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Variations on a Theorem of Ryser

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ABSTRACT

A famous theorem of Ryser asserts that a $v \times v$ zero-one matrix A satisfying $AA^T = (k - \lambda)I + \lambda J$ with $k \neq \lambda$ must satisfy $k + (v - 1)\lambda = k^2$ and $A^TA = (k - \lambda)I + \lambda J$; such a matrix A is called the incidence matrix of a symmetric block design. We present a new, *elementary* proof of Ryser's theorem and give a characterization of the incidence matrices of symmetric block designs that involves eigenvalues of AA^T. © Elsevier Science Inc., 1997

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1. INTRODUCTION

In the first volume of the *Proceedings of the American Mathematical Society*, Ryser [3] proved the following theorem.

RYSER'S THEOREM (Version 1). *Let V be a set of size v, and let S_1, $S_2, \dots, S_v$ be subsets of V. If there are distinct k and λ such that $|S_i| = k$ for all i and $|S_i \cap S_j| = \lambda$ whenever $i \neq j$, then $k + (v-1)\lambda = k^2$, each point of V is included in precisely k of the sets S_i, and each pair of distinct points of V is contained in precisely λ of the sets S_i.* ■

This paper explores variations on Ryser's theorem, in two different spirits. Ryser's original proof, and all other proofs that we have seen or concocted, resort to notions such as determinants, matrix inverses, linear independence, or eigenvalues and rely on results of linear algebra such as

if C is a square matrix such that the equation $Cx = 0$ has a nonzero solution, then $\det CC^T = 0$

or

if A is a square matrix such that equation $AA^Tx = 0$ has no nonzero solution, then there is a matrix B such that $BA = I$.

While use of algebraic techniques to prove a combinatorial theorem is surely not reprehensible, it is natural to wonder if such techniques are necessary. In this particular case, the answer is negative: in Section 2, we shall present an elementary proof of Ryser's theorem.

A *symmetric block design* is any pair $(V, \{S_1, S_2, \dots, S_v\})$ that satisfies the hypothesis (and the conclusion) of Ryser's theorem. The *incidence matrix* A of this design is the $v \times v$ matrix, with rows indexed by $i = 1, 2, \dots, v$ and columns indexed by the elements of V, such that the ith row of A is the incidence vector of S_i; equivalently, $A = (a_{ix})$ with

$$a_{ix} = \begin{cases} 1 & \text{if} \quad x \in S_i, \\ 0 & \text{if} \quad x \notin S_i. \end{cases}$$

Note that A is the incidence matrix of a symmetric block design if and only if A is a square zero-one matrix and there are distinct integers k, λ such that

$$AA^T = (k - \lambda)I + \lambda J,$$

where I and J denote as usual the identity and all ones matrix, respectively. In Section 3, we shall prove that these conditions can be weakened: A is the incidence matrix of a symmetric block design if and only if A is a zero-one matrix, A is nonsingular, A has constant row sums, AA^T has precisely two distinct eigenvalues, and AA^T is *irreducible*, meaning that it cannot be permuted to assume the form

$$\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B, D are square matrices of positive order. The "only if" part is trivial [in particular, $k + (v - 1)\lambda$ and $k - \lambda$ are the only eigenvalues of a $v \times v$ matrix $(k - \lambda)I + \lambda J$]; our proof of the "if" part relies heavily on the Perron-Frobenius theorem.

2. FIRST VARIATION

Here, we offer an elementary proof of the following generalization of Ryser's theorem:

RYSER'S THEOREM (Version 2). *Let A be a real $v \times v$ matrix. If there are distinct k and λ such that $AJ = kJ$ and $AA^T = (k - \lambda)I + \lambda J$, then $k + (v - 1)\lambda = k^2$, $JA = kJ$, and $A^TA = (k - \lambda)I + \lambda J$.*

