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Rep-tiling Euclidean space

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Summary. A *rep-tiling* $\mathcal{T}$ is a self replicating, lattice tiling of R^n. *Lattice tiling* means a tiling by translates of a single compact tile by the points of a lattice, and *self-replicating* means that there is a non-singular linear map ϕ: $R^n \to R^n$ such that, for each $T \in \mathcal{T}$, the image $\phi(T)$ is, in turn, tiled by $\mathcal{T}$. This topic has recently come under investigation, not only because of its recreational appeal, but because of its application to the theory of wavelets and to computer addressing. The paper presents an exposition of some recent results on rep-tiling, including a construction of essentially all rep-tilings of Euclidean space. The construction is based on radix representation of points of a lattice. One particular radix representation, called the *generalized balanced ternary*, is singled out as an example because of its relevance to the field of computer vision.

1. Introduction

The subject of this exposition, self-replicating tiling, has gained the interest of a wide spectrum of mathematicians. It is a recent addition to the large body of work on the geometry and symmetry of tilings, a topic surveyed, beginning with the mosaics in the Alhambra at Granada in Spain, in the book [15] by Grünbaum and Shephard. Self-replicating tiling also relates to fractal geometry. The boundaries of the tiles often have nonintegral Hausdorff dimension, and techniques have been developed for computing the dimension. Self-replicating tilings are connected with generalized number systems, a topic that dates back at least to Cauchy, who noted that allowing negative digits in the radix representation of an integer makes it unnecessary for a person to memorize the multiplication table past 5×5. Knuth [19] discusses numerous alternative positional number systems, in particular the balanced ternary system, whose base is 3 and whose digits are the "trits" $\{-1, 0, 1\}$. Self-replicating tilings arise in image processing and computer vision, especially in the addressing of points in the plane using a hexagonal, rather than a square, grid

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of pixels. In this case a system generalizing the balanced ternary system comes into play. Self-replicating tilings have recently been applied to the construction of wavelets. The standard wavelet bases are constructed using translations and expansions from simple functions with support on one tile in the usual cubic tiling of $\mathbb{R}^n$. Gröchenig and Madych [13], Lawton and Resnikoff [23] and Strichartz [31] use multiresolution analysis modeled on other self-replicating tilings of $\mathbb{R}^n$. In 1984 Shechtman, Blech, Gratias and Cahn [30] discovered the first substance (an aluminum-manganese alloy) whose electron diffraction pattern indicates both "long range order" and violation of the crystallographic restriction (five-fold rotational symmetry in this case). Long range order usually means periodicity, but periodicity is incompatible with five-fold symmetry. Although the arrangement of atoms in this and similar materials, now called *quasicrystals*, is still unknown, certain self-replicating tilings due to Penrose [27] and others have become a canonical model for their structure. Thurston [32] makes basic connections between self-replicating tilings, finite state machines and Markov partitions in dynamical systems. The "expansion function" of a self-replicating lattice tiling of $\mathbb{R}^n$ induces a self map of the torus, the torus being the quotient of $\mathbb{R}^n$ by the lattice group of isometries isomorphic to $\mathbb{Z}^n$. Radin [28], in attempting to find the extent of disorder possible in certain tilings, uses the expansion function of a self-replicating tiling to construct another dynamical system on a certain space of tilings. He makes connections between the symmetry of the tilings and ergodic theory and statistical mechanics.

The intent of this paper is not an exhaustive survey of the topics mentioned above, but an introductory exposition of the subject for an interested nonspecialist. After giving a definition of rep-tiling in Section 2, a correspondence between rep-tiles and radix systems is presented in Section 3. As a trivial example, the standard base 10 radix system corresponds to the tiling of the real line by unit intervals with the following self-replicating property: an expansion of each tile by a factor of ten results in a tiling of the line by (first level) tiles, each of which is the union of ten of the original (zero level) tiles. Continuing this process leads to the hierarchy upon which ordinary arithmetic is based: for each $m \geq 1$, the line is tiled by mth level tiles, each of which is tiled, in turn, by $(m-1)$st level tiles. The main result in Section 3 is a bijection between pure lattice rep-tilings of $\mathbb{R}^n$ and n-dimensional radix systems satisfying unique representation. Section 4 deals with a particular radix system relevant to computer vision. The corresponding tiling of space, in this case, is by permutohedra (hexagons in dimension 2, truncated octahedra in dimension 3, ...), and the radix system is a generalization of the balanced ternary. No necessary and sufficient conditions for unique representation are known, but Section 5 provides several sufficient conditions for a radix system to possess the desired unique representation property. The term "self-replicating", as used in the first paragraph of this paper, is generic in the sense that it has slightly different definitions depending on the context. In particular, concern in this paper is mainly with lattice tilings; generalizations, variations and open problems will be discussed briefly in Section 6. For most proofs, the reader willl be referred to the appropriate source.

2. Rep-tiling

In a tiling, all tiles are compact; the tiles cover $\mathbb{R}^n$; and the intersection of the interiors of any two distinct tiles is empty. Most tilings in this paper are lattice tilings with a certain self-replicating property. More precisely, a *lattice* in $\mathbb{R}^n$ is the set of all integer combinations of n linearly independent vectors, and a *lattice tiling* is a tiling $\mathcal{T}$ of $\mathbb{R}^n$ by translates of a single tile T by a lattice L. In other words, $\mathcal{T} = \{x + T \mid x \in L\}$. The common wall tilings by squares or by hexagons are examples of lattice tilings.

The self-replicating property goes back at least to 1964 when Golomb [12] defined a figure F to be *rep-k* if F can be tiled by k congruent similar figures. Three rep-4 figures are shown in Figure 1. Combining the notion of rep-k figure with the notion of tiling, Figure 2 shows three tilings of the plane where the tiles are the corresponding rep-4 figures in Figure 1. Each of the examples in Figure 2 is a tiling $\mathcal{T}$ having the property that there exists a similarity, i.e. a matrix of the form $A = bQ$ with Q orthogonal and b positive real, such that for each tile T the image $A(T)$ is, in turn, tiled by copies of tiles in $\mathcal{T}$. In the first and second examples the similarity is expansion by a factor of 2; in the third example the similarity is a $\pi/2$ rotation composed with expansion by a factor of 2. The first example is a lattice tiling, but the other two examples are not. In fact, the third example is not even *periodic*, which means there do not exist translations in two linearly independent directions that preserve the tiling.

A rep-tiling is slightly more general than the examples above in that arbitrary expansive matrices, not just similarities, are allowed. A matrix is called *expansive* if all eigenvalues have modulus greater than 1. If $A = bQ$ is a similarity, then A expansive is equivalent to $b > 1$. A *rep-tiling* is a tiling $\mathcal{T}$ by translates of a single

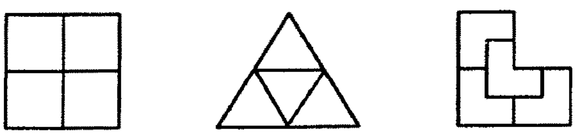

Figure 1. Rep-4 figures.

