# Regular Combinatorial Maps 

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#### Abstract

The classical approach to maps is by cell decomposition of a surface. A combinatorial map is a graph-theoretic generalization of a map on a surface. Besides maps on orientable and non-orientable surfaces, combinatorial maps include tcsscllations, hypcrmaps, higher dimensional analogues of maps, and certain toroidal complexes of Coxeter, Shephard, and Grünbaum. In a previous paper the incidence structure, diagram, and underlying topological space of a combinatorial map were investigated. This paper treats highly symmetric combinatorial maps. With regularity defined in terms of the automorphism group, necessary and sufficient conditions for a combinatorial map to be regular are given both graphand group-theoretically. A classification of regular combinatorial maps on closed simply connected manifolds generalizes the well-known classification of metrically regular polytopes. On any closed manifold with nonzero Euler characteristic there are at most finitely many regular combinatorial maps. However, it is shown that, for nearly any diagram $D$, there are infinitely many regular combinatorial maps with diagram $D$. A necessary and sufficient condition for the regularity of rank 3 combinatorial maps is given in terms of Coxeter groups. This condition reveals the difficulty in classifying the regular maps on surfaces. In light of this difficulty an algorithm for generating a large class of regular combinatorial maps that are obtained as cyclic coverings of a given regular combinatorial map is given.


## 1. Introduction

A polytope is the convex hull of a finite set of points in Euclidean space $E^{n}$. The set of all proper faces of an $n$-dimensional polytope $P$ form a cell complex of dimension $n-1$, called the boundary complex of $P$. A map on a surface, i.e., a cell decomposition of a surface, is a topological generalization of the boundary complex of a polyhedron. In a previous paper [29], a generalization of a map, called a combinatorial map, was formulated in terms of edge colored graphs. Basic definitions are reviewed in Section 2. A related concept, called a crystallization, was independently investigated in a topological setting by Ferri and Gagliardi $[12,13]$. It was rediscovered by Lins [20], where it is called a graph-encoded map. Also related is the very

TABLE I
Regular Polytopes

| Polytope | Dimension | Schläfli symbol |
| :--- | :---: | :---: |
| $n$-Gon | 2 | $\{n\}$ |
| Simplex | $\geqslant 2$ | $\{3,3, \ldots, 3\}$ |
| Hypercube | $\geqslant 2$ | $\{4,3, \ldots, 3\}$ |
| Cross polytope | 3 | $\{3, \ldots, 3,4\}$ |
| Dodecahedron | 3 | $\{5,3\}$ |
| Isocahedron | 4 | $\{3,5\}$ |
| 24-Cell | 4 | $\{3,4,3\}$ |
| 120-Cell | 4 | $\{5,3,3\}$ |
| 600-Cell |  | $\{3,3,5\}$ |

important work of Tits [27,28] on buildings and subsequent work of Ronan [24,25] on coverings and Buekenout [4] on diagrams. These investigations are carried out in the more general context of geometries and chamber complexes, of which maps may be considered a "thin" case.

This paper treats highly symmetric combinatorial maps. The regular polytopes-those with sufficiently transitive automorphism group-have been completely classified and are listed in Table I with their Schläfli symbol [6]. In [22] McMullen defines combinatorial equivalence and combinatorial regularity of polytopes in terms of the poset of faces. McMullen then proves that the set of combinatorially regular polytopes essentially coincides with the set of metrically regular polytopes.

Theorem 1.1 (McMullen). Every combinatorially regular polytope is combinatorially equivalent to a metrically regular polytope.

In this paper we take a completely combinatorial point of view and investigate regular combinatorial maps. The treatment has the advantage of avoiding metric technicalities. In Section 3 regularity of a combinatorial map is defined in terms of its automorphism group. Basic properties of regular combinatorial maps are presented in Section 3: Necessary and sufficient conditions for a combinatorial map to be regular are given both graph and group theoretically. The automorphism group $\Gamma(G)$ of a regular combinatorial map $G$ is shown to be generated by involutions, and $G$ is precisely the Cayley graph of $\Gamma(G)$ with respect to this set of generators. Dual combinatorial maps generalize the dual of a polytope and various dual constructions of Coxeter and Moser [7] and Wilson [32]. The section concludes with a proof that every (finite) combinatorial map can be covered by a (finite) regular combinatorial map.

Section 4 treats regular combinatorial maps on spheres. The main result is




Fig. 1. Maps on a torus. Opposite sides of the two squares and two hexagons are to be identified; (a) $\left\{4,4 \mid\left(r_{0} r_{1} r_{2} r_{1}\right)^{3}\right\}, \quad$ (b) $\left\{4,4 \mid\left(r_{0} r_{1} r_{2}\right)^{4}\right\}, \quad$ (c) $\left\{6,3 \mid\left(r_{0} r_{1} r_{2}\right)^{6}\right\}$, (d) $\left\{6,3 \mid\left(r_{0} r_{1} r_{0} r_{1} r_{2}\right)^{4}\right\}$.
the classification of regular combinatorial maps on closed simply connected manifolds (Theorem 4.1). This generalizes the classification of metrically regular polytopes and also McMullen's Theorem 1.1.

In Section 5 the classification of maps on surfaces other than the sphere is investigated. The systematic study of regular maps on surfaces, those possessing the greatest degree of symmetry, goes back at least to Brahana $[2,3]$. The problem of classifying the finite regular maps has been pursued along three lines:
(a) By genus. The maps on surfaces of genus $0,1,2,3$, and on nonorientable surfaces of Euler characteristic $1 \geqslant \chi \geqslant-4$ have becn completely classified $[2,3,11,14,16-18,26]$. The regular maps on the sphere are exactly the boundary complexes of the five regular polyhedra. There are 6 infinite families of regular maps on a torus. A representative of four of these families is shown in Fig. 1. The other two families are duals of these.
(b) By families. Coxeter and Moser [7] have compiled lists of regular maps (not known to be complete) formed by identifying points of a hyperbolic tessellation that are $r$-steps separated along some type of path (e.g., a petrie path).
(c) By the embedded graph. The general question is, given a graph $G$, when does there exist a regular map with underlying graph $G$ ? Biggs [1] proves a result in this direction for the complete graph on $n$ points.

The classification of regular rank 3 combinatorial maps includes the long standing classification problem for regular maps on surfaces. We show in

Section 5 that the classification is equivalent to the determination of the normal, finitely generated, torsion free subgroups of the rank 3 Coxeter groups. Although no explicit classification exists for maps on surfaces, we give an algorithm in Section 7 for generating a large class of regular finite combinatorial maps obtained as cyclic coverings of a given regular combinatorial map.

Section 6 concerns the number of regular combinatorial maps on a given manifold or with a given diagram. In general there are at most finitely many regular combinatorial maps on any closed manifold with nonzero Euler characteristic. However, for nearly any diagram $D$, there are infinitely many finite regular combinatorial maps with diagram $D$. In particular, this answers a question of Grünbaum [19]: for $1 / p+1 / q \leqslant \frac{1}{2}$, are there infinitely many regular surface maps consisting of $p$-gons, $q$ of them surrounding each vertex?

