SIAM J. DISC. MATH.
Vol. 6, No. 3, pp. 501–521, August 1993

REPLICATING TESSELLATIONS*

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Abstract. A theory of replicating tessellation of $\mathbb{R}^n$ is developed that simultaneously generalizes radix representation of integers and hexagonal addressing in computer science. The tiling aggregates tesselate Euclidean space so that the $(m+1)$st aggregate is, in turn, tiled by translates of the mth aggregate, for each m in exactly the same way. This induces a discrete hierarchical addressing systsem on $\mathbb{R}^n$. Necessary and sufficient conditions for the existence of replicating tessellations are given, and an efficient algorithm is provided to determine whether or not a replicating tessellation is induced. It is shown that the generalized balanced ternary is replicating in all dimensions. Each replicating tessellation yields an associated self-replicating tiling with the following properties: (1) a single tile T tesselates $\mathbb{R}^n$ periodically and (2) there is a linear map A, such that $A(T)$ is tiled by translates of T. The boundary of T is often a fractal curve.

Key words. tiling, self-replicating, radix representation

AMS(MOS) subject classifications. 52C22, 52C07, 05B45, 11A63

1. Introduction. The standard set notation $X + Y = \{x + y : x \in X, y \in Y\}$ will be used. For a set $T \subset \mathbb{R}^n$ denote by $T_x = x + T$ the translate of T to point x. Throughout this paper, Λ denotes an n-dimensional lattice in $\mathbb{R}^n$. A set T *tiles* a set R *by translation by* lattice Λ if $R = \bigcup_{x\in\Lambda} T_x$ and the intersection of the interiors of distinct tiles T_x and T_y is empty. Such a tiling is called *periodic*.

In this paper, $A : \Lambda \to \Lambda$ will be an endomorphism of Λ, often given by a nonsingular $n \times n$ matrix. Given A, a finite subset $S \subset \Lambda$, containing 0, is said to *induce a replicating tessellation* or simply a *rep-tiling* of Λ if (1)

$$S_m = \sum_{i=0}^{m} A^i(S)$$

tiles Λ by translation by the sublattice $A^{m+1}(\Lambda)$ for each $m \geq 0$, and (2) every point of Λ is contained in S_m for some m. The pair (A, S) will be called a *replicating tiling pair* or simply *rep-tiling pair*. This definition of rep-tiling is related to the rep-k tiles of Golomb [11], Dekking [5], Bandt [1], and others as described later in this introduction.

The definition of rep-tiling can be restated in terms of the Voronoi cells of the lattice. Recall that a lattice Λ determines a tessellation by polytopal Voronoi cells where the *Voronoi cell* of the lattice point x is defined by $\{y \in \mathbb{R}^n : |y - x| \leq |y - z| \text{ for all } z \in \Lambda\}$. Let V_m denote the union of the Voronoi cells corresponding to the lattice points of S_m. The definition of rep-tiling is equivalent to (1) V_m tiles $\mathbb{R}^n$ by translation by the sublattice $A^{m+1}(\Lambda)$, for each $m \geq 0$, and (2) every point of $\mathbb{R}^n$ lies in V_m for some m. The set S_m (or the corresponding V_m) is called the *m-aggregate* of the pair (A, S). If S induces a replicating tessellation, then the $(m + 1)$-aggregate is tiled by $|S|$ copies of the m-aggregate for each $m \geq 0$. More precisely, $S_0 = S$ and S_{m+1} is the disjoint union

$$S_{m+1} = \bigcup_{x\in A^{m+1}(S)} x + S_m$$

for all $m \geq 0$. Hence, we have the term "replicating."

*Received by the editors May 6, 1992; accepted for publication (in revised form) July 21, 1992.

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Given $A : \Lambda \to \Lambda$ and S, a *finite address* of a lattice point $x \in \Lambda$ is a finite sequence $s_0\, s_1 \ldots s_m$ such that $x = \sum_{i=0}^{m} A^i s_i$ where $s_i \in S$. The m-aggregate is then the set of lattice points whose address has at most $(m+1)$ digits.

PROPOSITION 1. *Given endomorphism $A : \Lambda \to \Lambda$, the set $S \subset \Lambda$ induces a rep-tiling of Λ if and only if every lattice point in Λ has a unique finite address.*

Proof. Condition (2) in the definition of a rep-tiling is equivalent to every lattice point having a finite address. Given condition (2), condition (1) in the definition is equivalent to the finite address being unique. This is proved as part of Proposition 2 in §2. □

Before proceeding with the theory, consider the following three examples. The first has applications to computer arithmetic and the representation of numbers by symbol strings [20], [21]. The third has applications to data addressing in computer vision and remote sensing [6], [18], [25].

Example 1 (Radix representation in $\mathbb{Z}$). The lattice Λ is the one-dimensional integer lattice $\mathbb{Z}$, and A is multiplication by an integer b. By Proposition 1, a finite subset S of $\mathbb{Z}$ induces a rep-tiling of $\mathbb{Z}$ if every integer x has a unique base b *radix* representation $x = \sum_{i=0}^{m} s_i b^i$, where $s_i \in S$. With $S = \{0, 1, \ldots, b-1\}$ and $b \geq 2$, the Fundamental Theorem of Arithmetic states that every nonnegative integer (but no negative integer) has such a unique radix representation. The m-aggregate, in this case, is the set of integers $\{0, 1, \ldots, b^m - 1\}$, and, clearly, each aggregate is tiled by b copies of the previous aggregate. With $b \leq -2$, *every* integer has a unique radix representation. With $b = 3$ and $S = \{-1, 0, 1\}$, the radix representation is called *balanced ternary*. Every integer has a unique representation in the balanced ternary system. Although $S = \{-1, 0, 4\}$ is also a complete set of residues modulo 3, the number -2 has no base 3 radix representation with coefficients in the set $S = \{-1, 0, 4\}$. Unique representation, in a more general setting, is a main topic of this paper.

Knuth [20] gives numerous reference to alternative positional number systems dating back to Cauchy, who noted that negative digits make it unnecessary for a person to memorize the multiplication table past 5×5. For a given positive integer base b, Odlyzko [22] gives necessary and sufficient conditions for a set S of positive real numbers to have the property that every real number can be represented in the form $\pm \sum_{i=-N}^{\infty} s_i b^{-i}$, $s_i \in S$. The unique representation of integers is investigated by Matula [21].

Example 2 (Radix representation in $\mathbb{Z}[i]$). Gilbert [7]–[9] extends radix representation to algebraic numbers. For example in the Gaussian integers $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$, let $\beta = -1 + i$. Every Gaussian integer has a unique radix representation of the form $\sum_{i=0}^{m} s_i \beta^i$, where $s_i \in S = \{0, 1\}$. (This will be proved in §7.) In the terminology of this paper, if A is complex multiplication by β, then $S = \{0, 1\}$ induces a rep-tiling of the square lattice in the plane.[1] The first aggregate is the union of two translates of the zero aggregate; the second aggregate is the union of two translates of the first aggregate; in general the $(i+1)$st aggregate is the union of two translates of the ith aggregate. Using Voronoi cells to represent the lattice points, Fig. 1 illustrates how the aggregates fit together like jigsaw pieces. By contrast, with the value $\beta = 1 + i$ replacing $-1 + i$, a rep-tiling is not induced because the Gaussian integer i has no radix representation with coefficients in S.

The base β arithmetic in the Gaussian integers resembles usual arithmetic except in the carry digits. For example, $1 + 1 = 0011$ because $\beta = -1 + i$ satisfies the polynomial $x^3 + x^2 - 2$, i.e., $2 = \beta^2 + \beta^3$. So $1 + 1$ results in carrying 011 to the next three places to

[1] Katai and Szabo [16] show that for base $\beta = -k + i$, where k is a positive integer, every Gaussian integer has a unique radix representation with coefficients in $S = \{0, 1, \ldots, k^2\}$.

