord with both endvertices on B. Choose w_1 on an arc B corresponding to a minimal chord, adjacent to an endvertex of B. (If p_k has no chords, choose w adjacent to an endvertex of p_k.) This vertex $w_1 \neq u$, v satisfies condition (2) of the lemma. □

Remark. The conclusion (1) in Lemma 1 implies that there is a $w \neq u$, v such that $G - w$ is a path. In fact, w can be chosen as either neighbor of v. Note also that, in this case, $G - v$ is a cycle containing u.

A 2-block that has only one cutpoint in *pruned*(G) will be called an *end-block* of G. *Pruned*(G) is clearly reconstructible from $G - x$, where x is any 1-vertex of G. This implies that the pruned center P of G and the number of 2-blocks in G are reconstructible.

Theorem 2. *The set of branches of G is reconstructible unless all the following hold:*

(1) *the pruned center of G is a vertex or an edge,*
(2) *G has exactly two branches,*
(3) *one branch contains all the 1-vertices of G and the other branch contains exactly one end-block.*

Proof. Assume that G contains a 1-vertex; otherwise G is known to be reconstructible since all separable graphs without a 1-vertex are reconstructible [3]. Take a $G - x \in \{G - v\}$ such that x is a 1-vertex. Such a $G - x$ can be identified in $\{G - v\}$ since the degree sequence is reconstructible. The proof of the reconstructibility of the set of branches is now divided into somewhat technical cases. When we say a case is "recognizable", we mean that the class of graphs covered by this case is recognizable, that is, the property determining the case is reconstructible. We will not prove the recognizability for easy cases.

Case 1. G contains exactly one 1-vertex x.

This case is recognizable since the degree sequence is reconstructible, that is, the property that G contains exactly one 1-vertex is reconstructible. Let the vertex adjacent to x be r. Without loss of generality, assume that r is a vertex on *pruned*(G); otherwise the unique 1-vertex of $G - x$ must be r, and G can be reconstructed from $G - x$ by adding a vertex x joined to r.

Case 1.1. G contains exactly one 2-block.

Recall that a 2-block is always nontrivial unless otherwise stated. In this Case 1.1 *pruned*$(G) = P$, and the graph G has only one branch, which is the edge rx.

Case 1.2. G contains more than one 2-block and r is on an end-block, denoted D, of *pruned*(G), but r is not the cutpoint of D in *pruned*(G).

Assume that *pruned*(G) has at least three 2-blocks. Otherwise, the case is excluded from Theorem 2. To show this case is recognizable, it is first shown that, from the graph $G - r$, which can be recognized in $\{G - v\}$ as the only graph that contains an isolated vertex, whether r is in an end-block and is not a cut-point of *pruned*(G), is recognizable. First, r is a cut-point of *pruned*(G) if and only if $G - r$ has more than one nontrivial (that is, not an isolated vertex) connected component. Hence, whether or not r is a cut-point of *pruned*(G) is recognizable. Next, assuming that r is not a cut-point of *pruned*(G), we will show that, whether or not r is in an end-block, is recognizable from $G - r$. Define a matching process between the end-blocks of *pruned*(G) and the end-blocks of *pruned*($G - r$) as follows. Match each end-block of *pruned*(G) with an isomorphic end-block of *pruned*($G - r$). There are possibly end-blocks of *pruned*($G - r$) with tree-growth rooted on a vertex which is not a cut-point in *pruned*($G - r$). A *tree-growth* is a maximal connected subgraph that has exactly one vertex in common with *pruned*(G)). Exclude these end-blocks from the matching process. The matching starts from the largest end-block of *pruned*(G) and proceeds to the smallest one. Observe the following two facts:

(1) If r is not in any end-block, then all end-blocks of *pruned*(G) are matched to end-blocks of *pruned*($G - r$), although *pruned*($G - r$) may have end-blocks left unmatched.