## 2. Combinatorial Maps

Let $I$ be a finite set. A combinatorial map over $I$ is a connected graph $G$, regular of degree $|I|$, whose lines are $|I|$-colored such that no two incident lines are the same color. A combinatorial map may be finite or infinite. Let the function $\tau: E(G) \rightarrow I$, from the line set of $G$ to $I$, be the coloring. The image of a line or set of lines under $\tau$ is called its type. The rank of $G$ is $|I|$. An isomorphism of two combinatorial maps is a type preserving graph isomorphism. Automorphism is similarly defined. For $J \subseteq I$ two points of $G$ are $J$-adjacent ( $J$-adj) if they are joined by a path colored in $J$. Points that are $\{i\}$-adj are adjacent in the usual sense. For $J \subseteq I$ let $G_{J}$ be the subgraph of $G$ obtained by deleting all lines of type not in $J$. Each connected component of $G_{J}$ is a combinatorial map over $J$ and is called a residue of type $J$. The only residue of rank $|I|$ is $G$ itself. The residues of rank 0 are the points of $G$. The residues of type $I-\{i\}$ are called $i$-faces of $G$. Two distinct faces $x$ and $y$ are called incident if $x \cap y \neq \varnothing$. Let $X$ denote the set of faces of $G$; let $\tau: X \rightarrow I$ be defined by $\tau(x)=i$ if $x$ is an $i$-face; and let $*$ denote the incidence relation on faces. Then the triple $S(G)=(X, \tau, *)$ is referred to as the incidence structure of $G$.
To any combinatorial map $G$ is associated an ( $|I|-1$ )-dimensional simplicial complex $\Delta G$ as follows. For each point $v$ in the point set $V(G)$ of $G$, let $\Delta v$ be a simplex of dimension $|I|-1$. Arbitrarily assign to each vertex of $\Delta v$ a distinct element of $I$. Call the set of elements assigned to a face $s$ of $\Delta v$ the type of $s$. Let $K$ be the disjoint union of the set $\{\Delta v \mid v \in V(G)\}$. In $K$ identify two simplexes $s \subseteq \Delta v$ and $s^{\prime} \subseteq \Delta v^{\prime}$ of same type $J$ if and only if $v$ and $v^{\prime}$ are $(I-J)$-adjacent. If $\sim$ denotes this identification, take $\Delta G=K / \sim$. Intuitively, $\Delta G$ can be thought of as being built from $(|I|-1)$-simplexes, one for each point of $G$, such that two $(|I|-1)$-simplexes share a common
codimension 1 face if the corresponding points are adjacent in $G$. The space $|G|=|\Delta G|$ is called the underlying topological space of $G$.

Example. Let $K$ be a cell complex with underlying topological space $|K|$. If $|K|$ is a connected manifold without boundary, then $K$ will be called a map on a manifold. In particular, if $|K|$ is a surface, then $K$ is called a map on a surface. Given a map $K$ on a manifold, a combinatorial map $G(K)$ is obtained as the dual 1 -skeleton of the barycentric subdivision of $K$. Each vertex of the barycentric subdivision can be labeled with the dimension of the cell it represents. A line of $G(K)$ is then colored $i$ if it joins two maximal simplexes whose labels differ only by $i$. An $i$-face of the incidence structure $S(G)$ corresponds to an $i$-cell of $K$. Note that $G$ and $K$ have the same underlying topological space. If $K$ is a map on a manifold, then the combinatorial map obtained is denoted $G(K)$.

The following definitions are discussed in greater detail in [29]. Let $G$ be a combinatorial map over $I$ and $R$ a rank 2 residue over $\{i, j\}$. If $R$ is finite then it is a cycle in $G$ consisting of lines alternately colored $i$ and $j$. Let $p(R)$ be half the length of this cycle. If $R$ is infinite then $p(R)=\infty$. If the automorphism group of $G$ acts transitively on the points of $G$, then the value of $p(R)$ is the same for all residues of type $\{i, j\}$. In general define $p_{i j}=$ lem $p(R)$ where the least common multiple is taken over all residues of type $\{i, j\}$. The diagram $D(G)$ of $G$ is obtained by representing each $i \in I$ as a node labeled $i$ and connecting nodes $i$ and $j$ by a line labeled $p_{i j}$. By convention the line is omitted when $p_{i j}=2$ and the line label is omitted when $p_{i j}=3$. The diagram is a generalization of the Schläfli symbol of a regular polytope.

Let $G_{1}$ and $G_{2}$ be combinatorial maps over disjoint sets $I_{1}$ and $I_{2}$ with point sets $V\left(G_{1}\right)$ and $V\left(G_{2}\right)$. The product $G_{1} * G_{2}$ is a combinatorial map over $I_{1} \cup I_{2}$ with point set $V\left(G_{1}\right) \times V\left(G_{2}\right)$. Two points $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are $i$-adj whenever $\left[u_{1}=v_{1}\right.$ and $u_{2} i$-adj $\left.v_{2}\right]$ or $\left[u_{2}=v_{2}\right.$ and $u_{1} i$-adj $\left.v_{1}\right]$. This is the standard product construction for graphs, together with the appropriate line coloring. A combinatorial map is called reducible if it is isomorphic to the product of two other combinatorial maps. Otherwise it is irreducible. It is not hard to see that if $D(G)$ is connected, then $G$ is irreducible.

A combinatorial map $G$ is called ordered if there is a partial order $>$ on the faces of its incidence structure $S(G)$ such that $x * y$ if and only if $x>y$ or $y>x$ for all faces $x$ and $y$. For many classical examples, like boundary complexes of polytopes and maps on surfaces, the faces are partially ordered by inclusion. In [29] it is shown that if $D(G)$ is linear, i.e., of the form

then $G$ is ordered. A more general result characterizing the spaces of incidence structures belonging to linear diagrams is due to Buekenhout [4].

Ramified coverings have been studied by both Tits [28] and Ronan $[24,25]$ in the more general setting of chamber systems. The concept of homotopy subsequently used in this paper was introduced by Tits. Numerous results on coverings of chamber systems, including a generalization of our Corollary 3.2, are proved by Ronan [24]. In the context of this paper, consider combinatorial maps $G$ and $G^{\prime}$ over $l$. For a non-negative integer $m$, an $m$-covering $G^{\prime} \rightarrow G$ is a function $f: V\left(G^{\prime}\right) \rightarrow V(G)$ that preserves $i$-adjacency for all $i \in I$ and is bijective when restricted to rank $m$ residues. By a covering we mean an $m$-covering for some $m>0$. The covering $f$ naturally induces a topological map $|f|:\left|G^{\prime}\right| \rightarrow|G|$, and an ( $|I|-1$ )-covering induces a topological covering of the underlying topological spaces. For $u \in V(G)$ the set $f^{-1}(u)$ is called the fiber above $u$. Any two fibers have the same cardinality, and if this cardinality is $d$, we say that $f$ is a $d$-fold covering. The group of automorphisms of $G^{\prime}$ preserving each fiber is called the group of covering transformations of $f$. Two coverings $f: G_{1}^{\prime} \rightarrow G_{1}$ and $g$ : $G_{2}^{\prime} \rightarrow G_{2}$ are called equivalent if there exists isomorphisms $\theta: G_{1}^{\prime} \rightarrow G_{2}^{\prime}$ and $\phi$ : $G_{1} \rightarrow G_{2}$ such that $\phi \circ f=g \circ \theta$.

In [29] it is shown that any combinatorial map is isomorphic to a Schreier coset graph of a group $W$ generated by involutions $\left\{r_{i} \mid i \in I\right\}$. Recall that if $H$ is a subgroup of $W$, the Schreier coset graph $G(W, H)$ is an $I$-colored graph defined as follows. The points of $G(W, H)$ arc the right cosets of $W / H$ and two points $u$ and $u^{\prime}$ are $i$-adj if and only if $u^{\prime}=u r_{i}$. If $H$ is the trivial subgroup of $W$, the Schreier coset graph coincides with the Cayley graph of $W$. If $G \simeq G(W, H)$ then $G(W, H)$ is called a Schreier representation of the combinatorial map $G$. The group $W$ can be taken to be $\left\langle r_{i}, i \in I \mid r_{i}^{2}=1\right\rangle$, in which case $G(W, H)$ is called the canonical representation.