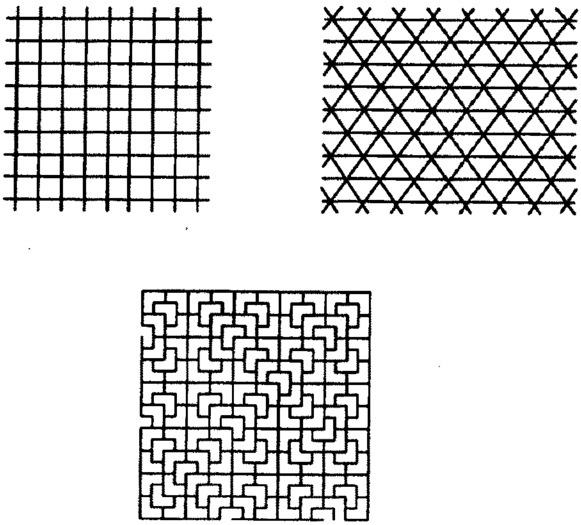

Figure 2. Rep-4 tilings.

tile T_0 such that

(1) T_0 is compact with nonempty interior, and
(2) there exists an expansive matrix A such that for each tile T the image $A(T)$ is, in turn, tiled by copies of tiles in $\mathscr{T}$.

It is sufficient to assume in condition (1) that T_0 has positive Lebesque measure (instead of nonempty interior), and it follows from the definition that T_0 is actually the closure of its interior and that the boundary of T_0 has Lebesque measure zero [20]. Each tile in a rep-tiling is a rep-k figure. It follows from the definition and the fact that distinct pairs of tiles intersect in a set of measure zero that $k = |\det A|$. In particular, det A must be an integer. If, in addition to conditions (1) and (2),

(3) $\mathscr{T}$ is a lattice tiling,

then $\mathscr{T}$ is called a *lattice rep-tiling*. The first example in Figure 2 is a lattice rep-tiling, but the other two rep-tilings are not lattice rep-tilings. In fact, the only example of a lattice rep-tiling given so far is the standard tiling of the plane by squares. What seems surprising at first is that there are infinitely many lattice rep-tilings in each dimension. A construction is given in Section 3. Lattice rep-tilings have been investigated independently by Kenyon [17], Gröchenig and Hass

[14], Gröchenig and Madych [13], Bandt [1] and Vince [34], and more recently by Lagarias and Wang [20]–[22] and Gelbrich [5].

3. Radix representation

A basic result in number theory states that every non-negative integer has a unique base $\beta \geq 2$ representation of the form

$$\sum_{i=0}^{m} d_i \beta^i, \tag{1}$$

where $d_i \in D = \{0, 1, \ldots, \beta - 1\}$. Here D is called the *digit set* and β is called the *radix*. The representation (1) has been generalized in several ways. In each case a central issue is unique representation.

(i) The radix need not be positive. In fact, for $\beta \leq -2$ every integer, including the negatives, has a base β representation with digit set $D = \{0, 1, \ldots, |\beta| - 1\}$. The digits can also be negative. Particularly nice is the *balanced ternary* system, where the radix is 3 and the digit set is $D = \{-1, 0, 1\}$. Every integer has a unique radix representation in the balanced ternary system.

(ii) In 1981 Gilbert [7]–[9] extended radix representation to the Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. For example, every Gaussian integer has a unique radix $\beta = -1 + i$ representation of the form (1), where $d_i \in D = \{0, 1\}$ — hence a binary system for the Gaussian integers. The radix β arithmetic in the Gaussian integers resembles usual binary arithmetic except in the carry digits. For example, $1 + 1 = 1100$ because $2 = \beta^3 + \beta^2$. So $1 + 1$ results in 0 with 110 "carried" three places to the left. Surprisingly, with $\beta = 1 + i$ instead of $-1 + i$, not every Gaussian integer has a representation; for example i does not.

(iii) Radix representation can likewise be extended to other number fields. Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial, irreducible over $\mathbb{Z}$, α a root of $f(x)$ in some extension field of the rationals, $\mathbb{Z}[\alpha]$ the ring obtained by adjoining α to $\mathbb{Z}$. In the case $f(x) = x^2 + 1$, the ring $\mathbb{Z}[\alpha]$ is the Gaussian integers given in (ii) above. Another interesting example is $f(x) = x^2 + x + 1$. In this case $\alpha = -\frac{1}{2} + (\sqrt{3}/2)i$, and $\mathbb{Z}[\alpha]$ can be viewed geometrically as the hexagonal lattice in the complex plane (the lattice points being the centers of the hexagons in the hexagonal tiling). If the radix is chosen as $\beta = \frac{5}{2} + (\sqrt{3}/2)i$ and the digit set is $D = \{0, 1, \omega, \omega^2, \ldots, \omega^5\}$, where $\omega = \frac{1}{2} + (\sqrt{3}/2)i$ (so that D consists of 0 and the sixth roots of unity), then every point of $\mathbb{Z}[\alpha]$ has a unique radix representation for the form (1) [18]. It will become apparent in Section 4 that this hexagonal system is a natural 2-dimensional generalization of the balanced ternary, the digit set D playing the role of the "trits".

Consider the following general framework for all the examples above. Let L be a lattice, viewed either geometrically as a set of points in Euclidean space, or algebraically as a finitely generated free Abelian group. Let $A: L \to L$ be a group endomorphism, and D a finite subset of L containing 0. The map A can, without loss of generality, be regarded as any square nonsingular matrix, as long as L is A-invariant. Indeed, if the basis for this matrix is chosen in L, then A is an integer matrix. The triple (L, A, D) is said to have the *unique representation property* if every element of L has a unique finite representation of the form

$$\sum_{i=0}^{m} A^i(d_i), \tag{2}$$

where $d_i \in D$. If L has a ring structure and A is the matrix that represents multiplication by an element β in the ring (the endomorphism $x \mapsto \beta x \ \forall x \in L$), then expression (2) reduces to the form of expression (1). In this lattice framework the triples (L, A, D) corresponding to the examples above — the balanced ternary, the Gaussian integers with base $-1+i$ and the hexagonal example — are, respectively:

(i) $L = \mathbb{Z}, \qquad A = (3), \qquad D = \{-1, 0, 1\};$

(ii) $L = \mathbb{Z}[i], \qquad A = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad D = \{0, 1\};$

(iii) $L = \mathbb{Z}\left[-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right], \qquad A = \begin{bmatrix} \frac{5}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{5}{2} \end{bmatrix}, \qquad D = \{0, 1, \omega, \omega^2, \ldots, \omega^5\}.$

In the last two examples, the matrix A is with respect to the standard basis. In the rest of this paper the examples above, all of which possess the unique representation property, will be referred to as *Examples 1, 2 and 3.*

The following proposition gives two necessary conditions for unique representation in (L, A, D) and a sufficient condition for uniqueness. The reader is referred to [34] for the somewhat technical proof of the second statement. A *digit set* for (L, A) is a complete set of residues for L modulo $A(L)$; in other words D is a digit set if it contains exactly one representative from each coset in the quotient group $L/A(L)$. The number of digits, i.e. the number of cosets, is, by standard algebraic techniques, equal to $|\det A|$.