(2) If r is from an end-block D', then, during the matching process defined above, there will be an end-block $D'' \cong D'$ (possibly $D'' = D'$), such that there is no end-block of *pruned*($G - r$) left to match D''. From the above observation, it can be seen that, whether or not r is on an end-block, is recognizable. Hence Case 1.2. is recognizable.

The above argument also shows that the 2-block D is reconstructible. In fact, for the graph $G - r$, where r is on an end-block but r is not a cutpoint of *pruned*(G), the first unmatched end-block D' of *pruned*(G) must be isomorphic to D.

Let u be the cutpoint of D in *pruned*(G). We now show that D, with u and r labeled, is reconstructible. Let $D_1 \neq D$ be an end-block of G and let u_1 be the cutpoint on D_1. By the remark following Lemma 1, there is a $w \in D_1$, $w \neq u_1$, such that $D_1 - w$ contains only one 2-block, or $D_1 - w$ is a path. There are four possibilities.

(i) $D_1 - w$ contains only one 2-block which contains u_1.

First, it is shown that we can choose a w' so that $D_1 - w'$ contains no path or contains a path of length at least two if we do not insist that $D_1 - w'$ contain only one 2-block. If $D_1 - w$ contains no *pendant edge* (which is an edge incident with a 1-vertex), then choose w' to be w. Assume that $D_1 - w$ contains a pedant edge. If $d_G(w) \neq 2$, we can choose the 1-vertex w_1 of this pendant edge as w'. If $d_G(w) = 2$, then let $w_2 \neq w_1$ be the other neighbor of w. In $G - w_2$, w is a 1-vertex on a path of length two. We can choose w_2 as w' if there is no path of length one in $G - w_2$. Otherwise let w_3 be a 1-vertex on a path of length one in $G - w_2$ and choose w_3 as w'. Now $G - w'$ contains no pendant edge. So in $G - w'$, the only end-block with a pendant edge rooted on it must be D. Hence consider a $G - w'$ that has only one end-block containing a root of a pendant edge, and that end-block has exactly $v(D) = |V(D)|$ vertices. This end-block must be D. Hence D, with u and r labeled, is reconstructed.

(ii) $D_1 - w$ is a path of length at least two. Again, D with u and r labeled, can be recognized from $G - w$.

Let $D_2 \neq D$ be a 2-block adjacent to D_1. Note that such a D_2 exists since *pruned*(G) has at least three 2-blocks. Note also that if D_2 is an edge and $D_1 - w$ is an edge, that is, D_1 is a triangle, then D with u and r labeled, can be recognized form $G - w$ since x can be recognized from $G - w$ as the only 1-vertex joining a vertex of degree greater than two. Hence in cases (iii) and (iv), D_2 will be assumed to be a non-trivial 2-block.

(iii) $D_1 - w$ is an edge, that is, D_1 is a triangle; D_2 is not an end-block of $G - w$ or D_2 is an end-block of $G - w$ but $D_1 - w$ is rooted on the cutpoint of D_2 in *pruned*$(G - w)$.

In this case, D with u and r labeled, can be recognized form $G - w$.

(iv) $D_1 - w$ is an edge, that is, D_1 is a triangle; D_2 is an end-block of $G - w$; and $D_1 - w$ is not rooted on the cutpoint of D_2 in *pruned*$(G - w)$.