## 3. Regularity

Let $G$ be a combinatorial map over $I$. If the automorphism group $\Gamma(G)$ acts transitively on the set of points of $G$ then we say that $G$ is regular. This is a strong requirement on the symmetry of $G$. For a map $K$ on a surface, each point of $G(K)$ corresponds to a triple $\left\{f_{0}, f_{1}, f_{2}\right\}$ consisting of pairwise incident 0,1 , and 2 faces. Hence to be regular, it is not only necessary that the automorphism group act transitively on 0,1 , and 2 faces, but it must act transitively on the set of triples $\left\{f_{0}, f_{1}, f_{2}\right\}$. For polytopes and maps on surfaces "regular" is standard terminology. The term "flag transitivity" is usual in examples arising from algebraic and finite simple groups. The combinatorial maps associated with the polytopes listed in Table I are all regular. Other familiar examples of regular combinatorial maps arise from
regular tessellations of the plane (Euclidean of hyperbolic) and regular maps on surfaces $[1,6,20]$. Example 4 in [29] and the examples of $[8,19]$ yield more cxotic regular combinatorial maps.

Let $T=i_{1} i_{2} \cdots i_{m}$ be a sequence of elements of $I$. Such a sequence is referred to as an $I$-sequence. A path of type $T$ in a combinatorial map $G$ is a path whose lines are colored successively $i_{1}, i_{2}, \ldots, i_{m}$. The combinatorial map $G$ is called homogeneous if it has the following property. For any $I$-sequence $T$ if some path of type $T$ is closed in $G$, then all paths of type $T$ are closed. In other words, homogeneous means that whether or not a path of type $T$ is closed is independent of the base point.

Theorem 3.1. Let $G$ be a combinatorial map with Schreier representation $G(W, H)$. The following statements are equivalent.
(1) $G$ is regular,
(2) $\Gamma(G)$ acts regularly on $V(G)$,
(3) $H$ is normal in $W$,
(4) $G$ is homogeneous.

Proof. (1) $\Leftrightarrow$ (2). If $f \in \Gamma(G)$ and $f(v)=v$ for some $v \in V(G)$, then by the connectivity of $G, f$ is the identity.
(1) $\Leftrightarrow$ (3) By Theorem 7.5 in [29] $\Gamma(G)$ acts transitively on $V(G)$ if and only if $H \unlhd W$.
$(1) \Leftrightarrow(4)$. Consider $\Gamma(G)$ as a permutation group acting on $V(G)$. Let $P$ be the permutation group on $V(G)$ generated by involutions $\left\{\rho_{i} \mid i \in I\right\}$, where $\rho_{i} v=v^{\prime}$ if and only if $v i$-adj $v^{\prime}$. Then $\Gamma(G) \simeq C_{\Sigma}(P)$, where $C_{\Sigma}(P)$ is the centralizer of $P$ in the full symmetric group $\Sigma$ on $V(G)$. By a standard result in the theory of permutation groups $C_{\Sigma}(P)$ acts transitively if and only if $P$ acts sharply. But the latter condition is equivalent to the homogeneity of $G$.

Corollary 3.2. If $G$ is a regular combinatorial map with Schreier representation $G(W, H)$ then $H \unlhd W$ and $\Gamma(G) \simeq W / H$.

Proof. By Theorem 7.5 of $[29] \Gamma(G) \simeq N_{W}(H) / H$. So the result is an immediate consequence of Theorem 3.1.

THEOREM 3.3. Let $G$ be a regular combinatorial map over $I, v$ any point of $V(G)$ and $R_{i}$ the unique automorphism taking $v$ to the point $i$-adj to it. Then each $R_{i}$ is an involution and $G$ is isomorphic to the Cayley graph of $\Gamma(G)$ with respect to the generators $\left\{R_{i} \mid i \in I\right\}$.

Proof. Let $G(W, H)$ be a Schreier representation of $G$ with respect to the set of generators $\left\{r_{i} \mid i \in I\right\}$. By Corollary 3.2 there is an isomorphism
$W / H \rightarrow \Gamma(G)$. Since an automorphism of $G$ is determined by its action on one point, $R_{i}$ is the image of $\mathrm{Hr}_{i}$. The theorem follows from the fact that $G$ is the Schreier coset graph of $W$ with respect to $H$.

Further examples of regular combinatorial maps can be obtained as duals of known regular combinatorial maps. The usual notion of duality for polyhedra reverses the role of vertices and faces. For instance, the cube and octahedron are dual to each other. We give a generalization that produces numerous duals from a single combinatorial map. Let $G$ be a combinatorial map over $I$ and $T=i_{1} i_{2} \cdots i_{m}$ an $I$-sequence. Call an $I$-sequence $T$ an involution if every path of type $T^{2}:=i_{1} i_{2} \cdots i_{m} i_{1} i_{2} \cdots i_{m}$ is closed. Let $\beta=$ $\left\{T_{i} \mid i \in I\right\}$ be an $I$-indexed set of involutions. Define a new $I$ labeled graph $G_{\beta}$ as follows. The point set of $G_{\beta}$ is $V(G)$, and two points $u$ and $u^{\prime}$ are $i$-adj in $G_{B}$ if and only if $u$ and $u^{\prime}$ are connected by a path of type $T_{i}$ in $G$. If $G_{\beta}$ is connected then it is a combinatorial map which is called the $\beta$-dual of $G$. The following proposition follows immediately from the definitions.

Proposition 3.4. If $G$ is a regular combinatorial map and $G_{\beta}$ is the $\beta$ dual of $G$, then $G_{\beta}$ is also regular.

Example 1. Let $\pi$ be a permutation of $I$ and take

$$
\beta=\left\{\pi^{-1}(i) \mid i \in I\right\} .
$$

In other words $G_{\beta}$ is obtained from $G$ by permuting the colors. This will be referred to as a permutation dual or $\pi$-dual of $G$. Note that the permutation duals of $G$ have the same underlying topological space as $G$. If the cube is considered as a combinatorial map over $\{0,1,2\}$, then the ( 02 )-dual is the octahedron.

Example 2. Suppose that $j$ and $k$ are non-adjacent nodes in the diagram of a regular combinatorial map $G$ over $I$. Then take $\beta=\left\{T_{i} \mid i \in I\right\}$, where

$$
\begin{aligned}
T_{i} & =j k & & \text { if } \quad i=j \\
& =i & & \text { otherwise }
\end{aligned}
$$


a

b

FIG. 2. Construction of a dual combinatorial map; (a) $G$, (b) $G_{\beta}$.


Fig. 3. Dual maps of the cube. (a) $\left\{6,3 \mid\left(r_{0} r_{1} r_{2}\right)^{4}\right\} g=0$; (b) $\left\{4,6 \mid\left(r_{0} r_{1} r_{2}\right)^{3}\right\} g=3$; (c) $\left\{3,6 \mid\left(r_{2} r_{1} r_{0}\right)^{4}\right\} g=0$; (d) $\left\{6,4 \mid\left(r_{2} r_{1} r_{0}\right)^{3}\right\} g=3$.