PROPOSITION. (1) *If (L, A, D) has the unique representation property then D is a digit set. If D is a digit set then representation of a point, if it exists, is unique.*

(2) *If (L, A, D) has the unique representation property then A is an expansive map.*

Proof of (1). Assume that (L, A, D) has the unique representation property. To show that no coset is represented twice, assume, by way of contradiction, that $d \equiv d' \bmod A(L)$ for some $d, d' \in D, d \neq d'$. This implies that $d' = d + A(x) = d + A(\sum_{i=0}^{m} A^i(d_i))$ for some $x \in L$ and some $d_i \in D$. Then $d' = d + \sum_{i=1}^{m+1} A^i(d_i)$, which contradicts uniqueness. To show that each coset has at least one representative in D, consider any $x \in L$. Then $x = \sum_{i=0}^{m} A^i(d_i) = d_0 + \sum_{i=1}^{m} A^i(d_i)$ for some $d_i \in D$ implies that $x \equiv d_0 \bmod A(L)$.

Concerning the second statement, let D be a digit set and assume uniqueness of representation is violated. Then $\sum_{i=0}^{m} A^i(d_i) = \sum_{i=0}^{m} A^i(d_i')$ for some d_i, $d_i' \in D$, and since A is invertible it may be assumed, without loss of generality, that $d_0 \neq d_0'$. But then $d_0 \equiv d_0' \bmod A(L)$, a contradiction. □

A triple (L, A, D) will be called a *radix system* if A is expansive and D is a digit set. Hence if (L, A, D) is a radix system, unique representation is, by part (1) of the proposition, reduced to showing that each lattice point has some representation. The two necessary conditions in the proposition, however, are not sufficient to insure that each lattice point has a representation, even in dimension 1. For example, with 3 as radix, $D = \{-1, 0, 4\}$ is a complete set of residues modulo 3, i.e. a digit set. However, -2 has no radix 3 representation. Conditions under which each integer has a unique radix representation have been investigated by Matula [25] and by Odlyzko [26]. There seems to be no known simple necessary and sufficient conditions to insure representation. We will return to this problem in Section 5.

Given a radix system (L, A, D), a set $T(A, D)$ is constructed as follows, where the sum is the Minkowski sum $\sum_{i=0}^{\infty} X_i = \{x_0 + x_1 + \cdots \mid x_i \in X_i\}$ in $\mathbb{R}^n$:

$$T(A, D) = \sum_{i=1}^{\infty} A^{-i}(D). \tag{3}$$

Translating $T(A, D)$ by the lattice L gives

$$\mathscr{T}(L, A, D) = \{x + T \mid x \in L\}. \tag{4}$$

It is not obvious that $\mathscr{T}(L, A, D)$ is a tiling of $\mathbb{R}^n$ or even that $T(A, D)$ is a tile (compact with nonempty interior). This turns out to be the case, however, when

(L, A, D) is a radix system. Moreover, the following theorem relates rep-tiling to unique representation. A rep-tiling is called *pure* if the origin lies in the interior of some tile.

THEOREM 1. (1) *If (L, A, D) is a radix system having the unique representation property, then $\mathcal{T}(L, A, D)$ is a pure lattice rep-tiling.*

(2) *If $\mathcal{T}$ is a pure lattice rep-tiling, then $\mathcal{T} = \mathcal{T}(L, A, D)$ for some radix system (L, A, D) having the unique representation property.*

Using a computer implementation of formulas (3) and (4), the tiling in Figure 3 is constructed from Gilbert's binary system for the Gaussian integers (Example 2) and produces a rep-tiling by the rep-2 "dragon" tile. The tiling in Figure 4 is constructed from the hexagonal Example 3 and produces a rep-tiling by the rep-7 "flowsnake." A rep-5 tiling appears in Figure 5. The three tiles (rep-4 Sierpinski triangle, rep-3 figure and rep-9 figure) appearing in Figures 6–8 also produce a rep-tiling by translation of the tile by the respective lattice. These examples show that individual tiles may be topologically complex, not necessarily connected or simply connected. In fact, the tile in Figure 8 has infinitely many connected components.

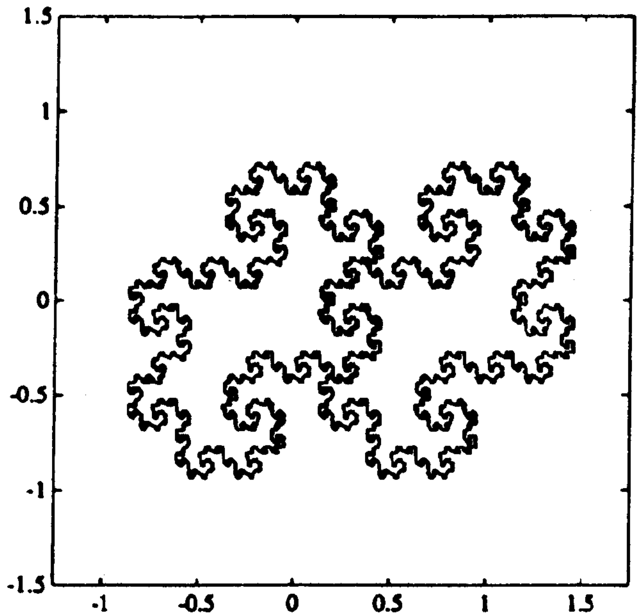

Figure 3. $\mathcal{T}\left(\mathbb{Z}^2, \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}\right)$.