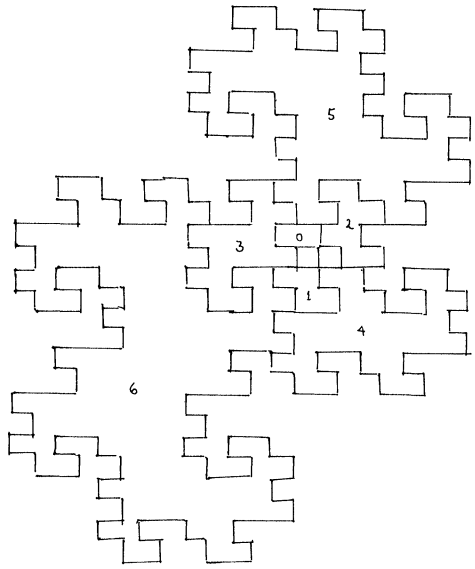

FIG. 1. *Radix representaion base* $-1+i$. *Copies of the first seven aggregates are indicated.*

the right. The ring structure of radix representation in algebraic number fields is further discussed in §7.

Example 3 (Hexagonal tiling). This example is a two-dimensional analogue of the balanced ternary of the first example. The lattice Λ is the hexagonal lattice in the plane shown in Fig. 2, and the endomorphism A is given by the matrix

$$A = \begin{pmatrix} \frac{5}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{5}{2} \end{pmatrix},$$

which is a composition of an expasnsion by a factor of $\sqrt{7}$ and an $\arctan(\sqrt{3}/2)$ rotation. The set S, consisting of the origin and the six points located at the sixth roots of unity, induces a replicating tessellation. The 0-aggregate consists of the seven cells in Fig. 3(a). The set of cells in Fig. 3(b) is a first aggregate and is the union of seven translates of the zero aggregate. The second aggregate in Fig. 3(c) is, in turn, the union of seven translates of the first aggregate. In general, the $(i+1)$st aggregate is the union of seven translates of the ith aggregate. The entire plane can be tessellated by translated copies of the ith aggregate for any i in such a way that aggregates in the tessellation are nested in the manner described above. Moreover, every hexagon lies in some aggregate. In the unique finite address $s_0 s_1 \ldots s_m$ of a cell, the digit s_i indicates the relative position of that particular cell in the ith aggregate level. Replicating hexagonal tiling is generalized to higher dimensions in §7. From a computer science point of view, hexagonal addressing is an efficient addressing system that allows for addition and multiplication of addresses based on simple sum, product, and carry tables [6], [18]. In fact, one firm has developed a planar database management system based on hexagons (Gibson and Lucas [6]).

We consider two main questions.

Question 1. Given $A : \Lambda \to \Lambda$ and a finite set of lattice points S, does S induce a rep-tiling of Λ?

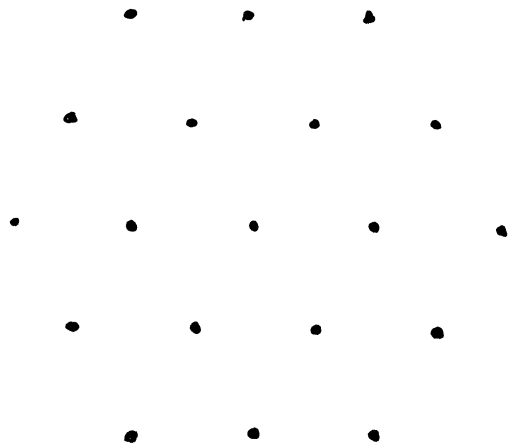

FIG. 2. *Hexagonal lattice.*

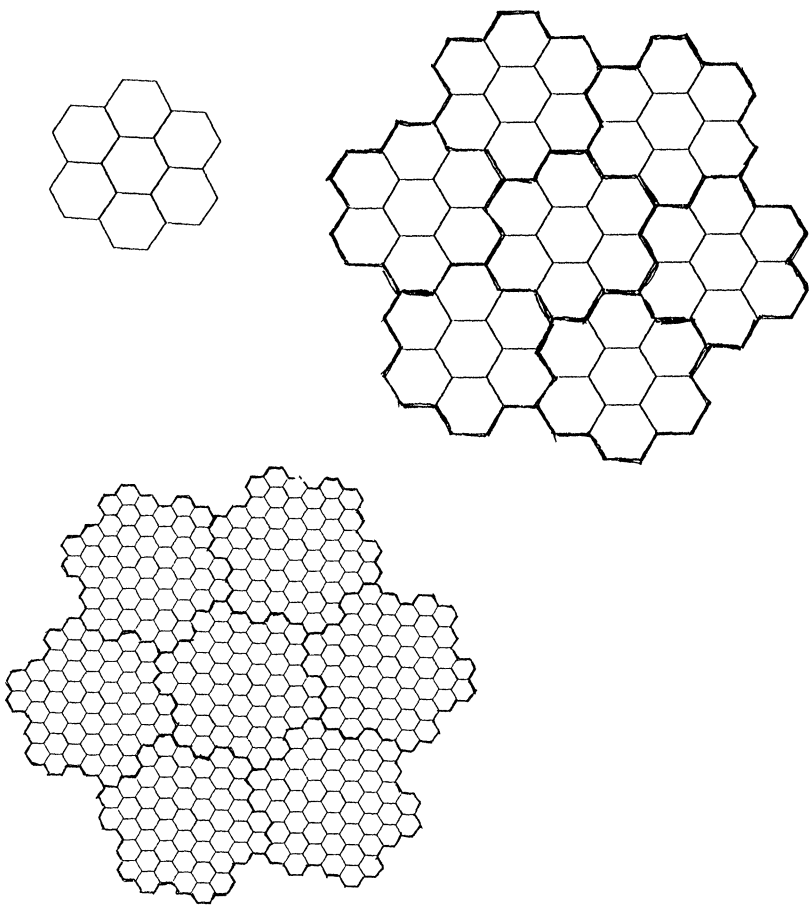

FIG. 3. *Zero, first, and second aggregates.*

Question 2. Given $A : \Lambda \to \Lambda$, does there exist some finite subset S of Λ, such that S induces a rep-tiling of Λ?

Necessary conditions for (A, S) to be a rep-tiling pair are that S be a fundamental domain for A (Proposition 2); in particular $|S| = |\det A|$. Also necessary is that A be a linear expansive map (Proposition 4), which means that the modulus of each eigenvalue of A is greater than 1. If A is linear expansive but (A, S) is not a rep-tiling pair, then, in general, not every lattice point has a finite address. However, every lattice point x does have an infinite repeating address that converges, in a certain sense, to x. This is proved in §5, where the A-adic integers are defined in analogy to the classical number theoretic p-adic integers; the A-adics are applied in §6. Section 6 contains three theorems giving various necessary and sufficient conditions for S to induce a rep-tiling, thus providing answers to Question 1. An efficient algorthm to determine whether (A, S) is a rep-tiling pair is based on one of these theorems. A fourth theorem in §6 states that, for a large class of matrices A, those with sufficiently large singular values (at least two in dimension 2), the set S of lattice points in the Voronoi region of a certain sublattice of Λ serves as a fundamental domain such that S induces a rep-tiling. This provides an answer to Question 2. The existence of an efficient algorithm, given $A : \Lambda \to \Lambda$, to decide whether or not there exists a finite set S that induces a replicating tessellation, is open.

A periodic tiling of $\mathbb{R}^n$ by translation of a single tile T by the lattice Λ is called *self-replicating* if there exists a linear expansive map $A : \Lambda \to \Lambda$, such that for each $x \in \Lambda$,

$$A(T_x) = \bigcup_{w \in S(x)} T_w$$

for some set $S(x) \subset \Lambda$. This self-replicating property originated with Golomb [11] who defined a figure to be *rep-k* if k congruent figures tile a similar figure. For example, a triangle is rep-k for k a perfect square. In this paper, tiling is restricted to lattice tiling, but similarity is generalized to allow any linear expansive map A. Giles [10] discusses the construction of rep-k figures whose boundary has Hausdorff dimension between 1 and 2, including the rep-7 Gosper "flowsnake" and the rep-16 Mandelbrot "square snowflake." The work of Dekking [4], [5], Bandt [1], Kenyon [17], and Gröchenig and Madyeh [12] all deal with the self-replicating property and use a construction similar in principle to Theorem 1. The notion of a self-replicating tiling of $\mathbb{R}^n$ is due to Kenyon [17], although the definition in [17] does not require that the tiling by translations of T be periodic, i.e., a lattice tiling. Kenyon shows that, on the line, the tiling is forced to be periodic, but not necessarily periodic in dimensions greater than one.