To form $G_{\beta}$ the graph $G$ is altered as shown in Fig. 2. It is apparent that $G_{\beta}$ is connected so that $G_{\beta}$ is a combinatorial map. This is a generalization of a special construction of Coxeter and Moser [7, p. 112] involving Petrie polygons.

Besides the octahedron and the cube itself, four other $\beta$-duals of the cube can be realized as maps on surfaces. These are shown in Fig. 3 where likenumbered edges are to be identified. In this figure, $G_{1}$ and $G_{2}$ are duals of the cube by an application of Example 2 and $G_{3}$ and $G_{4}$ are duals of $G_{1}$ and $G_{2}$ by an application of Example 1. The symbol $g$ denotes the genus of the surface. These also appear in [32] where they are called derivatives.

THEOREM 3.5. Every combinatorial map $G$ has a covering $G^{\prime} \rightarrow G$ by a regular combinatorial map $G^{\prime}$. If $G$ is finite, then $G^{\prime}$ can also be chosen finite.

Proof. Let $G(W, H)$ be a Schreier representation of $G$. Furthermore let $N=\bigcap_{a \in W} a^{-1} H a$ and $G^{\prime}=G(W, N)$. Since $N$ is normal in $W, G^{\prime}$ is regular by Theorem 3.1. The assignment $N a \mapsto H a$ defines a covering $G^{\prime} \rightarrow G$. The second statement in the theorem follows from the fact that $N$ is the kernel of the map $\phi: W \rightarrow \Sigma_{W / H}$ from $W$ to the symmetric group on the elements of $W / H$ given by $\phi w(H a)=H a w$.

## 4. Regular Combinatorial Maps on Simply Connected Manifolds

Recall that a Coxeter group over $I$ is a group generated by involutions with presentation

$$
\begin{equation*}
\left\langle r_{i}, i \in I \mid\left(r_{i} r_{j}\right)^{p_{i i}}=1, p_{i i}=1, p_{i j} \geqslant 2, i \neq j\right\rangle . \tag{4.1}
\end{equation*}
$$

We do not eliminate the possibility that $p_{i j}=\infty$ in which case the relation $\left(r_{i} r_{j}\right)^{p i j}=1$ is absent. By abuse of language we use the term Coxeter group to refer to the presentation (4.1). The diagram for a Coxeter group is

TABLE II
Finite Irreducible Coxeter Groups

| Group | Diagram | Order | Automorphism Group of |
| :---: | :---: | :---: | :---: |
| $A_{n}$ | --..-.......... | $(n+1)$ ! | Simplex |
| $B_{n}$ | .. 4 | $2^{n} n$ ! | Hypercube |
|  |  |  | Cross polytope |
| $D_{n}$ |  | $2^{n-1} n!$ |  |
| $E_{6}$ |  | 51980 |  |
| $E_{7}$ |  | 2903040 |  |
| $E_{8}$ | ! | 696729600 |  |
| $F_{4}$ | -. ${ }^{4} .-$ | 1152 | 24-Cell |
| $\mathrm{H}_{3}$ | 5. | 120 | Dodecahedron |
|  |  |  | Tcosahedron |
| $\mathrm{H}_{4}$ | -.-. 5. | 1440 | 120-Cell |
|  |  |  | 600-Cell |
| $I_{2}^{\prime}$ | . ${ }^{\text {r }}$. | $2 r$ | $r$-Gon |

constructed by representing each $r_{i}$ as a node labeled $i$ and connecting nodes $i$ and $j$ by a line labeled $p_{i j}$. By convention the line is omitted when $p_{i j}=2$ and the line label is omitted when $p_{i j}=3$. A Coxeter group is uniquely determined by its diagram. If the diagram $D$ of a Coxeter group $W$ is disconnected, then $W$ is the direct product of Coxeter groups corresponding to the components of $D$. Hence a Coxeter group is said to be reducible if its diagram is disconnected. Otherwise it is irreducible. Coxeter [5] classified all finite irreducible groups with presentation (4.1) and identified some as the automorphism groups of the regular polytopes. The finite irreducible Coxeter groups are listed in Table II. It has been shown [29] that every combinatorial map $G$ has a Schreier representation $G(W, H)$, where $W$ is a Coxeter group with the same diagram as $G$. This representation will be called the Coxeter representation.

Consider the special class of combinatorial maps of the form $G(W):=$ $G(W,\{1\})$, where $W$ is a Coxeter group and $\{1\}$ is the trivial subgroup. By Theorem 3.1 these combinatorial maps are regular. For example, if $W$ is the Coxeter group with diagram $\cdot r$. then for $r=3, G(W)$ is a tetrahedron, i.e., $G(W,\{1\})=G(K)$, where $K$ is a tetrahedron. If $r=4$ then $G(W)$ is a cube. If $r=5$ then $G(W)$ is a dodecahedron. If $r=6$ then $G(W)$ is the tessellation of the Euclidean plane into regular hexagons. If $r>6$ then $G(W)$ is the tessellation of the hyperbolic plane (open unit disk) into regular $r$-gons, 3 of them surrounding each vertex. For $W$ a finite irreducible Coxeter group we shall call a combinatorial map of the form $G(W)$ a Coxeter map. Among the Coxeter maps are all maps $G(K)$, where $K$ is the boundary complex of a regular polytope. However, by Theorem 4.2 of [29], the Coxeter maps corresponding to $E_{6}, E_{7}, E_{8}$ and $D_{n}, n \geqslant 4$ are not ordered, hence not associated with any polytope.

The next theorem is a generalization of the classification of regular polytopes and also of McMullen's Theorem 1.1. The proof has the advantage of avoiding metric technicalities.

Theorem 4.1. Let $G$ be a regular irreducible combinatorial map whose underlying topological space is a closed simply connected manifold. Then $G$ is a Coxeter map.

Proof. Let $G$ be a regular irreducible combinatorial map such that $|G|$ is a simply connected closed manifold. Let $G(W, H)$ be the canonical Schreier representation of $G$, i.e., $W=\left\langle r_{i}, i \in I \mid r_{i}^{2}=1\right\rangle$. For $J \subseteq I$ let $W_{J}$ be the subgroup of $W$ generated by $\left\{r_{i} \mid i \in J\right\}$ and let $H_{m}$ be the smallest normal subgroup of $W$ containing the subgroups $\left\{H \cap W_{J}| | J|=m\rangle\right.$. By [29, Corollary 6.3 and Theorem 7.8| $0=\pi^{2}(G) \simeq H / H_{2}$. Therefore $H=H_{2}$ and $G \simeq G\left(W, H_{2}\right)$. Since $G$ is regular $H_{2} \unlhd W$ by Theorem 3.1. Hence $G \simeq$ $G\left(W, H_{2}\right) \simeq G\left(W / H_{2},\{1\}\right)$. But $W^{\prime}=W / H_{2}$ is a Coxeter group. Since $G$


Fig. 4. A chiral map on the torus. Opposite sides of the square are to be identified.
and $W^{\prime}$ have the same diagram, $G$ irreducible implies $W^{\prime}$ irreducible by the arguments of $[29$, Theorem 4.1]. Because $|G|$ is a closed manifold, $\Delta G$ has finitely many maximal simplexes, and hence $W^{\prime}$ is finite. Therefore $G$ $G\left(W^{\prime}\right)$ is a Coxeter map.

We remark that for any Coxeter map $G$ the underlying topological space $|G|$ is actually homeomorphic to a sphere.