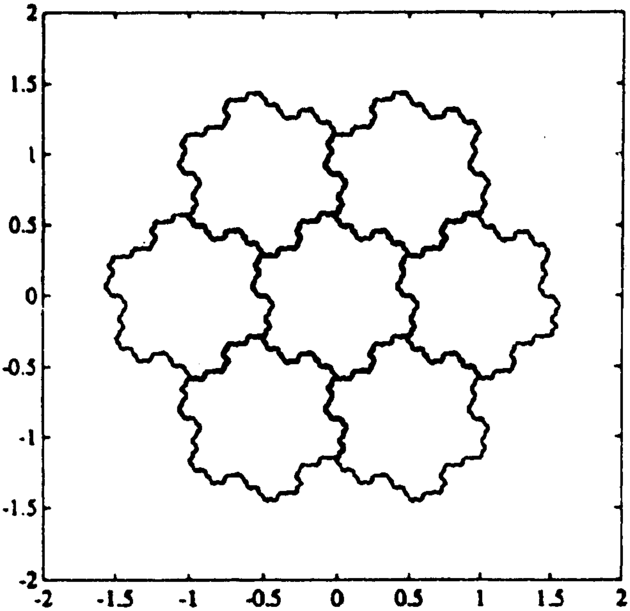

Figure 4. $\mathscr{T}\left(L, \begin{pmatrix} 5/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 5/2 \end{pmatrix}, \{(0, 1, \omega, \ldots, \omega^5)\}\right)$, where L is the hexagonal lattice.

It is not entirely surprising that the boundary of the tiles are fractal. Mandelbrot [24], Giles [10] and Gilbert [7] had constructed rep-k fractal figures (but not tilings). Dekking [3]–[4] and Bandt [1] subsequently gave systematic constructions for such "fractiles". Dekking constructs the boundary of a rep-tile in R^2 by a recursive string-rewriting procedure, and Bandt constructs the tile using summation (3).

Omitting the technicalities of the proof of Theorem 1, it is, nevertheless, not difficult to see, given the rep-tiling, how the digits in the radix representation arise, and conversely, given the digits, how formulas (3) and (4) for the tiling arise: The definition of lattice rep-tiling in Section 2 implies that for any such tiling there is a set $D = \{d_1, d_2, \ldots, d_k\}$ consisting of $k = |\det A|$ lattice points such that if T_0 is the tile whose interior contains the origin, then

$$A(T_0) = \bigcup_{k=1}^{k} (d_i + T_0), \tag{5}$$

where the digit d_1 can be chosen to be the origin 0. The set D is the required digit set.

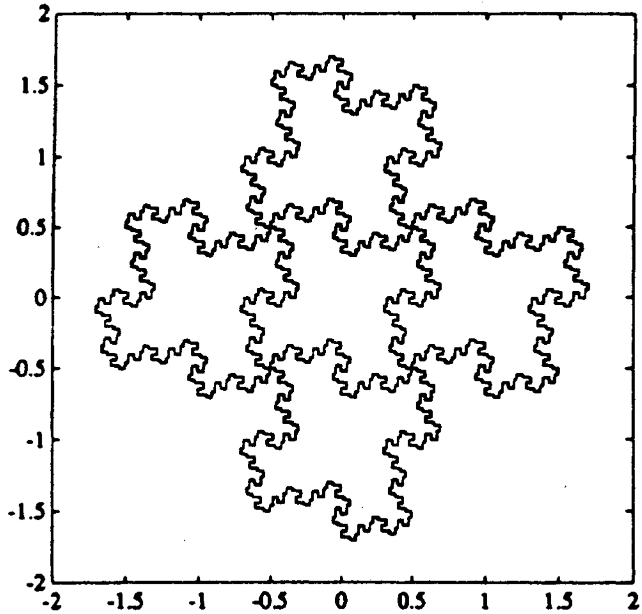

Figure 5. $\mathscr{T}\left(\mathbb{Z}^2, \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}, \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}\right)$.

Conversely, given the radix system (L, A, D), let

$$D_m = \sum_{i=0}^{m-1} A^i(D)$$

denote the set of all lattice points that can be represented with at most m digits. It is equivalent to formula (3) to express $T(A, D)$ as the limit (in the Hausdorff metric) of a nested sequence of sets:

$$T(A, D) = \lim_{m \to \infty} A^{-m}(D_m). \tag{6}$$

For a similarity A, the "evolution" of the set $T(A, D)$ in the limit (6) can be nicely visualized. Recall that the Voronoi cell centered at the lattice point $x \in L$ is defined as the set of points y such that y is at least as close to x as to any other point of the lattice: $V_x = \{y \in \mathbb{R}^n : |y - x| \le |y - z| \text{ for all } z \in L\}$. Let $\overline{D_m} = \bigcup_{x \in D_m} V_x$ be the union of the Voronoi cells V_x centered at the points in D_m. Then, by equation (6),

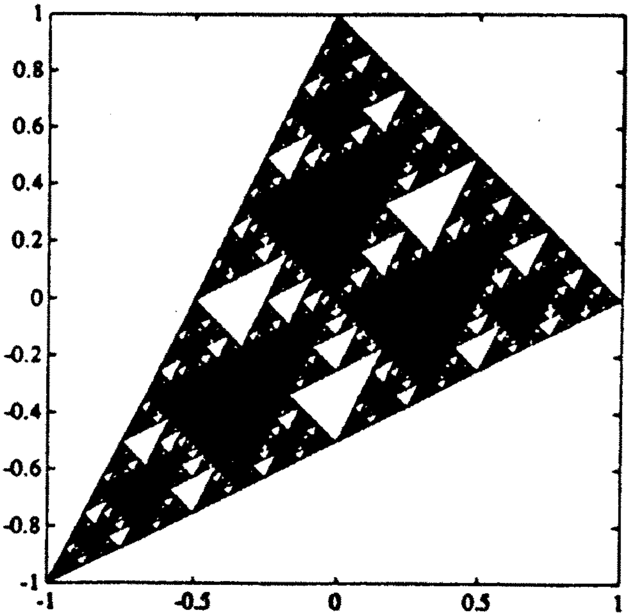

Figure 6. $T\left(\mathbb{Z}^2, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right\}\right)$.

$\overline{D_m}$ (scaled down by A^{-m}) is an mth approximation to T. Figure 9 shows the first approximations to the "dragon tile" $T(A, D)$ of Figure 3.