The main point here is that each rep-tiling pair (A, S) induces a self-replicating periodic tiling. The construction is as follows. Let

$$E_m = \sum_{i=1}^{m} A^{-i}(S).$$

Note that the E_m are nested and let

$$E = \bigcup_{m=1}^{\infty} E_m \quad \text{and} \quad T := T(A, S) = \overline{E},$$

where $\overline{E}$ denotes the closure of E.

THEOREM 1. *If (A, S) is a rep-tiling pair for Λ, then*

(1) *$T = T(A, S)$ is compact and is the closure of its interior.*

(2) *T tiles $\mathbb{R}^n$ periodically by translation by the lattice Λ.*

(3) *The tiling is self-replicating.*

The above construction of self-replicating tessellations is applied to the second and third examples in Figs. 4 and 5. Note that the tile in Fig. 4 is rep-2; the union of the two tiles (viewed at an angle $\pi/4$) is similar to the original tile. Also, the tile in Fig. 5 is rep-7; the union of the seven tiles is similar to the original tile. The proof of Theorem 1, given in §4 of this paper, is shorter and simpler than the proof of a similar theorem by Kenyon [17, Thm. 11], but the hypotheses in [17] are slightly less restrictive. Nevertheless, essentially very periodic self-replicating tiling can be obtained by the construction above (Theorem 2).

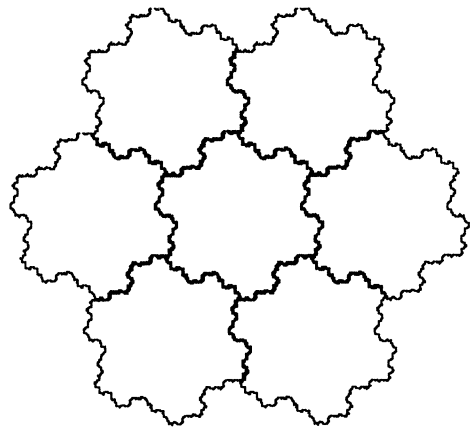

FIG. 4. *Self-replicating tessellation by tile $T(A, S)$, where $A = \begin{pmatrix} \frac{5}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{5}{2} \end{pmatrix}$ acts on the hexagonal lattice and S consists of the origin and the sixth roots of unity.*

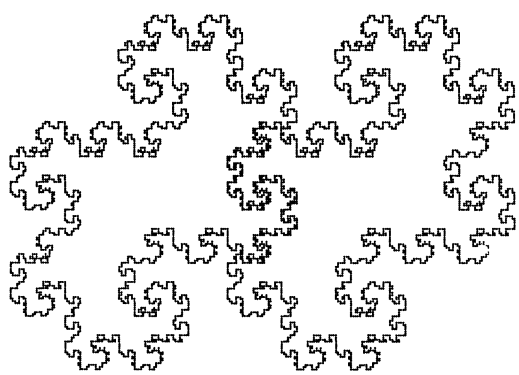

FIG. 5. *Self-replicating tessellation by tile $T(A, S)$ where A is multiplication by $-1 + i$ acting on the square lattice and S consists of the origin and the point $(1, 0)$.*

Section 7 of this paper examines the algebra, as well as the geometry, of replicating tessellations. A construction is given in which the lattice has a ring structure that

allows for both addition and multiplication of finite addresses. This construction generalizes radix representation where the base is an algebraic integer and the hexagonal tessellation used in image processing. It is shown that one important example, the generalized balanced ternary, provides replicating tessellations in all dimensions. Subjects not treated in this paper, but of related interest, include L codes and ambiguity [3], [14], [23].

2. Fundamental domain. The notation $AX = A(X)$ will be used hereafter. For a lattice Λ, both Λ and $A\Lambda$ are abelian groups under addition. Define a *fundamental domain* S to be a set of coset representatives of the quotient $\Lambda/A\Lambda$. Indeed, if V is the union of the Voronoi cells corresponding to the points of such a set S, then V is a fundamental domain (Dirichlet domain) for the group of isometries of $\mathbb{R}^n$ that are translations by vectors in $A(\Lambda)$. If S is a fundamental domain, then [15]

$$|S| = |\det A|.$$

In Example 1 of the Introduction, $A = (b)$ and $\Lambda/A\Lambda = \mathbb{Z}/b\mathbb{Z}$. So a fundamental domain S, in this case, is a complete set of residues modulo $|b|$ and $|S| = |b|$. In Example 3 of the Introduction, $|S| = det(A) = 7$, corresponding to the seven lattice points in the 0-aggregate.

PROPOSITION 2. *Let $A : \Lambda \to \Lambda$ be an endomorphism.*

(1) *If S induces a rep-tiling of Λ, then S must be a fundamental domain.*

(2) *If S is a fundamental domain, then* (i) $S_m = \sum_{i=0}^{m} A^i(S)$ *tiles Λ by translation by the sublattice $A^{m+1}(\Lambda)$ for all $m \geq 0$, and* (ii) *the finite address of a lattice point, if it exists, is unique.*

Proof. Condition (1) in the definition of rep-tiling, with $m = 0$, is equivalent to S being a fundamental domain. To show (2), assume S is a fundamental domain and, by way of contradiction, assume the existence of lattice point with two distinct finite addresses. Thus $\sum_{i=0}^{m} A^i s_i = \sum_{i=0}^{m} A^i t_i$ for some $s_i, t_i \in S$ and, without loss of generality, $s_0 \neq t_0$. But this implies that $s_0 \equiv t_0 (\mathrm{mod} A\Lambda)$, a contradiction. To show that $S_m = \sum_{i=0}^{m} A^i(S)$ tiles Λ by translation, note that $\{S_x : x \in A\Lambda\}$ tiles Λ by translates of S. Iterate to obtain successive tilings

$$\begin{aligned}
\Lambda &= S + A\Lambda \\
&= S + A(S + A\Lambda) = S + AS + A^2\Lambda = S_2 + A^2\Lambda \\
&\cdots \\
&= (S + AS + \cdots + A^m S) + A^{m+1}\Lambda = S_m + A^{m+1}\Lambda. \qquad \square
\end{aligned}$$

According to Proposition 2, if S is a fundamental domain then the finite address of a lattice point, if it exists, is unique. It is a consequence of topics in §5 that every lattice point has a unique infinite address, which coincides with the finite address in the case that all digits after a certain position are zero.

3. Equivalent tessellations. Matrix transformations $A : \Lambda \to \Lambda$ and $B : \Gamma \to \Gamma$ of lattices Λ and Γ, respectively, are said to be equivalent if there exists an invertible matrix Q, such that $B = QAQ^{-1}$ and $\Gamma = Q\Lambda$. Proposition 3 essentially states that questions about replicating tessellations are invariant under equivalence.