Consider the following two cases. (a) If $D_2 \cong D$, with the root u_1 of $D_1 - w$ corresponding to r, and the cutpoint u' of D_2 in *pruned*$(G - w)$ corresponding to u, then D, with u and r labeled, can be recognized from $G - w$, up to isomorphism. (b) If the above isomorphism does not exists, then G is reconstructible, which can be proved as follows. Let H be a reconstruction of G. Without lose of generality, assume that H is obtained from $G - w$ by adding to it the vertex w and two edges incident to w. We can either let w join the two vertices of $D_1 - w$, in which case $G = H$, or let w join x and r, in which case we will prove *pruned*$(G) \ncong$ *pruned*(H) and get a contradiction since *pruned*(G) is reconstructible. Note that D is not a triangle; otherwise, since we have $G - r \cong H - u_1$ (which are the only graphs containing isolated vertices in $\{G - e\}$ and $\{H - e\}$, respectively), D_2 must be a triangle. This contradicts the assumption that (a) is not the case. Similarly, D_2 is not a triangle. Define a pair of *outer-blocks* (B_1, B_2) to be a subgraph of G (or H) consisting of an end-block B_1 and a 2-block B_2 (trivial or nontrivial) that contains exactly two cutpoints in *pruned*(G), one between B_1 and B_2 and the other on B_2. Two pairs of *outer-blocks* (B_1, B_2) and (B_1', B_2') are isomorphic if there is an isomorphism between (B_1, B_2) and (B_1', B_2') such that cutpoints of (B_1, B_2) correspond to the cutpoints of (B_1', B_2'). Assume the number of pairs of outer-blocks isomorphic to (D_1, D_2) in G is k. Then the number of pairs of outer-blocks isomorphic to (D_1, D_2) in H must be $k - 1$ for the following reason: (1) Since (a) is not the case, (D_1, D_2) in G is not isomorphic to (D', D) in H, where D' is the triangle xrw in H; (2) D_2 cannot be in a pair of outer-blocks in H that is isomorphic to (D_1, D_2) since D_1 is a triangle while D_2 is not. Hence we have shown that *pruned*$(G) \ncong$ *pruned*(H), a contradiction. We have now shown that G is reconstructible in case (b).

Having proved that D, with r and u labeled, is reconstructible, we now proceed to show that G is reconstructible. Lemma 1 and the remark following it guarantee the existence of a $w \in D$, $w \neq u$, r such that either $D - w$ contains exactly one 2-block which contains u or $D - w$ is a path. Therefore a connected graph $G - w$ in $\{G - v\}$ exists satisfying one of the following two sets of conditions: (1) The number of 2-blocks in G and $G - w$ are the same; there is a unique 2-block D' of $G - w$ such that all the tree-growths are rooted on D' and D' is a proper subgraph of D; (2) The number of 2-blocks in $G - w$ is one less than the number of 2-blocks in G, and there is tree-growth T in $G - w$ containing all 1-vertices. If (1) is the case, then G can be reconstructed from $G - w$ by replacing D' and all tree-growth rooted on it with D and the edge rx rooted on D. If (2) is the case, then G can be reconstructed from $G - w$ by replacing T with D and the edge rx rooted on D. In either case, G is reconstructible.

Case 1.3. G contains more than one 2-block and r is not on any end-block other than being a cutpoint of an end-block.

A vertex of a tree is *peripheral* if it is an end of a longest path. An end-block of G is a *peripheral-block* if it corresponds to a peripheral vertex of the block-cutpoint tree of *pruned*(G). Take a peripheral-block D such that D intersects the least number of other peripheral-blocks, and D has the least number of vertices among all such peripheral-blocks. Let the cutpoint on D be u, and let the number of peripheral-blocks containing u be n_u. Note that n_u and $v(D)$ are reconstructible, because *pruned*(G) is reconstructible. When D is not a cycle, Lemma 1 and the remark following it states that there is a $w \in D$ such that $D - w$ contains only one 2-block which contains u. When D is a cycle, there is a $w \in D$ such that $D - w$ is a path. Therefore we can find a $G - w'$ in $\{G - v\}$ satisfying one of the four following conditions:

(1) (corresponding to the case that D is not a cycle) $G - w'$ contains a cutpoint u' that belongs to exactly n_u peripheral-blocks, and among these peripheral-blocks there is one, denoted B, having less than $v(D)$ vertices;
(2) (corresponding to the case that D is a cycle and $n_u \geq 2$) $G - w'$ contains a cutpoint u' that belongs to exactly $n_u - 1 > 0$ peripheral-blocks and there is a path of length $v(D) - 2$, denoted B, rooted on u';
(3) (corresponding to the case that D is a cycle, $n_u = 1$ and the 2-block adjacent to D is nontrivial) $G - w'$ contains a path of length $v(D) - 2$, denoted B, with its root u' on an end-block, but u' is not a cutpoint of this end-block in *pruned*(G);
(4) (corresponding to the case that D is a cycle, $n_u = 1$, and the 2-block adjacent to D is an edge) $G - w'$ has a tree-growth T containing a path of length at least $v(D) - 1$.