Since D is a set of coset representatives of $L/A(L)$, the set D_m is a set of coset representatives of the group $L/A^m(L)$. Therefore L is the disjoint union

$$L = \bigcup \{x + D_m \mid x \in A^m(L)\}, \tag{7}$$

which implies that

$$A^{-m}(L) = \bigcup \{x + A^{-m}(D_m) \mid x \in L\}.$$

Letting $m \to \infty$ shows that $\mathbb{R}^n$ is, indeed, covered by copies of the tile $T(A, D)$. (What is not clear is that the interiors of distinct tiles are disjoint, and, in fact, this may not be the case if (L, A, D) does not satisfy the unique representation property. We return to this question in Section 6.) In the case that (L, A, D) does satisfy the unique representation property, let $T_0 = T(A, D)$ and $T_m = \bigcup \{x + T(A, D) \mid x \in D_m\}$ for $m \geq 1$. Then by equation (7)

$$\mathscr{T}_m = \{x + T_m \mid x \in A^m(L)\}$$

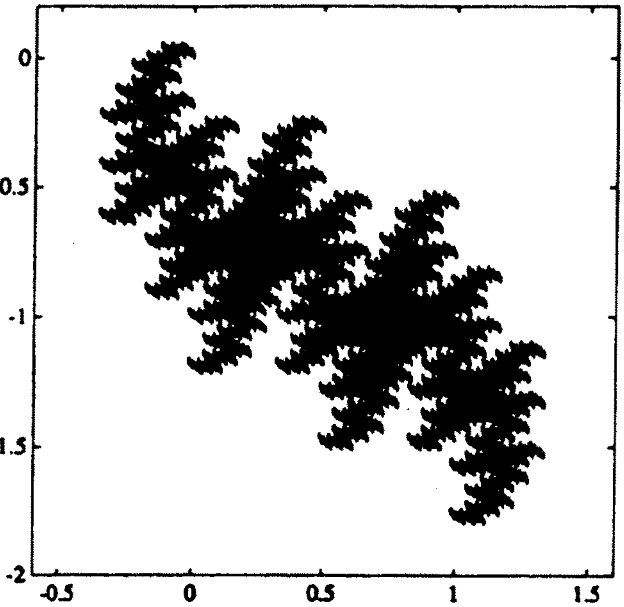

Figure 7. $T\left(L, \begin{pmatrix} 3/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 3/2 \end{pmatrix}, \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right\}\right)$, where L is the hexagonal lattice.

is a tiling of $\mathbb{R}^n$ for each m with the following hierarchical property: each tile in $\mathcal{T}_m$ is, in turn, tiled by $|\det A|$ tiles in $\mathcal{T}_{m-1}$.

There is another way to view the tiling; $T(A, D)$ arises as the attractor of a certain iterated function system. More precisely, the functions

$$w_i(x) = A^{-1}(d_i + x),$$

$i = 1, 2, \ldots, k$, are, in the terminology of Barnsley [2], an affine iterated function system, and its *attractor* is, by definition, the unique fixed point of the transformation W defined on the space of all compact subsets of $\mathbb{R}^n$ by

$$W(X) = \bigcup_{i=1}^{k} w_i(X) = \bigcup_{i=1}^{k} A^{-1}(d_i + X). \tag{8}$$

On one hand such an attractor is given explicity by the summation formula (3), and on the other hand it is, by comparing equations (5) and (8), the self-replicating tile for (L, A, D).

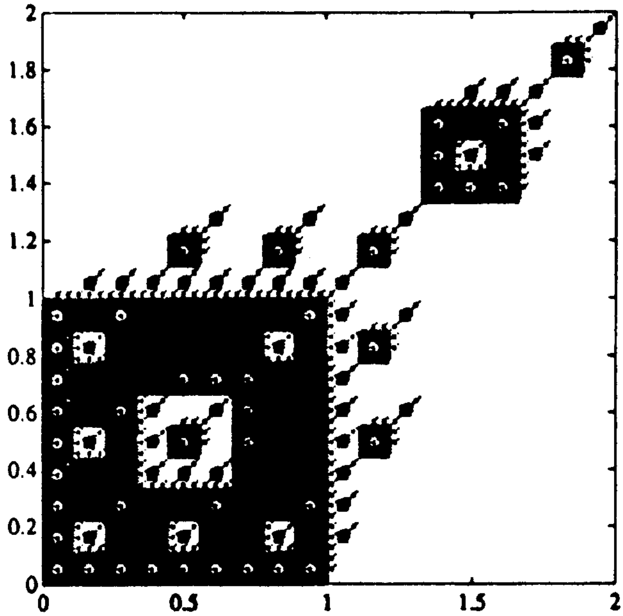

Figure 8. $T\left(\mathbb{Z}^2, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix} \right\}\right)$.

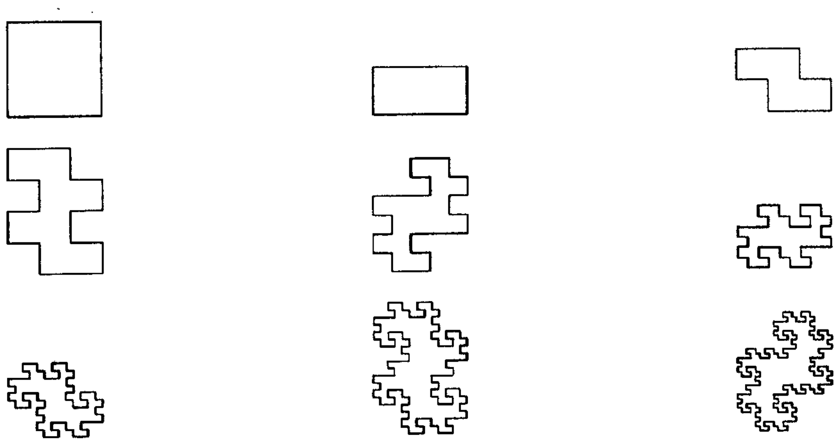

Figure 9. Approximations to the rep-2 dragon tile.

Theorem 1 naturally suggests the following two questions, which will be addressed in Section 5.

(1) Is it possible to determine whether a given radix system (L, A, D) has the unique factorization property?

(2) Given (L, A), does there exist at least one digit set D for which (L, A, D) has the unique representation property?

4. Generalized balanced ternary

Just as the decimal system is suitable for denoting integer points on the line, any radix system (L, A, D) with the unique representation property provides a system for addressing lattice points in Euclidean n-space, the *address* of a lattice point x given by the string of digits in the radix representation of x. The generalized balanced ternary, defined below, is a useful n-dimensional radix system that simultaneously generalizes the 1-dimensional balanced ternary and the 2-dimensional hexagonal system.

Let $\mathbf{A}_n^*$ denote the dual of the classical n-dimensional root lattice $\mathbf{A}_n$. For our purposes $\mathbf{A}_n^*$ is the lattice in $\mathbb{R}^n$ generated by a set $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ of vertices of a regular n-simplex with barycenter at the origin. So $\mathbf{A}_1^*$ is the integer lattice on the line, and $\mathbf{A}_2^*$ is the hexagonal lattice in the plane. The Voronoi region of $\mathbf{A}_1^*$, $\mathbf{A}_2^*$, and $\mathbf{A}_3^*$ is an interval, a hexagon, and a truncated octahedron, respectively. In general, the Voronoi region of $\mathbf{A}_n^*$ is a permutohedron, the n-dimensional polytope whose vertices (embedded in $\mathbb{R}^{n+1}$) consist of the $(n+1)!$ points obtained by permuting the coordinates of $(-n/2, (-n+2)/2, (-n+4)/2, \ldots, (n-2)/2, n/2)$.