PROPOSITION 3. *Assume that $A : \Lambda \to \Lambda$ and $B : \Gamma \to \Gamma$ are equivalent via matrix Q.*

(1) *S is a fundamental domain for A if and only if QS is a fundamental domain for B.*

(2) *$s_0\, s_1 \ldots s_m$ is the finite address of $x \in \Lambda$ if and only if $Qs_0\, Qs_1 \ldots Qs_m$ is the finite address of $Qx \in \Gamma$.*

(3) *S induces a rep-tiling of Λ if and only if QS induces a rep-tiling of Γ.*

Proof. Concerning (1), there is a partition $\Lambda = S + A\Lambda$ if and only if there is a partition $\Gamma = Q\Lambda = QS + QA\Lambda = QS + B\Gamma$. Concerning (2), $x = \sum_{i=0}^{m} A^i s_i$ if and only if $Qx = \sum_{i=0}^{m} QA^i s_i = \sum_{i=0}^{m} B^i(Qs_i)$. Statement (3) follows from statement (2). □

Remark. Since equivalence is essentially a change of basis for the matrix A, there exist equivalent matrices in several canonical forms. By changing to a basis of the lattice Λ itself, an equivalent integer matrix $B : \mathbb{Z}^n \to \mathbb{Z}^n$ is obtained. Hence, from the point of view of replicating tessellations, there is no loss of generality in assuming that A is an integral matrix acting on the cubic lattice $\mathbb{Z}^n$. In particular, the characteristic and minimal polynomials for A have integral coefficients. Similarly, we can obtain equivalent matrices in Jordan canonical form or rational canonical form. As an example, consider the matrix A associated with the hexagonal tiling in the Introduction. Then

$$B = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}, \qquad J = \begin{pmatrix} 2+\omega_1 & 0 \\ 0 & 2+\omega_2 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & -7 \\ 1 & 5 \end{pmatrix}$$

are equivalent integral, Jordan, and rational forms, where ω_1 and ω_2 are the complex third roots of unity and the lattice for R is $\mathbb{Z}^2$. The lattice for the Jordan canonical form is actually a two-dimensional real lattice in $\mathbb{C}^2$.

Recall that a linear expansive map A is one for which each eigenvalue is greater than 1.

PROPOSITION 4. *If (A, S) is a rep-tiling pair, then A must be a linear expansive map.*

Proof. Assume that A has an eigenvalue of modulus $\epsilon < 1$. By the remark above, the $n \times n$ matrix A may be assumed in Jordan canonical form. Assume J is an $m \times m$ Jordan block of A of the form $\epsilon I + N$ corresponding to eigenvalue ϵ, where N is the nilpotent matrix consisting of all 0's except 1's just below the diagonal. (Without loss of generality, assume that J is the topmost block of A.) Let T be the projection of the fundamental domain S on the first m coordinates. For $t \in T$, an easy calculation shows that each entry in the matrix J^k is $O(k^m \epsilon^k)$. Since $|t| < C$ for all $t \in T$ and some bound C, then also $|J^k t| = O(k^m \epsilon^k)$ and $|\sum_{i=0}^{\infty} J^i t_i| < \sum_{i=0}^{\infty} O(i^m \epsilon^i) < c$ for some constant c. Hence, the component consisting of the first m coordinates of any finite address is bounded. However, there exists lattice points where this component is arbitarily large.

Next, assume that A has an eigenvalue λ_0 of modulus 1. By the remark above, A may be assumed to be an integer matrix. Let $p(x) \in \mathbb{Z}[x]$ be a factor of the characteristic polynomial, irreducible over $\mathbb{Z}$, with root λ_0. Assume that $p(x)$ has a root λ with modulus not equal to 1. If $|\lambda| < 1$, then we are done by the paragraph above, so assume $|\lambda| > 1$. There exists a Galois automorphism ϕ of $\mathbb{Q}[x]$ fixing $\mathbb{Q}$ elementwise, such that $\phi(\lambda_0) = \lambda$. Now $\phi(\lambda_0)\phi(\overline{\lambda_0}) = 1$ implies $|\phi(\overline{\lambda_0})| < 1$. Again, an eigenvalue has modulus less than 1, a contradiction. Therefore, all roots of $p(x)$ have modulus 1. However, Polya and Szegö [24, p. 145] prove the following result due to Kronecker [19]: If $p(x)$ is an irreducible monic polynomial with integer coefficients such that all roots lie on the unit circle, then the roots of $p(x)$ are roots of unity. This implies that A^j has eigenvalue 1 for some positive integer j. Because A is an integer matrix, the corresponding eigenvector x can be taken to have integer coordinates. If $s_0 s_1 \ldots s_r$ is the finite address of x, then the finite address of $x = A^j x$ is $00\ldots0 s_1 s_2 \ldots s_r$, where the initial segment has j 0's. Now x has two finite addresses, which contradicts the assumption that the finite address is unique. □

The condition that A be expansive is necessary for (A, S) to be a rep-tiling pair, but it is not sufficient. There are matrices A satisfying the eigenvalue condition that admit *no* fundamental domain S for which S induces a rep-tiling. In dimension 1, for example, $A = (2)$ is such a matrix. (This is the only example in dimension 1.) Examples in all dimensions are given in the comments after Corollary 1 of Theorem 3 in §6. Nevertheless, Proposition 4 is sharp in the sense that, for any $\epsilon > 0$, there exists a matrix A_ϵ and a fundamental domain S_ϵ, such that A_ϵ has an eigenvalue of modulus a where $|a - 1| < \epsilon$, and such that (A_ϵ, S) is a rep-tiling pair. The generalized balanced ternary is proved, in the remark at the end of §7, to be such an example.

4. Self-replicating tilings. The proof of Theorem 1 appears in this section, as well as the proof of the converse, Theorem 2.

THEOREM 1. *If (A, S) is a rep-tiling pair for Λ and $T = T(A, S)$, then*

(1) *T is compact and is the closure of its interior;*

(2) *T tiles $\mathbb{R}^n$ periodically by translation by the lattice Λ; and*

(3) *the tiling is self-replicating.*

Proof. Since, by Proposition 4, all the eigenvalues of A^{-1} are less than 1, the set E_m is bounded, the bound independent of m; therefore, T is compact. Consider statement (2) of Theorem 1. Condition (1) in the definition of a rep-tiling pair (A, S) guarantees that E_m tiles $A^{-m}(\Lambda)$ by translation by Λ. Therefore, E tiles $\bigcup_{m=1}^{\infty} A^{-m}(\Lambda)$ by translation by Λ. The facts that $\bigcup_{m=1}^{\infty} A^{-m}(\Lambda)$ is dense in $\mathbb{R}^n$ and E is bounded imply $\mathbb{R}^n = \bigcup_{x\in\Lambda} T_x$. To show that the intersection of the interiors of distinct tiles is empty, it suffices to prove that $\mu(T_x \cap T_y) = 0$, where μ is Lebesgue measure. By condition (2) in the definition of a rep-tiling pair, there is an integer m such that $x, y \in S_m$. From the definition of E it follows that $A^{m+1}(E) = \bigcup_{w\in S_m} E_w$, which implies that $A^{m+1}(T) = \bigcup_{w\in S_m} T_w$. Now $(\det A)^{m+1}\mu(T) = \mu(A^{m+1}(T)) = \mu(\bigcup_{w\in S_m} T_w) \le \sum_{w\in S_m} \mu(T_w) = (detA)^{m+1}\mu(T)$ implies that $\mu(\bigcup_{w\in S_m} T_w) = \sum_{w\in S_m} \mu(T_w)$. This, in turn, implies that $\mu(T_x \cap T_y) = 0$. Concerning statement (3) in the theorem it follows as above, with $m = 1$, that $A(T) = \bigcup_{w\in S} T_w$. Then $A(T_x) = \bigcup_{w\in S_{Ax}} T_w$.