Let $G$ be a combinatorial map over $I$ and $J \subseteq I$. If $\Gamma(G)$ acts transitively on the set of residues of type $I-J$, we say that $G$ is $J$-regular. Thus an $I$ regular combinatorial map is regular in the usual sense. If $G$ is $\{i\}$-regular, then $\Gamma(G)$ acts transitively on the set of $i$-faces of $G$. It is not necessarily true that if $G$ is $J$-regular for every proper subset $J$ of $I$, then $G$ is regular. Extending the terminology used by Wilson $\lfloor 33\rfloor$ for maps on surface, a nonregular combinatorial map that is $J$-regular for every proper subset $J$ of $I$ will be called chiral. The map $K$ in Fig. 4 consisting of 5 vertices, 10 edges and 5 squares on a torus yields a chiral combinatorial map $G(K)$.

Theorem 4.2. (1) There does not exist a chiral map withnonorientable underlying topological space.
(2) There does not exist a chiral map with underlying topological space that is a simply connected manifold.

Proof. If $G$ is chiral, then $\Gamma(G)$ has exactly 2 orbits in its action on $V(G)$, and adjacent points belong to different orbits. Hence $G$ is bipartite. By [29, Theorem 6.1] $|G|$ is orientable.

To prove statement (2) let $G$ be a chiral map with canonical Schreier representation $G(W, H)$. Theorem 7.4 of [29] implies that $N_{W^{\prime}}(H)$ must be the "even subgroup" of $W$, i.e., the words of even length in the generators $\left\{r_{i} \mid i \in I\right\}$. Thus if $a\left(r_{i} r_{j}\right)^{k} a^{-1}$ is an element of $H$, then $\left(r_{i} r_{j}\right)^{k} \in H$ in case $a$ is even and $\left(r_{j} r_{i}\right)^{k}=r_{i}\left(r_{i} r_{j}\right)^{k} r_{j} \in H$ in case $a$ is odd. In either case both
$\left(r_{i} r_{j}\right)^{k}$ and its inverse $\left(r_{j} r_{i}\right)^{k}$ lie in $H$. If $a$ is odd $r_{m} a\left(r_{i} r_{j}\right)^{k} a^{-1} r_{m} \in H$ and if $a$ is even $r_{m} a\left(r_{i} r_{j}\right)^{k} a^{-1} r_{m}=r_{m} a r_{j}\left(r_{j} r_{i}\right)^{k} r_{j} a^{-1} r_{m} \in H$ for any $m \in I$. Since $|G|$ is a simply connected manifold, [29, Corollary 6.3 and Thcorem 7.8] imply that $H$ is generated by elements of the form $a\left(r_{i} r_{j}\right)^{k} a^{-1}$. Thus $r_{m} \in$ $N_{W}(H)$ for all $m \in I$. Therefore $H \leq W$ and $G$ is regular by Theorem 3.1. This contradicts the assumption that $G$ is chiral.

Call a combinatorial map over $I n$-regular if every residue of rank $n$ is regular. Note that the rank $n$ residues are not required to be isomorphic to each other. We close this section with the following open question: Classify the spherical combinatorial maps over $I$ that are $|I|-1$ regular. By spherical we mean that the underlying topological space is a sphere. Since all spherical rank 3 combinatorial maps are 2 -regular, we are interested in the cases where $|I| \geqslant 4$.

## 5. Regular Rank 3 Combinatorial Maps

Let $G$ be a regular combinatorial map and $G(W, H)$ the Coxeter representation of $G$, i.e., $W$ is a Coxeter group with the same diagram as $G$. Since $H$ is normal in $W$, let $\mathscr{R}$ be a set of relations that normally generate $H$. That is, $H$ is the normal closure of $\mathscr{R}$ in $W$. The diagram $D:=D(G)$ and the set $\mathscr{R}$ uniquely determine $G$. Hence, for a regular combinatorial map $G$ we use the notation $\{D \mid \mathscr{R}\}$.

If $D(G)$ is linear

then this notation is abbreviated $\left\{p_{1}, p_{2}, \ldots, p_{n} \mid \mathscr{R}\right\}$. If $\mathscr{R}$ is empty this symbol reduces to $\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$. The regular combinatorial maps, $\{r\}$ $r \geqslant 2,\{3,3, \ldots, 3\},\{3, \ldots, 3,4\},\{4,3, \ldots, 3\},\{3,5\},\{5,3\},\{3,4,3\},\{3,3,5\}$ and $\{5,3,3\}$ correspond to the regular polytopes. In this case the notation is just the classical Schläfli notation for the regular polytopes.

A regular tessellation of the plane (spherical, Euclidean, or hyperbolic) is an arrangement of regular $p$-gons, $q$ surrounding each vertex, fitting together to cover the plane without overlapping. The regular combinatorial maps $\{p, q\}$ correspond to the tessellations of the appropriate plane: spherical if $1 / p+1 / q>\frac{1}{2}$, Euclidean if $1 / p+1 / q=\frac{1}{2}$ or hyperbolic if $1 / p+1 / q<\frac{1}{2}$.

According to $[29$, Theorem 5.2] the regular combinatorial maps $\{p, q \mid \mathscr{R}\}$ are exactly the classical regular maps on surfaces. For instance the combinatorial maps in Fig. 1 are regular and have the symbols

$$
\begin{array}{ll}
\left\{4,4 \mid\left(r_{0} r_{1} r_{2} r_{1}\right)^{3}\right\}, & \left\{4,4 \mid\left(r_{0} r_{1} r_{2}\right)^{4}\right\}, \\
\left\{6,3 \mid\left(r_{0} r_{1} r_{2}\right)^{6}\right\}, & \left\{6,3 \mid\left(r_{0} r_{1} r_{0} r_{1} r_{2}\right)^{4}\right\}
\end{array}
$$

Let $G=\{D \mid \mathscr{R}\}$ be a regular rank 3 combinatorial map. Here the diagram has the form


Diagram 2

The classification of regular rank 3 combinatorial maps includes the classification of regular maps on surfaces, a long standing problem. We will not attempt a review of the vast literature on this subject, but refer the reader to [7]. We divide the rank 3 combinatorial maps $G$ into three classes according to the universal cover of the underlying topological space $|G|$ :
(1) spherical, if $\chi(|G|)>0$,
(2) Euclidean, if $\chi(|G|)=0$,
(3) hyperbolic, if $\chi(|G|)<0$.

If $G$ is spherical then $|G|$ is either the sphere or the projective plane. By Theorem 4.1 the irreducible regular combinatorial maps with $|G|$ homeomorphic to a 2 -sphere are exactly the boundary complexes of the regular polyhedra and permutation duals of these. In addition there is the family $\{p, 2\}, p \geqslant 2$ of reducible maps and their duals. If $|G|$ is the projective plane, then Theorem 5.1 implies that $G$ has a 2 -fold covering whose underlying topological space is a 2 -sphere. The only such possibilities for $G$ are $\left\{3,4 \mid\left(r_{0} r_{1} r_{2}\right)^{3}\right\},\left\{3,5 \mid\left(r_{0} r_{1} r_{2}\right)^{5}\right\}$ and duals. The associated maps can be obtained by identifying opposite points on the boundary complex of the octahedron and icosahedron.

THEOREM 5.1. Every non-orientable regular combinatorial map has a unique 2-fold covering by a regular orientable combinatorial map.