Proof. Writing $A = (a_{ix})$ and

$$d_x = \sum_i a_{ix}, \qquad d_{xy} = \sum_i a_{ix}a_{iy}, \qquad t = k + (v - 1)\lambda,$$

note that, as $\Sigma_x(\Sigma_i a_{ix}) = \Sigma_i(\Sigma_x a_{ix})$,

$$\sum_x d_x = vk, \tag{1}$$

and that, as $\Sigma_x(\Sigma_i a_{ix})(\Sigma_j a_{jx}) = \Sigma_i(\Sigma_j \Sigma_x a_{ix} a_{jx})$,

$$\sum_x d_x^2 = vt, \tag{2}$$

and that, as $\Sigma_x(\Sigma_i a_{ix}^2) = \Sigma_i(\Sigma_x a_{ix}^2)$,

$$\sum_x d_{xx} = vk, \tag{3}$$

and that, as $\Sigma_x\Sigma_y(\Sigma_i a_{ix}a_{iy}) = \Sigma_i(\Sigma_x a_{ix})(\Sigma_y a_{iy})$,

$$\sum_x \sum_y d_{xy} = vk^2, \tag{4}$$

and that, as $\Sigma_x\Sigma_y(\Sigma_i a_{ix}a_{iy})(\Sigma_j a_{jx}a_{jy}) = \Sigma_i[\Sigma_j(\Sigma_x a_{ix}a_{jx})(\Sigma_y a_{iy}a_{jy})]$,

$$\sum_x \sum_y d_{xy}^2 = v\left[k^2 + (v-1)\lambda^2\right], \tag{5}$$

and that, as $\Sigma_x\Sigma_y(\Sigma_i a_{ix}a_{iy})(\Sigma_r a_{rx})(\Sigma_s a_{sy}) = \Sigma_i(\Sigma_r\Sigma_x a_{ix}a_{rx})(\Sigma_s\Sigma_y a_{iy}a_{sy})$,

$$\sum_x \sum_y d_{xy}d_x d_y = vt^2. \tag{6}$$

We propose to show that the identities (1)–(6) imply the desired conclusions:

$$t = k^2, \tag{7}$$

$$d_x = k \qquad \text{for all } x, \tag{8}$$

$$d_{xy} = \begin{cases} k & \text{if} \quad x = y, \\ \lambda & \text{if} \quad x \neq y. \end{cases} \tag{9}$$

For this purpose, let us set

$$c_{xy} = \begin{cases} t(k-\lambda) & \text{if} \quad x = y, \\ 0 & \text{if} \quad x \neq y. \end{cases}$$

From (5), (6), (3), and (2), we have

$$\sum_x \sum_y \left(td_{xy} - \lambda d_x d_y - c_{xy}\right)^2 = 0,$$

which, since each $td_{xy} - \lambda d_x d_y - c_{xy}$ is a real number, implies

$$td_{xy} - \lambda d_x d_y - c_{xy} = 0 \qquad \text{for all } x \text{ and } y. \tag{10}$$

From (4) and (1), we have

$$\sum_x \sum_y (td_{xy} - \lambda d_x d_y - c_{xy}) = v(k - \lambda)(k^2 - t),$$

which, along with (10) and the assumption that $k \neq \lambda$, implies (7). Next, from (2) and (1), we have

$$\sum_x (d_x - k)^2 = v(t - k^2),$$

which, along with (7) and the assumption that each $d_x - k$ is a real number, implies (8). Finally, writing

$$b_{xy} = \begin{cases} k & \text{if} \quad x = y, \\ \lambda & \text{if} \quad x \neq y, \end{cases}$$

we obtain from (3), (4), (5)

$$\sum_x \sum_y (d_{xy} - b_{xy})^2 = 2v\lambda(t - k^2),$$

which, along with (7) and the assumption that each $d_{xy} - b_{xy}$ is a real number, implies (9). ■

In Version 2, the assumption that A is a *real* matrix can be dropped; see Ryser's second proof of his theorem [4, theorem 2.1, p. 103] or Marshall Hall's extension of the theorem [1, Theorem 10.2.3, p. 104]. However, this assumption is indispensable in our proof; we know of no elementary proof of the generalization of Version 2 where A can be a *complex* matrix.

3. SECOND VARIATION

LEMMA. *If M is a nonnegative irreducible symmetric matrix with exactly two distinct eigenvalues, then $M = uu^T + sI$ for some positive u and some s.*

Proof. Let n denote the order of M, If $n = 2$, then the conclusion follows by setting

$$u = \left[\left(\frac{d + m_{11} - m_{22}}{2}\right)^{1/2}, \quad \left(\frac{d - m_{11} + m_{22}}{2}\right)^{1/2}\right]^T,$$

$$s = \frac{m_{11} + m_{22} - d}{2}$$

with $M = (m_{ij})$ and $d = [(m_{11} - m_{22})^2 + 4m_{12}]^{1/2}$. Hence we may assume that $n \geqslant 3$.