Consider statement (1) in the theorem. To prove that T is the closure of its interior, it suffices to show that each point $x \in E$ in an interior point of T. Let F^0 denote the interior of a point set F. Assume that $0 \in T^0$. Since there is a nonnegative integer m such that $x \in E_m$, we have $z := A^m x \in A^m(E_m) = S_{m-1} \subset \Lambda$. Therefore, $x \in T^0$ if and only if $z \in (A^m T)^0$ if and only if $0 \in (-z + A^m T)^0$. Since $z + E \subseteq A^m E$, we have $T \subseteq -z + A^m T$. Therefore, if $0 \in T^0$, then $x \in T^0$. Recall that T is compact and that $\mathbb{R}^n = \bigcup_{x\in\Lambda} T_x$. Hence, to show $0 \in T^0$, it suffices to prove that $0 \notin T_x$ for all $x \in \Lambda - \{0\}$. Assume, by way of contradiction, that $0 \in T_x$ for some $x \in S_k - S_{k-1}$ for some fixed $k \ge 1$. Let $L_m = \{A^m S_m + A^{m-1} s_{m-1} + \cdots + s_0 \,:\, s_i \in S,\, s_m \ne 0\}$. We claim that $\lim_{m\to\infty} \min_{y\in L_m} |y| = \infty$. To see this, note that for any $R > 0$ there exists an integer m, such that $\bigcup_{i=0}^{m} L_i$ includes all points of Λ within a sphere of radius R. Since finite addresses are unique, the points of L_{m+1} lie outside the sphere. Next, choose B such that $|y| < B$ for all $y \in T$ and choose m_0 such that $|y| > 2B$ for all $y \in L_m$, $m \ge m_0$. Let $\alpha = \sup_{x\in\mathbb{R}^n} |Ax|/|x|$ and $\epsilon = B/\alpha^{m_0-k}$. From the choice of x, there is a $y \in E$, such that $|z| < \epsilon$, where $z = x + y$. Now $A^{m_0-k} z \in L_{m_0} + E$ implies that $|A^{m_0-k} z| > 2B - B = B$, which in turn implies that $|z| > B/\alpha^{m_0-k} = \epsilon$, a contradiction. □

The periodic, self-replicating tiling by a single tile $T(A, S)$ given by Theorem 1 is said to be *induced* by the rep-tile pair (A, S). The next result states that essentially every periodic self-replicating tiling is induced by a rep-tiling pair.

THEOREM 2. *Consider a periodic, self-replicating tiling by a single tile T. If* (1) *T is compact and is the closure of its interior, and* (2) *the origin is contained in the interior of T, then the tiling is induced by a rep-tiling pair.*

Proof. Let A and Λ be the linear expansive map and the lattice, respectively, associated with the periodic, self-replicating tiling. Let S be the finite subset of Λ, such that $A(T) = \bigcup_{x\in S} T_x$. We claim $A(\Lambda) \subseteq \Lambda$. To see this, let x be any point of Λ and note that $\bigcup_{w\in S(x)} T_w = A(T_x) = Ax + A(T) = Ax + \bigcup_{w\in S} T_w = \bigcup_{w\in Ax+S} T_w$. Because this equality involves only finitely many compact tiles, $Ax + S = S(x)$. In particular, $Ax \in S(x) \subset \Lambda$.

We next prove that (A, S) is a rep-tiling pair. To show that S is a fundamental domain for A, consider the Lebesgue measure on both sides of the equation $A(T) = \bigcup_{x\in S} T_x$. This gives $|S| = \det(A)$. Since S has the correct number of elements, it suffices to show that no two elements of S are congruent mod$(A\Lambda)$. Assume, by way of contradiction, that $s = s' + Ax$ for some $x \in \Lambda - \{0\}$. Then s lies in the interior of T_s and hence in the interior of $A(T)$. Also, s' lies in the interior of $T_{s'}$, and hence s lies in the interior of $Ax+A(T) = A(T_x)$. However, the intersection of the interiors of T and T_x is empty, and hence, the same is true for $A(T)$ and $A(T_x)$, a contradiction. By Proposition 2, condition (1) in the definition of a rep-tiling pair is satisfied.

Iterating $A(T) = \bigcup_{x\in S} T_x$, we obtain $A^m(T) = \bigcup_{x\in S_{m-1}} T_x$. Because 0 lies on the interior of T, for any lattice point x there is an integer m such that $T_x \subset A^m(T) = \bigcup_{w\in S_{m-1}} T_w$. This implies that $x \in S_{m-1}$, proving condition (2) in the definition of a rep-tiling pair.

It remains to prove that $T = T(A, S)$. By the formula in the paragraph above, $S_{m-1} \subset A^m(T)$, which implies $E_m = A^{-m}(S_{m-1}) \subset T$ for all $m \geq 0$. This, in turn, implies $T(A, S) \subseteq T$. Since $T(A, S)$ tiles $\mathbb{R}^n$ by translation by Λ, the interior points of T satisfy $T^0 \subseteq T(A, S)$. Therefore $T = \overline{T^0} \subseteq T(A, S)$. □

Theorem 2 is false without the assumption that 0 is contained in the interior of some tile T. For example, the tiling of $\mathbb{R}$ by translates of the unit interval $T = [0, 1]$ is not induced by a rep-tile pair. This tiling is indeed induced by the pair (A, S) where $A = (3)$ and $S = \{0, 1, 2\}$, but (A, S) is not a rep-tiling pair because the negative integers have no base 3 radix representation with digit set S, i.e., -1 belongs to no aggregate. However, the tiling of $\mathbb{R}$ by translates of $T = [-\frac{1}{2}, \frac{1}{2}]$ *is* induced by the rep-tile pair (A, S) where $A = (3)$ and $S = \{-1, 0, 1\}$. It is an open question whether every periodic, self-replicating tiling is induced, up to a translation, by a rep-tiling pair.

5. A-adic integers. It is assumed here that S is a fundamental domain for the matrix $A : \Lambda \to \Lambda$ and that A is expansive. This implies, in particular, that $|\det A|$ is an integer greater than or equal to 2.

LEMMA 1. *If A is a linear expansive map, then*

$$\bigcap_{i=0}^{\infty} A^i\Lambda = \{0\}.$$

Proof. If all eigenvalues have modulus greater than 1, then examination of the Jordan canonical form shows that $A^m x \to \infty$ for all nonzero x. □

For $x \in \Lambda$, let $\nu = \nu(x)$ denote the greatest integer ν, such that $x \in A^\nu\Lambda$. By Lemma 1, ν is finite except when $x = 0$, in which case we set $\nu(0) = \infty$. Then

$$|x| = \frac{1}{|\det A|^{\nu(x)}}$$

has the property that $|x| = 0$ if and only if $x = 0$ and thus defines a norm on Λ, and $d(x, y) = |x - y|$ defines a metric we call the *A-adic metric*. Two lattice points are close in the corresponding topology if their difference lies in $A^m\Lambda$ for large m. If $A = (p)$ is a one-dimensional matrix, then this reduces to the classical p-adic metric where two integers are close if their difference is divisible by a large power of p. The completion of Λ with respect to the A-adic metric will be called the *A-adic integers* and denoted $\overline{\Lambda}$. (Alternatively, the A-adic integers can be defined as an inverse limit of the system $(\{\Lambda/A^k\Lambda\}, \{f_{jk}\})$, where $f_{jk} : \Lambda/A^k\Lambda \to \Lambda/A^j\Lambda$, $j < k$ is defined by $f_{jk}\bar{x}_k = \bar{x}_j$, where $x_j \equiv x_k \mathrm{mod} A^j\Lambda$.) Note that $\Lambda \subseteq \overline{\Lambda}$. If S is any set of coset representatives for $\Lambda/A\Lambda$, then, just as for the case of the ordinary p-adic integers, there is a unique canonical representation of each A-adic integer in the form $\sum_{i=0}^{\infty} A^i s_i$, where $s_i \in S$, which will be abbreviated $s_0\, s_1\, s_2 \ldots$ and called the *A-adic address*. The partial sums in this canonical form converge to the A-adic integer in the A-adic metric.

A simple recursive algorithm to determine the A-adic address $s_0 s_1 s_2 \ldots$ of a lattice point is obtained by iteration using the partition $\Lambda = \{S + x : x \in A\Lambda\}$, from the assumption that S is a fundamental domain. This process is analogous to finding the base b digits in the radix representation of a given integer.