If (1) is the case, then, since $v(D)$ is reconstructible, B, having fewer vertices than D, is not a peripheral-block of *pruned*(G). There is a bijection between the set of peripheral-blocks of *pruned*(G) and the set of peripheral-blocks of *pruned*$(G - w')$ such that all pairs of corresponding peripheral-blocks except one pair, denoted (D', B), are isomorphic with cut-point corresponding to cut-point under the isomorphism. Since *pruned*(G) is reconstructible, G can be reconstructed from $G - w'$, by replacing B and the path p rooted on B (if there is such a path) by D'. If (2) is the case, compare the set of 2-blocks of *pruned*$(G - w')$ with the set of 2-blocks of *pruned*(G). Let D' be the peripheral-block of *pruned*(G) that is missing from *pruned*$(G - w)$. Now G can be reconstructed from $G - w'$ by replacing B by D'. If (3) is the case, then B can always be distinguished from rx since r is not on any end-block other than being a cutpoint of an end-block. Hence G can be reconstructed from $G - w'$ by replacing B by a block D' exactly as in cases (1) and (2). Finally, if (4) is the case, let T be the unique tree-growth in $G - w'$ containing a path of length at least $v(D) - 1$. There is a 1-vertex y of $G - w'$ which is a peripheral vertex of T. Now G can be reconstructed by replacing a path of length $v(D) - 2$ containing y by a cycle of length $v(D)$.

Case 2. G contains more than one 1-vertex, all 1-vertices in the same branch B.

Case 2.1. *pruned*$(G) = P$, that is, G contains only one 2-block and B itself is a tree.

To show that Case 2.1. is recognizable, it is sufficient to show that the following is recognizable: all the 1-vertices are in the same branch. This can be shown as follows. If G has exactly two 1-vertices, x and y, and both $G - x$ and $G - y$ contain an edge as the unique branch, then 1-vertices are not in the same branch. Otherwise all the 1-vertices are in the same branch if and only if, in every $G - x$ where x is a 1-vertex, all 1-vertices are in the same branch.

To show that the branch B, with root labeled, is reconstructible, let the root of B be u. If P is a cycle, then G is a cactus which is reconstructible [4]. If P is not a cycle, by Lemma 1 and the remark following it, there is vertex $w \in P$, such that $P - w$ contains at most one 1-vertex and contains only one 2-block which contains u. Hence there must be a $G - w$ in the set $\{G - v\}$ satisfying the following conditions: (1) $w \in P$; (2) $G - w$ contains exactly one 2-block; (3) $G - w$ contains a branch B' which is a tree-growth and contains at least two 1-vertices. (It is possible that $G - w$ contains another branch which is a path and may or may not share the same root with B'. But B' can always be recognized in $G - w'$ as the only branch containing at least two 1-vertices.) The B' in $G - w$ is the B in G. Note that the conditions (1), (2) and (3) can be recognized since P is reconstructible. From any graph $G - w$ satisfying (1), (2) and (3), the branch B can be reconstructed since $B' = B$.

Case 2.2. G contains more than one 2-block.