Proof. Let $G$ be a non-orientable regular combinatorial map with canonical Schreier representation $G(W, H)$. Let $H^{\prime}$ be the subgroup of $H$ consisting of elements that can be expressed as even words in the generators $\left\{r_{i} \mid i \in I\right\}$, and let $G^{\prime}=G\left(W, H^{\prime}\right)$. By Theorem 6.1 and 7.5 of [29], the canonical covering $f: G^{\prime} \rightarrow G$ given by $H^{\prime} a \mapsto H a$ for all $a \in W$ is a 2-fold covering by an orientable combinatorial map $G^{\prime}$. To show that $G^{\prime}$ is regular we must prove that $H^{\prime} \unlhd W$. Let $h^{\prime} \in H^{\prime}$ and $a \in W$. Then $a h^{\prime}=h a$ for some $h \in H$ because $H \unlhd W$. Therefore $a h^{\prime} a^{-1} h^{-1}=1$. If the parity of $h$ and $h^{\prime}$ are opposite, then we have an odd word equal to an even word in $W$. Since this is impossible, $h \in H^{\prime}$. The uniqueness of the covering follows from the fact that any subgroup $N$ of index 2 in $H$, other than $H^{\prime}$, must contain an odd word, in which case $G(W, N)$ would not be orientable.

Wilson [31] proved Theorem 5.1 geometrically for the case of maps on
surfaces and also gave an example of two non-orientable maps with the same 2 -fold orientable cover.

If $G=\{D \mid \mathscr{R}\}$ is Euclidean and finite, then $|G|$ is cither a torus or a Klein bottle. By applying Lemma 5.2 below to the rank 3 case we have $1 / p+1 / q+$ $1 / r=1$. Hence the possibilities for ( $p, q, r$ ) in Diagram 2 are limited to $(2,4,4),(2,3,6),(3,3,3)$ and permutations of these. The normal subgroups of the Coxeter groups with these diagrams can be realized as well understood [23] discrete groups of isometries of the Euclidean plane. This leads to a complete (and well known in the case of maps on surfaces [7]) classification of the regular combinatorial maps $G$, where $|G|$ is a torus,

$$
\begin{aligned}
& \left\{4,4 \mid\left(r_{0} r_{1} r_{2}\right)^{2 m}\right\}, \quad\left\{4,4 \mid\left(r_{0} r_{1} r_{2} r_{1}\right)^{m}\right\}, \\
& \left\{3,6 \mid\left(r_{0} r_{1} r_{2}\right)^{2 m}\right\}, \quad\left\{3,6 \mid\left(r_{0} r_{1} r_{2} r_{1} r_{2}\right)^{2 m}\right\}, \\
& \left\{\cdot \mid\left(r_{0} r_{1} r_{2}\right)^{2 m}\right\}, \quad\left\{. \quad \mid\left(r_{0} r_{1} r_{2} r_{1}\right)^{m}\right\}, \quad m \geqslant 1
\end{aligned}
$$

and permutation duals. Theorem 5.1 can be applied to check that no regular combinatorial maps cxist on the Klein bottle.

Lemma 5.2. If $G$ is a finite regular combinatorial map over I then

$$
\chi(|G|)=|V(G)| \sum_{J \subsetneq I} \frac{(-1)^{(|I-J|-1)}}{p_{J}}
$$

where $p_{J}$ is the number of points in any residue of type $J$ in $G$ and $\chi$ is the Euler characteristic.

Proof. The simplexes of type $J$ in $\Delta G$ correspond to the residues of type $I-J$ in $G$. If one simplex of type $J$ is counted for each chamber of $\Delta G$, then each simplex is counted exactly $p_{I-J}$ times. Hence there are $p_{I} / p_{I-J}$ simplexes of type $J$ in $\Delta G$. The formula for $\chi(|G|)$ follows.

Coxeter and Moser [7] have compiled long lists of regular hyperbolic maps on surfaces, but no general classification exists. The following result indicates the difficulty in obtaining such a classification.

Theorem 5.3. There is a ane-to-one correspondence between rank 3 finite regular hyperbolic combinatorial maps with diagram $D$ and torsion free, finitely generated, normal subgroups of the Coxeter group with diagram $D$.

Proof. Let $W$ be the rank 3 Coxeter group with diagram not among


We claim that the function given by $f: H \mapsto G(W, H)$ is the desired correspondence. Since $H$ is a normal subgroup of $W, G(W, H)$ is regular, and since $H$ is torsion free, $G(W, H)$ also has diagram $D$. To show that $G(W, H)$ is finite, let $H^{\prime}$ be the even subgroup of $H$ and $W^{\prime}$ the' even subgroup of $W$. As in the proof of Theorem $5.1, H \unlhd W$ implies $H^{\prime} \unlhd W^{\prime}$. But $W^{\prime}$ can be represented faithfully as a Fuchsian group acting on the hyperbolic plane (open unit disc). By a theorem on finitely generated Fuchsian groups due to Greenberg [14] $H^{\prime} \unlhd W^{\prime}$ implies $(W: H)=$ $\left(W^{\prime}: H^{\prime}\right)<\infty$.

By [29, Corollary 7.3] $f$ is injective. To show that $f$ is surjective let $G$ be a finite regular hyperbolic combinatorial map. Then $D(G)$ is not among the diagrams listed above. By [29, Theorem 7.4] $G \simeq G(W, H)$, where $H \unlhd W$ and $W$ and $G$ have the same diagram. Because $G$ is finite, $|G|$ is a closed surface. By [29, Theorems 6.2 and 7.8$] H \simeq \pi_{1}(|G|)$. But it is well known that the fundamental group of a closed surface is finitely generated and torsion free.

## 6. Regular Maps

Let $Y$ be a topological space with $\chi(Y) \neq \varnothing$, where $\chi$ is the Euler characteristic. It is not hard to show that there are at most finitely many finite regular combinatorial maps with underlying space $Y$. In the case of rank 3 maps we also give a bound on the cardinality of the vertex set of $G$ in terms of the Euler characteristic of the underlying topological space. In contrast, we next show that for any diagram $D$, with the exception of those on a known short list, there are infinitely many finite regular combinatorial maps with diagram $D$. This answers a question posed by Grünbaum for maps on surfaces. Grünbaum asked whether for $1 / p+1 / q<\frac{1}{2}$ there are infinitely many finite regular maps consisting of $p$-gons, $q$ of them incident at each vertex [19]?

Theorem 6.1. For any nonzero integer $n$ there are at most finitely many finite regular maps $G$ with $\chi(|G|)=n$.

Proof. If $\chi(|G|)=n$ then by Lemma 5.2, $|V(G)|=n /( \pm 1+x)$, where $x=$ $\sum_{\phi \neq J^{\prime} I}(-1)^{(|-J|-1)} / p_{J}$. For all but finitely many values of the $p_{J},|x|<\frac{1}{2}$. This implies that $|V(G)|<2|n|$.

Theorem 6.1 does not necessarily hold if $n=0$. For example, there are infinitely many regular combinatorial maps $G$, where $|\boldsymbol{G}|$ is a torus.

Thenrfm 6.2. If $G$ is a finite regular, hyperbolic rank 3 combinatorial
map, then $|V(G)| \leqslant-84 \chi(|G|)$. Equality holds if and only if $G$ has the diagram


Proof. If $G$ has the form of Diagram 2, then $\chi(|G|)=\frac{1}{2}|V(G)|(1 / p+$ $1 / q+1 / r-1)$ by Lemma 5.2. Also $(1 / p+1 / q+1 / r-1) \leqslant-\frac{1}{42}$.

The simplest example for which equality holds in Theorem 6.2 is $G=$ $\left\{7,3 \mid\left(r_{0} r_{1} r_{2}\right)^{8}\right\}$. Theorem 6.2 is, of course, related to Hurwitz's classic theorem bounding the order of a group of orientation preserving conformal automorphisms of a Riemann surface. It is not known for which values of $\chi$ there exists a hyperbolic map with $-84 \chi$ points.