ALGORITHM A. The ith entry s_i, $i = 0, 1, \ldots$, in the A-adic address of a lattice point $x_0 \in \Lambda$ is the unique element of S, such that

$$s_i \equiv x_i \pmod{A\Lambda},$$

where

$$x_{i+1} = A^{-1}(x_i - s_i).$$

The A-adic address $s_0 s_1 \ldots$ of a lattice point is called *finite* if $s_i = 0$ for all i sufficiently large. The A-adic address of every lattice point is finite if and only if (A, S) is a rep-tiling pair. For $A = (3)$ and $S = \{-1, 0, 4\}$, which is in the first example of the Introduction, Algorithm A yields A-adic addresses:

$$1 = (4)(-1) = 4 + (-1)3,$$
$$-2 = 444\ldots.$$

Since -2 has no finite address, S does not induce a replicating tessellation on $\mathbb{Z}$. For the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^2$$

with

$$S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \ldots \begin{pmatrix} 6 \\ 0 \end{pmatrix} \right\}$$

the A-adic address of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \ldots.$$

Again, this shows that S does not induce a replicating tessellation of $\mathbb{Z}^2$.

If a lattice point x has an A-adic address with repeating string $s_{i+1} \dots s_{i+q}$, we say that x has a *repeating address*. Although a lattice point may not have a finite A-adic address, the next result shows that every lattice point has a repeating A-adic address.

LEMMA 2. *The* A*-adic address of any point in* Λ *is repeating.*

Proof. According to Algorithm A, when x_i takes on a value a second time the address repeats. Hence it is sufficient to show that the sequence $\{x_i\}$ is bounded. Iterating the formula in the algorithm gives $x_m = A^{-m}x - \sum_{i=0}^{m-1} A^{-m+i}s_i$. If $1/\alpha$ is the eigenvalue of A with the least modulus, then α is the eigenvalue of A^{-1} with the greatest modulus. Choose a real number a such that $1 > a > |\alpha|$. A calculation using the Jordan canonical form suffices to show that all entries of the matrix $(1/aA^{-1})^m$ tend to 0 as $m \to \infty$. This implies that $|A^{-m}x| < a^m|x|$ for m sufficiently large. Hence, there is a constant c such that $|A^{-m}x| < ca^m|x|$, where c is independent of m. This implies, for any m, a bound $|x_m| < c|x| + \frac{tc}{1-a}$, where $t = \max\{|s| : s \in S\}$. □

Note that the proof above gives an upper bound on the number of iterations in Algorithm A necessary to determine whether or not a given lattice point x has a finite address. For example, in dimension 1 with $A = (b)$ the bound is $|x| + \max\{|s| : s \in S\}/(|b| - 1)$.

6. Necessary and sufficient conditions for replicating tessellations. This section contains several necessary and sufficient conditions for the existence of replicating tessellations, thus providing some answers to the two main questions posed in the Introduction. Again it is assumed throughout that S is a fundamental domain for A and that A is expansive. Note that the matrix $(I - A^m)$ is nonsingular for any positive integer m. Otherwise, 1 would be an eigenvalue of A^m, and hence A would have an eigenvalue of modulus 1.

THEOREM 3. *Given* $A : \Lambda \to \Lambda$, *the following statements are equivalent*:

(1) S *induces a rep-tiling of* Λ.

(2) $(I - A^{m+1})^{-1}S_m$ *contains no nonzero lattice point for* $m = 0, 1, \dots$.

(3) $(I - A^{m+1})^{-1}S_m$ *contains only lattice points with finite address for* $m = 0, 1, \dots$.

Proof. (3) $\Rightarrow$ (1) Assume S does not induce a replicating tessellation on Λ. According to Lemma 2 some lattice point y has a repeating address where the repetition is not zeros. If there is an initial segment y_0 of length q before the address begins repreating, then $y - y_0 \in A^q\Lambda$ and $x = A^{-q}(y - y_0) \in \Lambda$ consists of that portion of y that repeats from the beginning. Let $s_0, s_1, \dots, s_m$ be the repeating digits in the address of x. Then $(I - A^{m+1})x = \sum_{i=0}^{m} A^i s_i$, and therefore $x \in (I - A^{m+1})^{-1}S_m$, where x does not have finite address.

(2) $\Rightarrow$ (3) Clearly, if $(I - A^{m+1})^{-1}S_m$ contains a lattice point without finite address, then it contains a nonzero lattice point.

(2) $\Rightarrow$ (1) Finally, assume that $(I - A^{m+1})^{-1}S_m$ contains a nonzero lattice point x. Then $(I - A^{m+1})x = \sum_{i=0}^{m} A^i s_i$ with $s_i \in S$. The lattice point y whose infinite address consists of the digits $s_0, s_1, \dots, s_m$ repeated satisfies the same equation $(I - A^{m+1})y = \sum_{i=0}^{m} A^i s_i$. Since $I - A^{m+1}$ is nonsingular, $x = y$ has a repeating (not finite) address, and therefore, S does not induce a rep-tiling of Λ. □

Theorem 3 implies, in particular, that if (A, S) is a rep-tiling pair, then S cannot contain any nonzero element of $(I - A)\Lambda$. In dimension 1, if $A = (b)$, then S can contain no integer divisible by $b - 1$, a result given in [21]. For example, with $S = \{-2, 0, 2\}$ there exist integers with no finite base 3 radix representation. Moreover, we have the following result.

COROLLARY 1. *Given* $A : \Lambda \to \Lambda$, *if* $\det(I - A) = \pm 1$, *then* A *admits no fundamental domain* S *such that* S *induces a rep-tiling of* Λ.

Proof. If $\det(I-A)=\pm 1$, then $(I-A)\Lambda=\Lambda$. Therefore, S must contain a nonzero element of $(I-A)\Lambda$. $\square$

Examples of such matrices acting on the cubic lattice that admit no fundamental domain include all matrices of the form

$$\begin{pmatrix} 0 & -m \\ 1 & m \end{pmatrix} \qquad \text{and} \qquad 2I+H,$$

where m is an integer and H is strictly upper trianglular.

Theorem 4, based on Theorem 3, states that only lattice points within a bounded region need be tested for the existence of a finite address. This leads to an efficient algorithm to determine, given $A:\Lambda\to\Lambda$ and a fundamental domain S, whether or not S induces a replicating tessellation of Λ.

LEMMA 3. *The sets* $(I-A^{m+1})^{-1}S_m$, $m=0,1,\ldots$ *are contained in some ball centered at the origin whose radius is independent of* m.

Proof. Let $H=A^{-1}$ and let $H=Q^{-1}\overline{H}Q$, where $\overline{H}$ is the Jordan canonical form of H and Q is an appropriate nonsingular matrix. Suppose that c is a constant, such that $\overline{H}QS$ is contained in a "box" $B(c)=\{(y_1,y_2,\ldots,y_n):|y_i|\le c,\ i=1,2,\ldots,n\}$. Furthermore, let $C=\frac{c}{a}(1/(1-a))^n$, where a is the modulus of the largest eigenvalue of H. Note that $a<1$, since it is assumed that the moduli of all eigenvalues of A are greater than one. Each entry in $\overline{H}^m$ approaches 0 as $m\to\infty$. Hence, for m large enough, say $m>m_0$, we have $B(C)\subseteq(\overline{H}^{m+1}-I)B(2C)$. For $m\le m_0$, there is a constant K, such that $(H^{m+1}-I)^{-1}(B(c)+\overline{H}B(c)+\cdots+\overline{H}^mB(c))\subseteq B(K)$. Let B' be the larger of the two boxes $B(2C)$ and $B(K)$ and let $B=Q^{-1}B'$. In the statement of the lemma, take any ball containing B.