To show that this case is recognizable, it is sufficient to show that the property that all the 1-vertices are in the same branch is recognizable. Consider the case that G contains more then two 1-vertices, or contains exactly two 1-vertices such that one of them is adjacent to a 2-vertex. Then all the 1-vertices of G are in the same branch if and only if, in each $G - x \in \{G - v\}$, where x is a 1-vertex, all the 1-vertices of $G - x$ are in the same branch. On the other hand, assume that G contains exactly two 1-vertices, denoted by x and y, and neither of them is adjacent to a 2-vertex. If both x and y are adjacent to a vertex v, not in *pruned*(G), then x and y are in the same branch. And this case can be recognized from $G - v$, since $G - v$ contains exactly two isolated vertices and *pruned*$(G) =$ *pruned*$(G - v)$.

If both x and y are adjacent to vertices of *pruned*(G), then the discussion is divided into three cases.

(1) At least one of x and y is adjacent to a vertex of the pruned center. In this case, by definition, x and y are in different branches. This case holds if and only if, for some $G - v \in \{G - v\}$ where v is a 1-vertex, there is a 1-vertex rooted on the pruned center.

In cases (2) and (3), assume that no 1-vertex is rooted on the pruned center.

(2) Both pendant edges are rooted on an end-block D, but not on the cutpoint u of D in *pruned*(G).

In this case, the two 1-vertices, x and y, are clearly on the same branch. This case can be recognized as follows.

(i) If (2) is the case, then the unique pendant edge in $G - x$ is rooted on an end-block D_x, but not on the cutpoint of D_x in *pruned*(G); in $G - y$, the unique pendant edge is rooted on an end-block, D_y, but not on the cutpoint of D_y in *pruned*(G); and $D_x \cong D_y$.
(ii) Conversely if G is a graph satisfying all conditions described in (i), then (2) is not the case for G if and only if there is w, such that $G - w$ has an isolated vertex and contains a 2-block $D' \cong D_x$, and there is a pendant edge rooted on D' but not rooted on its cutpoint in *pruned*$(G - w)$.

(3) Neither x nor y is adjacent to vertices of P, and there is no end-block D such that both x and y are adjacent to vertices of D other than its cut-point in *pruned*(G). In this case, whether or not all the 1-vertices are in the same branch, can be recognized as follows.
   (i) Let x and y be in the same branch B, and let D be any end-block of G in B. Let the cutpoint of D in *pruned*(G) be u, and let $v \in D$, $v \neq u$, be a root of a pendant edge, if such a root exists in D. Lemma 1 and the remark following it guarantees that there is $w \in D$, $w \neq u, v$, such that $D - w$ has only one 2-block which contains u, or $D - w$ is a path. $G - w$ contains no isolated vertex and, in $G - w$, all the 1-vertices are in the same branch, although the pruned center of $G - w$ and G may be different.
   (ii) On the other hand, if x and y are in different branches of G, then, for any end-block D of G, by Lemma 1 and the remark following it, there is a $w \in D$, where w is neither the cutpoint of D in *pruned*(G), nor a vertex joining x or y, such that $G - w$ satisfies the following conditions: (a) $G - w$ contains no isolated vertex; (b) either $D - w$ has exactly one 2-block which contains the cutpoint of D in *pruned*(G), or $D - w$ is a path. For *any* such D and w, there are 1-vertices in $G - w$ that lie in different branches. (Recall that no 1-vertex is adjacent to a vertex of *pruned*(G).)

From observations (i) and (ii), we conclude that all 1-vertices of G are in the same branch if and only if there is a connected graph (no isolated vertex) $G - w \in \{G - v\}$ satisfying the following conditions: (1) w is on an end-block D of G; (2) $D - w$ has only one 2-block which contains the cutpoint of D in *pruned*(G), or

$D - w$ is a path; (3) all the 1-vertices of $G - w$ are in the same branch. Hence Case 2.2 is recognizable, provided that whether or not a connected graph $G - w$ satisfies the following conditions (a) or (b) is recognizable: (a) w is on an end-block D of G, such that $D - w$ has only one 2-block which contains the cutpoint of D in *pruned*(G); (b) w is on an end-block D of G and $D - w$ is a path. Now $G - w$ satisfies (a) if and only if *block*(*pruned*$(G - w)$), the block-cutpoint tree of *pruned*$(G - w)$, is equal to *block*(*pruned*(G)), and the set of end-blocks of $G - w$ is not equal to the set of end-blocks of G. And $G - w$ satisfies (b) if and only if *block*(*pruned*$(G - w) \subset$ *block*(*pruned*(G)) and the number of 1-vertices plus the number of isolated vertices in $G - w$ is at most one more than the number of 1-vertices of G. Hence whether or not a connected graph $G - w$ satisfies the condition (a) or (b) is recognizable.