The lattice $\mathbf{A}_n^*$ is isomorphic, as an Abelian group, to the quotient $L = \mathbb{Z}[x]/(f)$, where $f(x) = 1 + x + \cdots + x^n$. Let $\omega = \bar{x}$, where the bar denotes the coset in L containing x. Since $\omega^{n+1} = 1$, the ring L acts somewhat like adjoining an $(n+1)$st root of unity to $\mathbb{Z}$. The isomorphism between $\mathbf{A}_n^*$ and L is obtained by mapping generators $\mathbf{v}_i \mapsto \omega^i$, and extending linearly. In fact, because L is a ring, the lattice $\mathbf{A}_n^*$ also inherits a ring structure. Addition in $\mathbf{A}_n^*$ is usual vector addition. By abuse of language no distinction will be made between $\mathbf{A}_n^*$ and L.

Let $\beta = \bar{2} - \omega \in L$ be the base for radix representation in $\mathbf{A}_n^*$. The matrix representing β, with respect to the generators $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$, is

$$A_\beta = \begin{pmatrix} 2 & 0 & \cdots & 0 & -1 \\ -1 & 2 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Let the digit set be $D = \{\varepsilon_0 + \varepsilon_1 \omega + \cdots + \varepsilon_n \omega^n : \varepsilon_i \in \{0, 1\}, \text{ not all } \varepsilon_i = 1\}$. Then $(\mathbf{A}_n^*, \mathbf{A}_\beta, D)$ has the unique representation property [18], and is called the *generalized balanced ternary* (GBT). The 1- and 2-dimensional cases are exactly the balanced ternary and hexagonal systems. In computer vision the pixel locations in an image can be thought of as the lattice points at the centers of Voronoi cells that tile the plane. A geometric advantage of pixel locations on a hexagonal grid, rather than the usual square grid, is that the hexagons are a reasonably accurate approximation to a circle. A computer software advantage of GBT addressing is that high throughput rates are achieved by performing addition and multiplication, as well as conversion of address to planar locations and vice-versa, in terms of the bit strings $\varepsilon_0 \varepsilon_1 \cdots \varepsilon_n$ that represent the digits [18][29]. One firm [6] has developed a planar database management system based on the 2-dimensional GBT. Figure 10 shows all planar locations with addresses of at most three GBT digits and also the product $25.255 = 604$. For convenience, the addresses in this figure are given using base 7 digits instead of binary string digits. This is possible because there is a ring isomorphism

$$\Theta : \mathbf{A}_n^* \to \mathbb{Z}$$

that, in dimension 2, takes a radix β representation in $\mathbf{A}_2^*$ with digits D to a radix 7 representation in $\mathbb{Z}$ with digits $\{0, 1, 2, 3, 4, 5, 6\}$. In general, the isomorphism is given as follows. Let $q = 2^{n+1} - 1$; then

$$\Theta : \sum_{i=0}^{m} d_i \beta^i \to \sum_{i=0}^{m} \theta(d_i) q^i,$$

where $\theta : D \to \{0, 1, \ldots, q-1\}$ is given by

$$\theta : \sum_{i=0}^{n} \varepsilon_i \omega^i \mapsto \sum_{i=0}^{n} \varepsilon_i 2^i.$$

The proof that this is an isomorphism follows from the facts that $\omega^{n+1} = 1$ and $\omega \equiv 2 \bmod (\beta L)$.

5. Unique representation

This section contains three remarks concerning when a radix system (L, A, D) has the unique representation property. Since D is a digit set, uniqueness follows

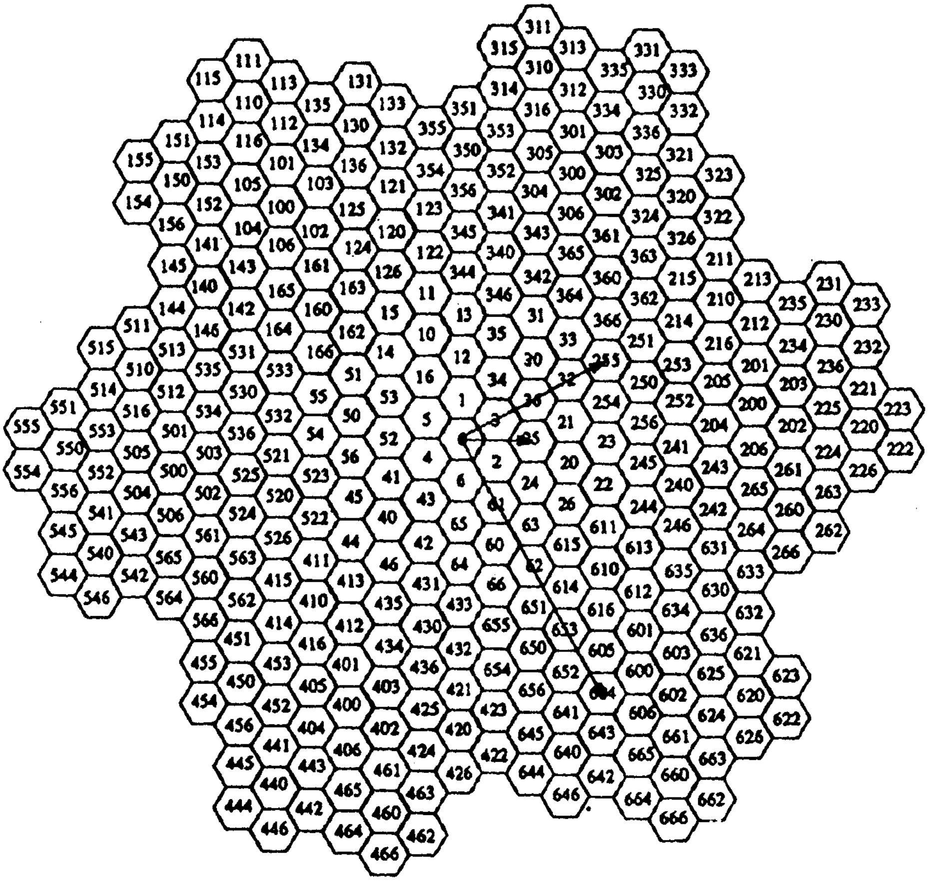

Figure 10. Addresses in generalized balanced ternary radix system.

from the proposition in Section 3. The issue is whether every lattice point has some representation. Proofs of the results in this section appear in [34].

REMARK 1. Assume in this remark that $A = bQ$ is a similarity; for the general case see [34]. Given a radix system (L, A, D), there is an efficient algorithm to determine whether (L, A, D) represents each lattice point. This algorithm is based on the following simple routine that is analogous to finding the base β digits of an integer.