Now $(I-A^{m+1})^{-1}S_m\subseteq B$ if and only if $S_m\subseteq(I-A^{m+1})B$. Multiplying by H^m gives the sufficient condition $B(c)+\overline{H}B(c)+\cdots+\overline{H}^mB(c)\subseteq(\overline{H}^{m+1}-I)B'$. This is true by definition for $m\le m_0$. For $m>m_0$, it suffices to examine the situation on each Jordan block of $\overline{H}$ of the form $J=\alpha I+N$, where α is an eigenvalue of H and N is the nilpotent component of the Jordan block. An upper bound on the modulus of any coordinate of $B(c)+JB(c)+\cdots+J^mB(c)$ is

$$c\sum_{k=0}^{m}\sum_{j=0}^{n}\binom{k}{j}|\alpha|^{k-j}\le c\sum_{j=0}^{n}\sum_{k=0}^{\infty}\binom{j+k}{j}a^k=c\sum_{j=0}^{n}\left(\frac{1}{1-a}\right)^{j+1}\le\frac{c}{a}\left(\frac{1}{1-a}\right)^n=C.$$

Therefore, we have $B(c)+\overline{H}B(c)+\cdots+\overline{H}^mB(c)\subset B(C)\subset(\overline{H}^{m+1}-I)B(2C)\subset(\overline{H}^{m+1}-I)B$. $\square$

THEOREM 4. *There exists a ball B centered at the origin, with radius depending only on A and S, such that (A,S) is a rep-tiling pair if and only if each lattice point in B has a finite address.*

Proof. If (A,S) is a rep-tiling pair, then every lattice point in B has a finite address because every lattice point does. The converse follows from Theorem 3 and Lemma 3. $\square$

For particular cases, it is possible to give an explicit value for the radius of the ball B. An efficient algorithm to determining whether or not (A,S) is a rep-tiling pair is obtained by applying Algorithm A to each of the finite number of lattice points in B. Then (A,S) is a rep-tiling pair if and only if each of these A-adic addresses is finite. Two examples are considered.

Similarities. Consider the case where $A = bU$, where U is an isometry and the real number b is greater than 1 in absolute value. Call such a matrix a *similarity*. The techniques of Theorem 4 yield the following version.

COROLLARY 2. *Given a similarity $A : \Lambda \to \Lambda$, a set S induces a rep-tiling of Λ if and only if every lattice point in the ball of radius* $\max\{|s| : s \in S\}/(|b| - 1)$ *centered at the origin has a finite address.*

Applying this result to the one-dimensional case gives the following corollary, which is proved by other means in [21].

COROLLARY 3. *Every integer has a unique base b radix representation with digit set S if and only if every integer in the interval*

$$\left[-\frac{\max\{|s| : s \in S\}}{|b| - 1}, \frac{\max\{|s| : s \in S\}}{|b| - 1}\right]$$

has such a representation.

Diagonalizable matrices. If matrix A is diagonalizable, for example, if the minimal polynomial of A is irreducible over the integers, then the following result is obtained by the methods of Theorem 4. Let Q be a nonsingular matrix and D a diagonal matrix such that $D = QAQ^{-1}$. By a *box* is meant a set of the form $\{(z_1, z_2, \ldots, z_n) : |z_i| \leq c_i, i = 1, 2, \ldots, n\}$ for some constants c_i. Let B_{QS} be the smallest box containing QS.

COROLLARY 4. *With notation as above, for the diagonalizable matrix A, the set S induces a rep-tiling of Λ if and only if each lattice point in $Q^{-1}(B_{QS})$ has a finite address.*

Example. Consider the matrix

$$A = \begin{pmatrix} 4 & -1 \\ -1 & 6 \end{pmatrix}$$

acting on the square lattice $\mathbb{Z}^2$. A fundamental domain S has cardinality $\det(A) = 23$; let S be the set of 23 circled latticed points in Fig. 6(a). Calculation shows that $Q^{-1}(B_{QS})$ is the rectangle indicated in the figure. It is routine to check that all lattice points within this rectangle are in the first aggregate (having, in fact, finite addresses of length at most two). By Corollary 4, this constitutes a proof that S induces an aggregate tessellation of $\mathbb{Z}^2$.

Theorem 5 essentially states that if, in adding two elements of S there is only one carry digit in the finite address of the sum, then S induces a replicating tessellation. More specifically, the sum and difference of any two vectors in the fundamental domain lie in the first aggregate.

THEOREM 5. *Given $A : \Lambda \to \Lambda$ and a fundamental domain S, if*

(1) *some aggregate contains a basis for the lattice Λ and*

(2) $S \pm S \subseteq S + A(S)$,

then S induces a rep-tiling of Λ.

Proof. By Proposition 2 it is sufficient to show that every lattice point has a finite address. Since some aggregate S_m contains a basis, every element of Λ is the sum of a finite number of elements with finite address of the form $\pm s_0 s_1 \ldots s_m$, possibly with summands repeated many times. The proof is by induction on the number k of summands. It is clearly true for $k = 0$. Assume that every lattice point that is the sum of $k - 1$ terms has a finite address and let x be the sum of k elements of S. By induction, the sum of the first $k - 1$ of these k terms has the form $x = s_0 s_1 \ldots s_q$, where $s_i \in S$. Let $y = t_0 t_1 \ldots t_m$ be the kth term, where $t_i \in \pm S$. Since $S \pm S \subseteq S + AS$, addition of x and y is performed on the respective addresses from the left, where the number of carries to

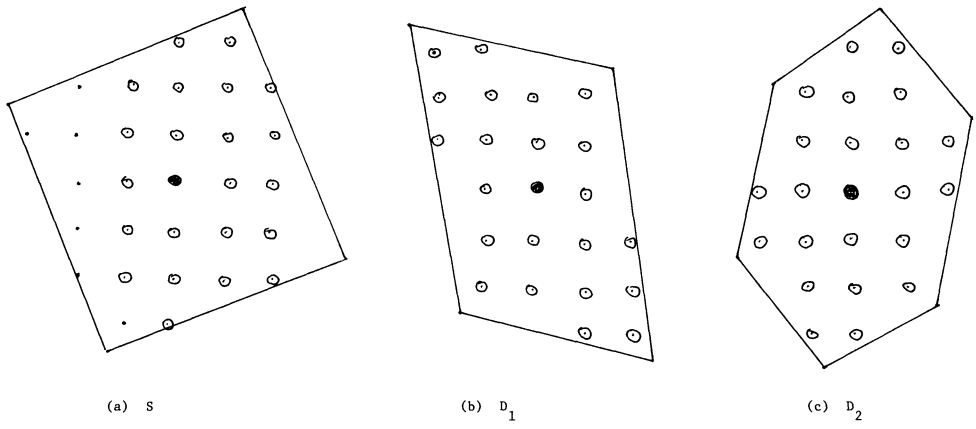

FIG. 6. *Fundamental domains for matrix A in the lattice* $\mathbb{Z}^2$.

the next digit to the right (e.g., place $i+1$) is one less than the number of summands at place i. It is not hard to deduce that the number of carries never exceeds $m+1$, and at place $\max(m,q)+i$ the number of carries does not exceed $m-i+1$. Hence, all digits after place $\max(m,q)+m+1$ are zero. □

For an n-dimensional lattice Λ, let G be the group of isometries of $\mathbb{R}^n$ generated by the n translations, taking the origin to each of n basis vectors of Λ. A *Dirichlet domain* is a subset $F \subset \mathbb{R}^n$, such that $\mathbb{R}^n$ is the disjoint union of the images of F under G. It is well known that there is a Dirichlet domain $\mathbf{V}_\Lambda$ whose closure is the Voronoi region of the lattice Λ. Call this Dirichlet domain $\mathbf{V}_\Lambda$ the *Voronoi domain* of Λ. The radius of the largest ball centered at the origin and contained in the Voronoi region is called the *packing radius* of Λ, and the radius of the smallest ball centered at the origin containing the Voronoi region is called the *covering radius* of Λ. Note that the packing radius is half the length of a minimum norm vector in Λ.

Theorem 6 states that, for a large class of matrices $A : \Lambda \to \Lambda$, there exists some fundamental domain S such that S induces a rep-tiling of Λ. In fact, a number of distinct viable fundamental domains can be obtained as the sets of lattice points contained in Voronoi domains of certain sublattices of Λ.