This concludes the proof that Case 2.2 is recognizable. To show that the set of branches is reconstructible, it is sufficient to show that B is reconstructible, since all other branches can be reconstructed from $G - x$, where x is a 1-vertex. Case 2.2 is now divided into four subcases which can be recognized by observing a $G - x$, where x is a 1-vertex.

Case 2.2.1. B contains a peripheral-block, and G contains at least three branches or contains exactly two branches B and B_1, where B_1 contains at least two end-blocks.

Consider $\{G - w | P_{G-w} = P\}$, where P_{G-w} is the pruned center of $G - w$. Note that whether or not $G - w$ and G have the same pruned center can be recognized by comparing the longest paths in their block-cutpoint trees and then comparing their pruned centers. For any $G - w \in \{G - w | P_{G-w} = P\}$, denote by B_w the unique branch of $G - w$ containing at least two 1-vertices, if such a branch exists. Choose a w so that B_w is maximum (with respect to number of vertices) in the set $\{B_w | P_{G-w} = P\}$. To show that there exists a w such that $P_{G-w} = P$ and that B_w exists, choose w as follows. Let D be a nonperipheral end-block not in B, or, in case all the end-blocks not in B are peripheral, let D be a peripheral-block not in B. Let the cut-point of D in *pruned*(G) be u and let $w \neq u$ be a vertex of D such that $D - w$ contains only one 2-block which contains u or $D - w$ is a path. For this w, $G - w$ satisfies the above requirement. Now the unique branch of $G - w$ containing at least two 1-vertices must be B, and thus B is reconstructed.

Case 2.2.2. B does not contain a peripheral-block.

In this case, G has at least three branches. Let D be peripheral-block; let u be the cut-point of D in *pruned*(G); and let $w \neq u$ be a vertex of D such that $D - w$ contains exactly one 2-block which contains u or $D - w$ is a path. That such a $G - w$ is recognizable can be proved as follows: As shown in the last part of the discussion on Case 2.2-(3)-(ii), whether or not w is in an end-block D such that $D - w$ contains only one 2-block which contains its cut-point u (in *pruned*(G)) or $D - w$ is a path is recognizable. Hence it only remains to prove that, whether or not D is peripheral, can be recognized. This is done by comparing the number of and length of longest paths in *block*(*pruned*$(G - w)$) and *block*(*pruned*(G)) and, in case they are the same, by further comparing the set of peripheral-blocks of $G - w$ with the set of peripheral-blocks of G.

Now the branch B can be reconstructed as follows. Let the unique branch of $G - w$ containing at least two 1-vertices be B_1. Then B_1 must contain B. This fact, along with the uniqueness of B_1, can be proved by a routine check of the following two cases: (a) there are more than two branches of G containing peripheral-blocks; (b) there are exactly two branches of G containing peripheral-blocks. To locate B in B_1, let x be a 1-vertex and let $B - x$ be the unique branch of $G - x$ containing a 1-vertex. Define an embedding f: $V(B - x) \to V(B_1)$ (which is an isomorphism between the graph $B - x$ and a subgraph of B_1) such that (i) if b is the root of $B - x$, then $f(b)$ is adjacent to a vertex not in $V(f(B - x))$; (ii) there is a vertex of $f(B - x)$ adjacent to a 1-vertex in $V(B_1) - V(f(B - x))$ and (iii) all the other vertices of $f(B - x)$ are not adjacent to any vertex not in $V(f(B - x))$. Now $f(b)$ must be the root of B. Hence B is located in B_1.