Given a point $x = x_0 \in L$, define a sequence $\{x_i\}$ of lattice points and a sequence $\{d_i\}$ of digits recursively by the formulas

$$x_{i+1} = A^{-1}(x_i - d_i),$$

where d_i is the unique element of D such that

$$d_i \equiv x_i \quad \mod A(L).$$

It is clear that, if $x_m = 0$ for some m, then $x_i = d_i = 0$ for all $i \geq m$. And, if there is a repetition in the sequence $\{x_i\}$ other than 0, then $x_i \neq 0$ for all i. Let $r(D) = \max\{|d| : d \in D\}/(b - 1)$; let B be a ball centered at the origin with radius $|x| + r(D)$; and let α be the number of lattice points of L in B. Then it can be shown that x has a representation in (L, A, D) if and only if $x_m = 0$ for some $m \leq \alpha$. In this case, the digits in the radix representation of x are $d_0, d_1, \ldots, d_{m-1}$. Consider, for example, $(\mathbb{Z}, 3, \{-1, 0, 4\})$. If $x = 2$, then

$$\begin{aligned} x_0 &= 2 & d_0 &= -1 \\ x_1 &= 4 & d_1 &= 4 \\ x_2 &= -1 & d_2 &= -1 \\ x_3 &= 0 & d_3 &= 0, \end{aligned}$$

and 2 has base 3 representation $(-1)(4)(-1)$. If $x = -2$, then $x_0 = -2$ and $x_1 = -2$ so that -2 has no finite representation. Call this procedure for determining whether a given lattice point has a (finite) radix representation, *Algorithm A*.

ALGORITHM. *Every lattice point in L has a representation in the radix system (L, A, D) if and only if every lattice point in a ball of radius r(D) has a finite radix representation. The latter condition can be efficiently tested using Algorithm A.*

As a first example, apply this algorithm to Example 3, the 2-dimensional GBT. In this case, $r(D) = 1/(\sqrt{7} - 1)$. Since 0 is the only lattice point in the ball of radius $r(D)$, it is immediate that (L, A, D) has the unique representation property. Not quite so trivial, consider $L = \mathbb{Z}[i]$, $\beta = 1 + i$ and $D = \{0, 1\}$. In this case $b = \sqrt{2}$, so that $r(D) = 1/(\sqrt{2} - 1)$. There are exactly 21 Gaussian integers in the ball of radius $r(D)$ to which Algorithm A is applied. However, Algorithm A applied to the Gaussian integer i (which lies in the ball) results in the sequence $i, i \ldots$. The repetition indicates that i has no finite radix representation. On the other hand with $\beta = -1 + i$, instead of $1 + i$, it can be checked using Algorithm A that all 21 lattice points in the ball do have finite radix representations. Therefore $(\mathbb{Z}[i], -1 + i, \{0, 1\})$ does have the unique representation property.

REMARK 2. A sufficient condition for representation of each lattice point can be stated in terms of the radix arithmetic.

SUFFICIENT CONDITION 1. *Every lattice point in L has a representation in the radix system (L, A, D) if*

(1) *each element in a basis for L has a radix representation, and*

(2) $D + D \subseteq D + A(D)$.

The second condition states that, in the radix arithmetic, a carry for addition can go at most one place to the left. Example 2 in Section 3, where digits are carried three places to the left, shows that this may not be the case. For the generalized balanced ternary, however, it can be shown that this is the case in all dimension.

REMARK 3. Given a lattice L and linear expansive map $A: L \to L$, does there exists *at least one set* D of digits such that (L, A, D) has the unique representation property? The answer, in general, is no. The simplest counterexample is the binary system for the integers: $L = \mathbb{Z}$ and $A = (2)$. In fact, this is the only example in dimension 1. In dimension 2 there are infinitely many counterexamples. Let $L = \mathbb{Z}^2$ and

$$A = \begin{pmatrix} 0 & -m \\ 1 & m \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 2 & m \\ 0 & 2 \end{pmatrix},$$

where m is an integer. In either case there is no set of digits so that $(\mathbb{Z}^2, A, D)$ has the unique representation property. In fact it can be proved that, if $\det(I - A) = \pm 1$, then there exists no appropriate set D of digits.

In the positive direction it can be shown that, if A has sufficiently large singular values, then there exists a digit set D such that (L, A, D) has the unique representation property. Recall that the singular value decomposition of a real matrix A is $A = U \circ \operatorname{diag}(\sigma_1, \ldots, \sigma_n) \circ V^T$, where U and V are orthogonal matrices. The real numbers $\sigma_1, \ldots, \sigma_n$ are called the *singular values* of A. If $A = bQ$ is a similarity, then all singular values of A are equal to the expansion factor b.

SUFFICIENT CONDITION 2. *If all the singular values of A are greater than* $3\sqrt{n}$, *then there exists a set of digits D such that (L, A, D) has the unique representation property. In dimensions* 1 *and* 2 *the bound* $3\sqrt{n}$ *can be improved to* 2.

The digit set D in this result is set of lattice points contained in the image under A of the half open unit cube (a fundamental domain for the cubic lattice) centered at the origin. As an example consider the square lattice $L = \mathbb{Z}^2$ and the linear map

$$A = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}.$$

Both singular values of A are $2.2361 > 2$. Hence those lattice points $\{(0,0), (1,0), (0,1), (-1,0), (0,-1)\}$ that lie in the image under A of the half open square $(-\frac{1}{2}, \frac{1}{2}] \times (-\frac{1}{2}, \frac{1}{2}]$ constitute a digit set D for which (L, A, D) is the one that appears in Figure 5.

A generalization of Sufficient Condition 2 produces a family of digit sets D for each pair (L, A), each digit set being the image under A of the fundamental domain of a certain lattice [34]. Strichartz [31] proves a result similar to Sufficient Condition 2 for the case that A is a similarity. Gröchenig and Haas [14] construct a digit set D for each pair (L, A) in dimension 2 (except when A has two irrational real eigenvalues) that guarantees that $\mathcal{T}(L, A, D)$ is a lattice rep-tiling, And in some cases, they show that the rep-tile $T(A, D)$ is connected.

6. Variations and generalizations

This section discusses two open problems related to topics in this paper, and a generalization from lattice rep-tiles to crystallographic rep-tiles. The subject of self-replicating tilings using more than one prototile is fascinating, but too vast to be discussed here.