By the Perron-Frobenius theorem [2, Theorem 9.2.1, p. 285], the characteristic equation of any nonnegative irreducible matrix has a simple root; in particular, the characteristic equation of M has a simple root, r. Every real symmetric matrix of order k has k linearly independent eigenvectors [2, Theorem 29.4, p. 76]; in particular, M has n linearly independent eigenvectors. Since only one of these n eigenvectors corresponds to r, the remaining $n - 1$ eigenvectors must correspond to the other root, s. In other words, the rank of $M - sI$ is 1. Hence $M - sI = ab^T$ for some real vectors a and b. Since M is symmetric, a and b are multiples of each other, and so $M - sI = \pm uu^T$ for some real vector u. Since M is ireducible, no component of u is zero. For any choice of three components u_i, u_j, u_k of u, the three products u_iu_j, u_iu_k, u_ju_k are off-diagonal entries of M; since M is nonnegative, the three products are nonnegative, and so u_i, u_j, u_k must have the same sign. Hence all components of u have the same sign; replacing u by $-u$ if necessary, we conclude that u is a positive vector and, since M is nonnegative, $M - sI = uu^T$. ∎

THEOREM. *A is the incidence matrix of a symmetric block design if and only if A is a zero-one matrix, A is nonsingular, A has constant row sums, AA^T is irreducible, and AA^T has precisely two distinct eigenvalues.*

Proof. As noted in the introduction, the "only if" part is trivial. To prove the "if" part, we use the Lemma with AA^T in place of M to find that $AA^T = uu^T + sI$ for some positive vector u and some s. Since A is zero-one, the diagonal elements of AA^T equal the row sums of A; since A has constant row sums, it follows that all diagonal elements of AA^T are the same. In turn, since u is a positive vector, it follows that all components of u are the same. Hence $AA^T = sI + tJ$ for some t; since A is nonsingular, $s \neq 0$. We conclude that A is the incidence matrix of a symmetric block design with $k = s + t$, $\lambda = t$. ∎

This theorem is best possible in the sense that none of its five conditions,

(a) A is a zero-one matrix,
(b) A is nonsingular,
(c) A has constant row sums,
(d) AA^T is irreducible,
(e) AA^T has precisely two distinct eigenvalues,

is implied by the four others:

To see that (a) cannot be dropped, consider

$$A = \begin{pmatrix} a & b & b & \cdots & b \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}$$

with $a = 2 - (v - 1)c$, $b = 1 + c$, $c = 2v(v + 2)/(v^3 + v^2 - 2v - 1)$. Since

$$\frac{a^2 + (v - 1)b^2 - 1}{a + vb} = \frac{a + vb}{v + 2},$$

the rank of $AA^T - I$ is 1; hence 1 is an eigenvalue of AA^T, and its multiplicity is $v - 1$. The other eigenvalue of AA^T, corresponding to the eigenvector $[a + vb, v + 2, v + 2, \ldots, v + 2]^T$, is $a^2 + (v - 1)b^2 + (v - 1)(v + 2)$; hence A is nonsingular.

To see that (b) cannot be dropped, consider any zero-one matrix A, other than the all ones or the all zeros matrix, such that all the rows of A are the same.

To see that (c) cannot be dropped, take the incidence matrix B of a symmetric block design with $k = \lambda^2 + 3\lambda + 1$ and $v = \lambda^3 + 6\lambda^2 + 10\lambda + 4$. (If $\lambda = 0$ then $B = I$; if $\lambda = 1$, then the design is the projective plane of order four. We do not know for what other values of λ such designs exist.) Then let e denote the all ones vector, and consider

$$A = \begin{pmatrix} 1 & e^T \\ e & B \end{pmatrix}.$$

Since

$$\frac{v + 1 - k + \lambda}{k + 1} = \frac{k + 1}{\lambda + 1},$$

the rank of $AA^T - (k - \lambda)I$ is 1; hence $k - \lambda$ is an eigenvalue of AA^T, and its multiplicity is $v - 1$. The other eigenvalue of AA^T, corresponding to eigenvector $[k + 1, \lambda + 1, \lambda + 1, \ldots, \lambda + 1]^T$, is $v + 1 + v(\lambda + 1)$; hence A is nonsingular.

To see that (d) cannot be dropped, consider

$$A = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

such that B is the incidence matrix of a symmetric block design.

To see that (e) cannot be dropped, consider

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

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