Case 2.2.3. G contains exactly two branches B and B_1, where B_1 contains exactly one end-block, and the pruned center P is a 2-block.

Let the two roots of B and B_1 be b and b_1, respectively. By Lemma 1 and the remark following it, there is a $G - w \in \{G - v\}$ satisfying the following condition (C0): $w \in P$, $w \neq b$, b_1, and $P - w$ contains exactly one 2-block which contains b or $P - w$ is a path. We will show that such a $G - w$ can be recognized in $\{G - v\}$. Such a $G - w$ must satisfy conditions C1 through C6 that follow. Moreover it will be shown that, first, for any $G - w' \in \{G - v\}$, $G - w'$ satisfies C0 if and only if $G - w'$ satisfies these six conditions, and second, whether or not $G - w'$ satisfies these six conditions is recognizable. This will prove that whether or not $G - w$ satisfying C0 can be recognized in $\{G - v\}$.

C1. $G - w$ has the following branches: (1) a branch B' that contains at least one (nontrivial) 2-block and at least two 1-vertices; (2) a branch B_1' that does not share a common vertex with B' if the 2-block of B_1' containing the root of B_1' is nontrivial. In addition B_1' contains at least one (nontrivial) 2-block and at most one 1-vertex (Note that B_1' consists of B_1 and possibly a subgraph of P; B_1 contains no 1-vertex; although $P - w$ may have two 1-vertices, by Lemma 1 one of them has to be b_1 thus cannot be a 1-vertex in $G - w$.); (3) possibly a branch which is a (nontrivial) 2-block (the unique 2-block in $P - w$) that has its root in common with B'; (4) possibly a branch that is a path. Moreover $G - w$ can have no branch other than those mentioned.

C2. The number of (nontrivial) 2-blocks in $G - w$ is no more than the number of (nontrivial) 2-blocks in G.

C3. $0 \leq v(B') - v(B)$ and $0 \leq v(B_1') - v(B_1) < v(P) - 2$. (Note: The reason for $v(P) - 2$ is that B_1 shares a vertex with P and w is not in B_1'.)

A set of 2-blocks $C_0, C_1, \ldots, C_n$ of G is called a chain between C_0 and C_n if for any $i > 0$, C_{i-1} and C_i have exactly one vertex in common.

C4. Let $k - 1$ be the number of 2-blocks (trivial or nontrivial) in a chain (of 2-blocks) between P and a peripheral-block in G. In $G - w$, there is a chain C of k 2-blocks (trivial or nontrivial) containing a peripheral-block D in B'.

(D is actually a peripheral-block D in B.) Let u be a vertex in C such that the distance between u and the pruned center of $G - w$ is minimum. (Here u is actually b.) Since B contains 1-vertices, it must be true that there is vertex of $V(C - u)$ adjacent to a vertex not in C.

C5. If $P - w$ contains exactly one (nontrivial) 2-block, then $u \in C$ (as described in C4) is contained in a (nontrivial) 2-block not in C.

C6. If $P - w$ contains no (nontrivial) 2-block, then (a) the number of (nontrivial) 2-blocks in $G - w$ is one less than that of G; (b) the number of 1-vertices in $G - w$ and G are equal or the number of 1-vertices in $G - w$ is one more than the number of 1-vertices in G, in which case there is a path rooted at b or b_1.

To show that the above conditions are sufficient for $G - w$ to satisfy the condition C0, we will show that if a $G - w$ satisfies the conditions C1 through C6, then $w \in P$, $w \neq b$, b_1 and $P - w$ contains exactly one 2-block which contains b, or $P - w$ is a path. Observe the following facts.