Two problems

In some of the questions below, the tile, rather than the tiling, is the fundamental object of study. A *rep-tile* is a compact set T in $\mathbb{R}^n$ of positive Lebesque measure with the property that there exists an expansive matrix A such that the image $A(T)$ is tiled by translates of T. As before, there is a set D consisting of $|\det A|$ digits defined by the equation $A(T) = \bigcup_{d \in D}(d + T)$, and $T = T(A, D)$ is given by formula (3) in Section 3. Note, however, that no lattice is mentioned in the definition of rep-tile. No tiling is mentioned either, but the following theorem holds [20]. Here $D_\infty = \bigcup_{i=1}^{\infty} D_m$, where $D_m = \sum_{i=0}^{m-1} A^i(D)$, and $\Delta(D_\infty) = D_\infty - D_\infty$.

THEOREM 2. *If $T(A, D)$ is a rep-tile, then there exists a set of translations $\mathcal{L} \subseteq \Delta(D_2)$ such that $\mathcal{L} + T(A, D)$ tiles $\mathbb{R}^n$.*

QUESTION 1. Is $T(L, A, D)$ a tiling even when the radix system (L, A, D) does not represent each lattice point?

Theorem 1 does not answer the question, because it does not even assert that L is a lattice, let alone the lattice L. It also does not assert that the tiling is a rep-tiling. (However, if $\mathscr{L} = L$ then it is easy to show that $T(L, A, D)$ is indeed a rep-tiling.) In fact, there is a counterexample to Question 1, even in dimension 1. Let $L = \mathbb{Z}$, $A = (3)$, $D = \{-2, 0, 2\}$. Then $T(A, D) = [-1, 1]$, so that adjacent tiles overlap on a unit interval.

Theorem 1 may, however, help in understanding this counterexample. If the lattice L in a radix system (L, A, D) has an A-invariant proper sublattice L' containing D, then clearly $\mathscr{L} \subseteq \Delta(D_\infty) \subseteq L' \subset L$. Since, by Theorem 1, $\mathscr{L} + T(A, D)$ tiles $\mathbb{R}^n$ it is impossible that $L + T(A, D)$ tiles $\mathbb{R}^n$. This is the case in the counterexample above, where the A-invariant sublattice containing D is $2\mathbb{Z}$. Of course, if L does have an A-invariant sublattice containing D, it is always possible to "mod out" and regard A as acting on the minimum (with respect to inclusion) such sublattice L'. So it is reasonable to conjecture, as was done in [14], that if L has no A-invariant sublattice containing D, then $T(A, D)$ does tile $\mathbb{R}^n$ by translation by L. The conjecture is true in dimension 1; however Lagarias and Wang [20] recently gave the following counterexample in dimension 2:

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\},$$

where the tile $T(A, D)$ is shown in Figure 11. There is, however, a tiling (not a

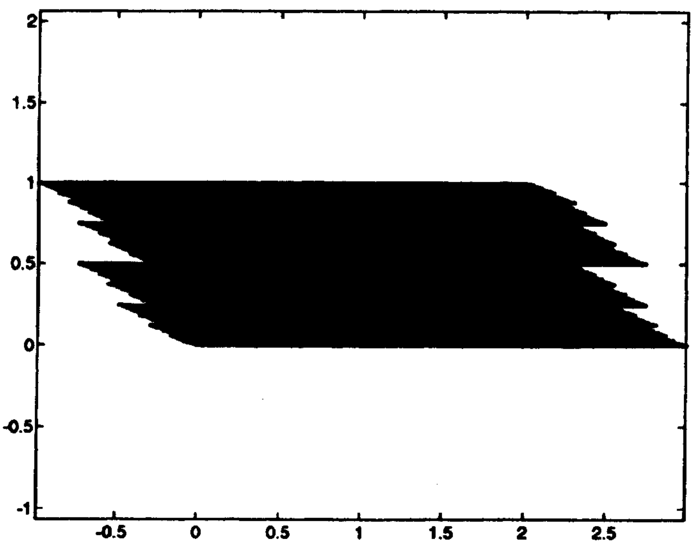

Figure 11. T does not give a tiling by translation by $\mathbb{Z}^n$.

rep-tiling) of $\mathbb{R}^2$ by translates of $T(A, D)$ to the lattice $3\mathbb{Z} \oplus \mathbb{Z}$. This motivates the weaker conjecture:

CONJECTURE 1. *If (L, A, D) is a radix system, then there is some lattice tiling using only translates of $T(A, D)$.*

Lagarias and Wang have announced a proof of Conjecture 1 in dimension 2 and in all dimensions when A is a similarity [21], but the general case appears to remain open. Concerning the original Question 1, Gröchenig and Haas [14] provide an algorithm that determines, given radix system (L, A, D), whether or not $\mathscr{T}(L, A, D)$ is a tiling.

Given any rep-tile T, Theorem 1 guarantees a tiling by translates of T but gives no limits on the degree of "disorder" the tiling can possess. This motivates the following question.

QUESTION 2. Does there exist a rep-tile that admits no periodic tiling? More generally, does there exist a region that tiles $\mathbb{R}^n$ by translation, but admits no periodic tiling?

In dimension 1, if a bounded region T tiles $\mathbb{R}$ by translation, then every tiling by translation is periodic [22]. The same result is true in dimension 2 if T is a topological disk with piecewise-C^2 boundary [11]. Likewise the result holds for convex polytopes in any dimension [33]. However, the Penrose tiles [27] are a set of two prototiles that admit infinitely many tilings of $\mathbb{R}^2$, but no periodic tilings. For one tile the answer to both questions, in general, is open. If translations are not the only motions allowed, then there are examples of both non-convex (Schmitt, unpublished) and convex (J. H. Conway, unpublished) polyhedra that tile $\mathbb{R}^3$ by Euclidean motions but only aperiodically.

Crystallographic rep-tiles

The group $\mathbb{Z}^n$, viewed as a discrete group of isometries acting on $\mathbb{R}^n$ with compact quotient, is just one of many such *crystallographic groups*. There are 7 in dimension 1 (frieze groups), 17 in dimension 2 (wallpaper groups); 230 in dimension 3; 4783 in dimension 4; and, in general, there are finitely many crystallographic groups in each dimension (proved by Bieberbach in 1910). The lattice rep-tilings are rep-tilings by all images of a single tile T under the action of the group $\mathbb{Z}^n$. Gelbrich [5] generalizes by calling a rep-tiling $\mathscr{T}$ *crystallographic* if $\mathscr{T} = \{\gamma(T) \mid \gamma \in \Gamma\}$, for some tile T and some crystallographic group Γ. He proves in dimension 2 that, for a crystallographic group and a given expansion factor k, there are finitely many isomorphism types of crystallographic rep-tiles that are homeomorphic to a disk. Here an isomorphism is an affine bijection that preserves tiles at every level in the hierarchy. Using an algorithm for determining the tiles, he shows, for example, that, in the case of a lattice rep-tile (crystallographic group generated by two independent translations), there are three classes of tiles for $k = 2$ and seven classes for $k = 3$.

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