Theorem 6.3. If $D$ is a connected diagram not among those listed in Table II, then there are infinitely many finite regular combinatorial maps with diagram $D$.

Proof. An arbitrary group $B$ is called residually finite if for any $x \in B$, $x \neq 1$, there is a homomorphism $f$ onto a finite group such that $f(x) \neq 1$. It is an easy exercise to verify that if $\left\{x_{1}, x_{2}, \ldots, x_{m}\right\}$ is any finite set of elements of residually finite group $B$ then there is a homomorphism $f$ from $B$ to a finite group such that $f\left(x_{i}\right) \neq 1$ for $i=1,2, \ldots, m$. Let $W$ be a Coxeter group with diagram $D$. It has been shown [9] that $W$ has a faithful representation in $G L(n, R)$, where $n$ is the number of nodes in $D$. Also, by a fundamental theorem in the theory of linear groups [30], every finitely generated matrix group is residually finite. Thus $W$ is residually finite. Assume that $D$ is connected and not among the diagrams listed above. Let $T$ be the set of elements consisting of generators $r_{i}$ and the powers of $r_{i} r_{j}$ for all $i$ and $j$. Then there is a homomorphism $f_{1}: W \rightarrow B_{1}$ with $B_{1}$ a finite group and such that $f_{1}(x) \neq 1$ for $x \in T$. Let $N_{1}=\operatorname{ker} f_{1}$. Since $D(W)$ is, by assumption, not in the list of excluded diagrams, $W$ is infinite. Since ( $W: N_{1}$ ) $<\infty, N_{1}$ is also infinite. So there is a $t_{1} \in N_{1}$ and a homomorphism $f_{2}: W \rightarrow B_{2}$ with $B_{2}$ finite and such that $f_{2}(x) \neq 1$ for $x \in T \cup\left\{t_{1}\right\}$. Let $N_{2}=\operatorname{ker} f_{2}$. Continuing in this fashion, an infinite sequence of distinct normal subgroups $N_{1}, N_{2}, \ldots$, of $W$ is defined. Let $G_{i}=G\left(W, N_{i}\right)$. Then for each $i, G_{i}$ is a finite regular combinatorial map with diagram $D$.
Let $f: G \rightarrow G^{\prime}$ be an $m$-covering of combinatorial maps over $I$. Such ramified coverings are defined in Section 2 . Note that a 2 -covering preserves the diagram: $D(G)=D\left(G^{\prime}\right)$. An automorphism of a combinatorial map is considered a trivial covering. A regular combinatorial map $G$ is called simple if there is no non-trivial 2 -covering $f: G \rightarrow G^{\prime}$ with $G^{\prime}$ regular. Thus every regular combinatorial map is a 2 -cover of a simple regular combinatorial map.

Conjecture 6.4. There are at most finitely many simple regular maps with a given diagram.

## 7. Cyclic Coverings

Although the classification of regular combinatorial maps seems intractible, there are methods that generate many regular combinatorial maps from a single given regular combinatorial map. The method of this section relies on coverings. Although the techniques differ, a similar approach has been successfully applied by Wilson [32] who has obtained nearly all known regular maps on surfaces from a very few parametrized families of maps. An $m$-covering is called cyclic (abelian) if the group of covering transformations (see Section 2) is a cyclic (abelian) group. By a regular m-covering we mean an $m$-covering $f: G^{\prime} \rightarrow G$ of regular combinatorial maps. In this section an algorithm is described that generates all finite regular combinatorial maps that are cyclic covers of a given finite regular combinatorial map.

For a given regular combinatorial map $G$ over $I$ let $V(G)=\left\{u_{1}, u_{2}, \ldots, u_{M}\right\}$ and let $E=\left\{e_{1}, e_{2}, \ldots, e_{N}\right\}$ be the set of lines in $E(G)$ not in some given spanning tree $S$ of $G$. To each line $\{u, v\}$ in $E$ an orientation is arbitrarily assigned: $+e=(u, v)$ and $-e=(v, u)$. For each $j, 1 \leqslant j \leqslant M$ and any $I$ sequence $T$ define a polynomial

$$
g_{T j}(X)=g_{T j}\left(X_{1}, X_{2}, \ldots, X_{N}\right)=\sum_{k=1}^{N} a_{T j k} X_{k},
$$

where

$$
\begin{aligned}
a_{T j k} & =+1 & & \text { if the path of type } T \text { based at } u_{j} \text { contains }+e_{k}, \\
& =-1 & & \text { if the path of type } T \text { based at } u_{j} \text { contains }-e_{k}, \\
& =0 & & \text { otherwise. }
\end{aligned}
$$

For an $n$-tuple $B=\left(b_{1}, b_{2}, \ldots, b_{N}\right)$ of integers reduced modulo an integer $d$ define

$$
w_{T j}(B)=d /\left(g_{T j}(B), d\right) .
$$

Take $T_{i}, 1 \leqslant i \leqslant N$ to be the type of the unique cycle in $S \cup\left\{+e_{i}\right\}$ and let $\mathscr{E}=\left\{T_{1}^{\alpha_{1}} T_{2}^{\alpha_{2}} \cdots T_{N}^{\alpha_{N}} \mid 0 \leqslant \alpha_{1} \leqslant d-1\right\}$. The $N$-tuple $B$ is called a $(d, m)$ solution for $G$ if the following conditions hold.
(1) $\operatorname{gcd}\left(b_{1}, b_{2}, \ldots, b_{N}\right)=1$,
$w_{T j}(B)=w_{T j^{\prime}}(B)$ for all $j, j^{\prime} \in I$ and all $T$,
$w_{T j}(B)=1$ if $T$ is the type of a path in $G m$-homotopic to 0.

It is not difficult to show that if conditions (2) and (3) hold for all $T \in \mathscr{E}$, then they hold for all $I$-sequences. The common value $w_{T}(B):=w_{T j}(B)$ is called the winding number of $T$ for the solution $B$.

Let $B$ be a $(d, m)$-solution for $G$ and construct a new $I$-labeled graph $G_{B}$ as follows: $V\left(G_{B}\right)=V(G) \times D$, where $D=\{1,2, \ldots, d\}$. Two points $(u, r)$ and ( $u^{\prime}, r^{\prime}$ ) are $i$-adj in $G_{B}$ if and only if $u i$-adj $u^{\prime}$ in $G$ and either

$$
\begin{aligned}
& \left\{u, u^{\prime}\right\} \in S \text { and } r=r^{\prime}, \text { or } \\
& \left(u, u^{\prime}\right)= \pm e_{k} \text { and } r^{\prime} \equiv r \pm b_{k}(\bmod d) \text { for some } k
\end{aligned}
$$

Let $f_{B}: G_{B} \rightarrow G$ be the projection on the first coordinate $(u, r) \mapsto u$.

Theorem 7.1. Let $G$ be a regular combinatorial map and $\mathscr{B}$ the set of all $(d, m)$-solutions for $G$. With the notation as above, the set of projections $f_{B}: G_{B} \rightarrow G, B \in \mathscr{B}$ is precisely the set of all regular d-fold cyclic m-coverings of $G$.