(1) Assume that w is not in P, but $G - w$ satisfies C1 through C4. If $P_{G-w} = P$, then we have $\nu(B_1') - \nu(B_1) < 0$ when $w \in B_1$, and $\nu(B') - \nu(B) < 0$ when $w \in B$, contradicting C3. Consider the case $P_{G-w} \neq P$. If $w \in B_1$, then $\nu(B_1') - \nu(B_1) < 0$ or $\nu(B_1') - \nu(B_1) > \nu(P) - 2$, contradicting C3. If $w \in B$ and $B_1 \subseteq B_1'$, then B_1' must contain P (note that B_1' cannot be equal to B_1 since $P_{G-w} \neq P$), contradicting C3. If $w \in B$ and $B_1 \subset B'$, take B_1 to be the chain C and take the root of B_1 (in G) to be u. There is no vertex of $V(C - u)$ adjacent to any vertex not in C, contradicting C4.

(2) Assume that $w \in P$ and that $G - w$ satisfies C1 through C6. If $P - w$ contains more that one (nontrivial) 2-block, then $G - w$ has more (nontrivial) 2-blocks than G does, contradicting C2. If $P - w$ contains exactly one (nontrivial) 2-block, then this 2-block must contain b because of C5. If $P - w$ contains no 2-block, then $P - w$ is a path because of C6.

We now show that, whether or not a $G - w$ satisfies C1 through C6, is recognizable. Whether a $G - w$ satisfies C1 through C4 is clearly recognizable. To show that whether a $G - w$ satisfies C5 and C6 is recognizable, it is sufficient to show that if $G - w$ satisfies C1 through C4, then whether or not $G - w$ satisfies C5 and C6 is recognizable. This can be shown as follows: As shown in (1), w must be in P. Whether $P - w$ contains exactly one (nontrivial) 2-block or no (nontrivial) 2-block can be recognized by comparing the number of (nontrivial) 2-blocks in $G - w$ with the number of (nontrivial) 2-blocks in G. Now b can be recognized from $G - w$ as follows. In $G - w$, consider a chain C of k 2-blocks (trivial or nontrivial) containing a peripheral-block D in B'. Then b must be the vertex in C such that the distance between b and the pruned center of $G - w$ is minimum. Similarly b_1 can be recognized by considering a chain C_1 of k 2-blocks (trivial or nontrivial) containing a peripheral-block D_1 in B_1'.

Finally, to show that B is reconstructible, let the only branch containing at least two 1-vertices in $G - w$ be B'. This B' must contain B. Exactly as in Case 2.2.2, B can be located in B'.

Case 2.2.4. G contains exactly two branches B and B_1, where B_1 contains exactly one end-block, and the pruned center P is a vertex or an edge.

This is the case excluded in Theorem 2.

Case 3. G contains more than one 1-vertex, not all in the same branch. Since Case 2 is recognizable, so is Case 3.

Let $\mathbf{B}' = \{B|B \text{ is a branch of some } G - y \text{ where } y \text{ is a 1-vertex}\}$. The following is an algorithm to obtain $\mathbf{B}$ from $\mathbf{B}'$.

Initialize m to the number of vertices in a largest graph in $\mathbf{B}'$. Repeat the following procedure until $m = 1$.

If there is no branch of m vertices in $\mathbf{B}'$, then $m \leftarrow m - 1$.

Let B be a branch with m vertices in $\mathbf{B}'$. Clearly $B \in \mathbf{B}$.

Let $\mathbf{B}' = \mathbf{B}' - \{B - y | y \text{ is } 1\text{-}vertex \text{ in } B\} - \{(k-1)B\}$, where k is the number of 1-*vertices* not in B. In another words, k is the number of times that B repeats in $\mathbf{B}'$.

At termination we clearly have $\mathbf{B}' = \mathbf{B}$. □

In the exceptional case of Theorem 2, the center P must be either a vertex or an edge. In this case, the reconstruction of the set of branches is equivalent to the reconstruction of the graph itself since, when the two branches are known, G can be reconstructed by identifying the roots of the two branches (in case the pruned center is a vertex), or by adding an edge between the two roots (in case the pruned center is an edge).

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