Before giving a proof several comments are in order.
Remark 1. Theorem 7.1 remains valid when the word "cyclic" is replaced by "abelian" if the following slight modifications are made in the definitions. Define $\bar{d}=\left(d_{1}, d_{2}, \ldots, d_{q}\right)$ as a $q$-tuple of relatively prime positive integers and $\bar{d}=d_{1} d_{2} \cdots d_{q}$. A $(d, m)$-solution over $G$ is an $N$-tuple $B=$ $\left(b_{1}, b_{2}, \ldots, b_{N}\right)$, where each $b_{i}=\left(b_{i 1}, b_{i 2}, \ldots, b_{i q}\right)$ is itself a $q$-tuple of integers modulo $d_{i}$ and where $B$ satisfies conditions (1), (2), and (3) componentwise. The $I$-labeled graph $G_{B}$ has point set $V(G) \times D$, where $D=D_{1} \times$ $D_{2} \times \cdots \times D_{q}$ and $D_{i}=\left\{1,2, \ldots, d_{i}\right\}$. The definition of $j$-adj is the same as for the cyclic case. Because the proof for the abelian case is virtually the same as for the cyclic case, but notationally cumbersome, we omit it.

Remark 2. If $G$ is a map on a surface over $\{0,1,2\}$, a covering $G^{\prime}$ of $G$ need not be associated with a map on a surface. If we want to find coverings of $G$ that are also maps on surfaces we must add the condition that the covering is not ramified over edges, i.e., the diagram is linear. In terms of the winding number we require
(4) $w_{T}(B)-1$ for $T-0202$.

Assume $G$ has diagram

and $G^{\prime}$ is a covering of $G$ that is also a map on a surface. Let $w_{2}$ and $w_{0}$ be the winding numbers of $(01)^{p}$ and (12) $)^{q}$, respectively. Then $G^{\prime}$ has diagram



Fig. 5. Regular coverings of the tetrahedron; (a) $K$, (b) $G(K)=\{3,3\}$, (c) $K_{1}$, (d) $K_{2}$, (e) $K_{3}$.
where $p^{\prime}=w_{2} p$ and $q^{\prime}=w_{0} q$. It is a direct consequence of Lemma 5.2 that

$$
\chi\left(\left|G^{\prime}\right|\right)=d \chi(|G|)\left(\frac{1}{2}-1 / p^{\prime}-1 / q^{\prime}\right) /\left(\frac{1}{2}-1 / p-1 / q\right) .
$$

As an example consider the tetrahedron $G=\{3,3\}$ of Fig. 5a. The set of lines $E \subset E(G)$ is shown in Fig. 5b. All 2-fold coverings of $G$, i.e., all $(2,0)$ solutions for $G$ that are also maps on surfaces, can be found by solving the following system of congruences. These congruences are equivalent to conditions (1)-(4).

$$
\begin{aligned}
x_{1}+x_{2}+x_{10} & \equiv x_{5}+x_{6} \equiv x_{3}+x_{8} \equiv x_{11} \equiv x_{12} \equiv x_{13} \equiv 0 \quad(\bmod 2) \\
x_{2} & \equiv x_{6}+x_{7} \equiv x_{3}+x_{4} \equiv x_{9} \quad(\bmod 2) \\
x_{1}+x_{4}+x_{5} & \equiv x_{7} \equiv x_{8} \equiv x_{9}+x_{10} \quad(\bmod 2)
\end{aligned}
$$

The solutions yield three possible coverings of $G$ corresponding to maps on surfaces of genus 0 and 3:

$$
\begin{array}{ll}
G_{1}^{\prime}=\left\{3,6 \mid\left(r_{0} r_{1} r_{2}\right)^{4}\right\}, & g=0, \\
G_{2}^{\prime}=\left\{6,3 \mid\left(r_{0} r_{1} r_{2}\right)^{4}\right\}, & g=0, \\
G_{3}^{\prime}=\left\{6,6 \mid\left(r_{0} r_{1}\right)^{3}\left(r_{1} r_{2}\right)^{3}\right\}, & g=3 .
\end{array}
$$

These are shown in Fig. $5 \mathrm{c}, \mathrm{d}$, e, where like-numbered edges are to be identified.

Proof of Theorem 7.1. Assume $B$ is a $(d, m)$ solution for $G$ and $u_{j 0}$ is a base point for $G$. The projection $f_{B}: G_{B} \rightarrow G$ is exactly the covering of [29, Theorem 6.5] corresponding to the permutation representation $f_{*}$ :
$\pi^{m}(G) \rightarrow \Sigma_{d}$, where $\left(f_{*} \sigma\right) \equiv r+g_{T j 0}(B)(\bmod d)$ and $\sigma$ is a path of type $T$ based at $u_{j 0}$. Condition (3) in the definition of a $(d, m)$-solution is necessary and sufficient for $f_{*}$ to be well defined. Condition (1) is necessary and sufficient for $f_{*}$ to be transitive, i.e., $f_{*}\left(\pi^{m}\right)$ acts transitively on $D$. Therefore by [29, Theorem 6.5] $f_{B}$ is a $d$-fold $m$-covering of combinatorial maps. Condition (2) is necessary and sufficient for $G_{B}$ to be homogeneous and hence regular by Theorem 3.1. It remains to show that $f_{B}$ is cyclic. By condition (1) the image of $\pi^{m}(G)$ under $f_{*}$ is generated by $r \mapsto r+1$ $(\bmod d)$. Since $G_{B}$ is homogeneous, the covering transformation $F_{k}$ taking $(u, 0)$ to $(u, k)$ takes $(u, r)$ to $(r, r+k)$ for all $r$, with addition mod $d$. Hence the covering transformation $F_{1}$ generates the group of covering transformations.

Conversely assume $f: G^{\prime} \rightarrow G$ is an arbitrary regular $d$-fold cyclic $m$ covering and $u$ is a base point of $G$. By [29, Theorem 7.7] the group of covering transformations of $f$ is transitive on each fiber. Since $f$ is cyclic, the points of $G^{\prime}$ in the fiber above $u$ can be labeled $(u, 0),(u, 1), \ldots,(u, d-1)$ so that $F(u, r)=(u, r+1)$ for some covering transformation $F$ and all $r$, where addition is $\bmod d$. Let $f_{*}: \pi^{m}(G) \rightarrow \Sigma_{d}$ be the permutation representation of $\pi^{m}(G)$ corresponding to $f$ by [29, Theorem 6.5]. By the homogeneity of $G^{\prime}$, if $\sigma$ is any closed path based at $u,\langle\sigma\rangle$ its $m$-homotopy class and $f_{*}\langle\sigma\rangle(0)=k$, then $f_{*}\langle\sigma\rangle(r) \equiv r+k$. Now let $E=\left\{e_{1}, e_{2}, \ldots, e_{N}\right\}$ be the set of lines of $E(G)$ not in some given spanning tree $S$ of $G$. Let $\sigma_{k}$ be the unique cycle in $S \cup\left\{e_{k}\right\}$ containing $+e_{k}$ and based at $u$. Then there exists a $b_{k}$ such that $f_{*}\left\langle\sigma_{k}\right\rangle(r) \equiv r+b_{k}$. Now $B=\left(b_{1}, b_{2}, \ldots, b_{N}\right)$ is a $(d, m)$-solution for $G$ and the algorithm yields a covering $f_{B}$ equivalent to $f$.

This paper has used a completely combinatorial generalization to investigate those maps with the greatest degree of symmetry-to study their automorphism groups, their classification, the algorithmic generation of such regular maps and certain questions concerning the diagram and underlying topological space. We point out that no attempt has yet been made to investigate, withing this framework, two topics often studied in connection with maps, enumeration, and graph embedding.

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