LEMMA 4. *$\Lambda \cap \mathbf{V}_{A\Lambda}$ is a fundamental domain for $A : \Lambda \to \Lambda$.*

Proof. Let $D = \Lambda \cap \mathbf{V}_{A\Lambda}$. By definition, $\{\mathbf{V}_{A\Lambda} + Ax : x \in \Lambda\}$ is a partition of $\mathbb{R}^n$. Hence $\{D_x : x \in A\Lambda\}$ is a partition of Λ. This is equivalent to saying that D is a fundamental domain for $A : \Lambda \to \Lambda$. □

LEMMA 5. *Given $A : \Lambda \to \Lambda$, let $D = \Lambda \cap \mathbf{V}_{A\Lambda}$. Assume that*

(1) *the set of minimum norm vectors in $A\Lambda$ contains a basis for $A\Lambda$, and*

(2) *all singular values of A are greater than $3R/r$, where r is the packing radius and R is the covering radius of $A\Lambda$.*

Then D induces a rep-tiling of Λ. In the one- and two-dimensional cases, the bound $3R/r$ can be improved to 2.

Proof. The proof uses Theorem 5 by showing that (1) D contains a basis for Λ, and (2) $D \pm D \subseteq D + AD$. To prove (1) we show that if $v_1, \ldots, v_n$ constitute a basis of minimum norm vectors of $A\Lambda$, then $A^{-1}v_1, \ldots, A^{-1}v_n$ is a basis for Λ contained in D. The condition on the singular values of A implies that $|A^{-1}x| < r/3R|x| < \frac{1}{2}|x|$ for all $x \in \mathbb{R}^n$. Therefore, $|A^{-1}v_i| < \frac{1}{2}|v_i| \le r$, which implies that $A^{-1}v_i \in D$ for all i.

Concerning the second condition, if $x \in D \pm D$, then $|x| \leq 2R$. By Lemma 4, we know that D is a fundamental domain for A, and hence, $x = s + Ay$, where $s \in D$ and $y \in \Lambda$. It now suffices to show that $y \in D$. But $y = A^{-1}(x - s)$ implies $|y| < r/3R(|x|+|s|) \leq r/3R(2R+R) = r$. Therefore, $y \in D$. The improvement in dimensions 1 and 2 is obtained by showing directly that $|Ay| \leq 2r$, and hence $|y| < r$ if all singular values of A are greater than 2. □

A similarity of the form bU, where U is an isometry and $b > 3\sqrt{n}$ ($b > 2$ in dimensions 1 and 2), satisfies the hypotheses of Lemma 5 if the lattice Λ itself has a basis of minimum norm vectors. Applying Lemma 5 in dimension 1 gives: if $|b| > 2$, then every integer has a unique base b radix representation with digits in $D = \{-\lfloor |b| - 1/2 \rfloor, \ldots, \lfloor |b|/2 \rfloor\}$. This is also proved in [21]. As another example, let D consist of the 9 lattice points with coordinates 0 or ± 1. Applying Lemma 5 to

$$A = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^2$$

implies that D induces a rep-tiling of $\mathbb{Z}^2$.

The first assumption in Lemma 5, concerning the minimum norm vectors, may very well fail in general. To remedy this situation, merely transform $A\Lambda$ to a lattice Λ_0 known to be generated by the minimum norm vectors.

THEOREM 6. *Given $A : \Lambda \to \Lambda$, let Q be any nonsingular matrix such that lattice $\Lambda_0 = Q(A\Lambda)$ is generated by its minimum norm vectors. If all singular values of QAQ^{-1} are greater than $3R/r$, where r is the packing radius and R is the covering radius of Λ_0, then $D = \Lambda \cap Q^{-1}\mathbf{V}_{\Lambda_0}$ induces a rep-tiling of Λ. In the one- and two-dimensional cases, the bound $3R/r$ can be improved to 2.*

Proof. Let $A_0 = QAQ^{-1}$. By definition, $A : \Lambda \to \Lambda$ and $A_0 : Q\Lambda \to Q\Lambda$ are equivalent. By Proposition 3 of §3, (A, D) is a rep-tiling pair for Λ if and only if (A_0, QD) is a rep-tiling pair for $Q\Lambda$. But $QD = Q\Lambda \cap \mathbf{V}_{\Lambda_0} = Q\Lambda \cap \mathbf{V}_{QA\Lambda} = Q\Lambda \cap \mathbf{V}_{A_0 Q\Lambda}$. The theorem now follows directly from Lemma 5 applied to A_0. □

Note that, in Theorem 6, if B is the matrix whose columns are a basis for Λ_0, then we can take

$$Q = BA^{-1},$$
$$D = \Lambda \cap AB^{-1}\mathbf{V}_{\Lambda_0},$$

in which case $QAQ^{-1} = BAB^{-1}$.

COROLLARY 5. *Given $n \times n$ matrix $A : \Lambda \to \Lambda$, let C be the Voronoi domain of the cubic lattice (the closure of C is a unit cube centered at the origin) and let $D = \Lambda \cap AC$. If all singular values of A are greater than $3\sqrt{n}$, then D induces a replicating tessellation of Λ. In the one- and two-dimensional cases, the bound $3\sqrt{n}$ can be improved to 2.*

Proof. Let Λ_0 be the cubic lattice so that B is the identity matrix. Then $R/r = \sqrt{n}$, $Q = A^{-1}$, $QAQ^{-1} = A$, and $D = \Lambda \cap Q^{-1}\mathbf{V}_{\Lambda_0} = \Lambda \cap AC$. The corollary now follows directly from Theorem 6. □

Corollary 5 can be applied directly to the square lattice in $\mathbb{R}^2$ to obtain the following result concerning radix representation in the Gaussian integers. If β is a Gaussian integer, not equal to 2 or $1 \pm i$, then there exists a fundamental domain D such that every Gaussian integer has a unique radix representation of the form $\sum_{i=0}^{m} s_i \beta^i$, where $s_i \in D$. Here D is a square Voronoi region centered at the origin.

A reasonable choice for Λ_0 in Theorem 6, besides the cubic lattice used in Corollary 5, is one having a small ratio R/r. One such lattice in all dimensions n is A_n^*, the dual to the root lattice A_n (generated by the roots of certain Lie algebra). A basis for A_n^* in $\mathbb{R}^n$ is any n of the $n+1$ vertices of an n-simplex centered at the origin. A particular choice for these $n+1$ vertices $b_0, b_1, \ldots, b_n$ is

$$b_i = \left(-\frac{c_0}{n}, -\frac{c_1}{n-1}, \ldots, -\frac{c_{i-1}}{n-i+1}, c_i, 0, \ldots, 0\right),$$

where $c_i = (((n-i)(n+1))/(n-i+1)n)^{\frac{1}{2}}$. Note that the b_i are unit vectors. Let $\mathbf{V}_n$ denote the Voronoi domain of the lattice A_n^*. The closure of $\mathbf{V}_2$ and $\mathbf{V}_3$ are a regular hexagon and truncated octahedron, respectively. In general, the n-dimensional Voronoi region is an n-dimensional permutahedron, congruent to a polytope with $(n+1)!$ vertices obtained by taking all permutations of the coordinates of the point $(-n/2, (-n+2)/2, (-n+4)/2, \ldots, (n-2)/2, n/2)$ in $\mathbb{R}^{n+1}$ [2]. It is known [2] that the packing radius of this lattice is $\frac{1}{2}$ and the covering radius is $\frac{1}{2}\sqrt{(n+2)/3}$. So, in applying Theorem 6, take

$$\begin{aligned} D &= \Lambda \cap AB^{-1}(\mathbf{V}_n), \\ QAQ^{-1} &= BAB^{-1}, \\ 3R/r &= \sqrt{3(n+2)}, \end{aligned}$$

where B is the matrix whose columns are the basis vectors b_i.

Example. The matrix

$$A = \begin{pmatrix} 4 & -1 \\ -1 & 6 \end{pmatrix}$$

discussed earlier in this section satisfies the hypotheses of Corollary 5 and also the hypotheses of Theorem 6 when $\Lambda_0 = A_n^*$. Applying each of these results, two fundamental domains D_1 and D_2 are obtained, each of which induces a rep-tiling of $\mathbb{Z}^2$. These fundamental domains are indicated by circled dots in Figs. 6(b) and 6(c). Note that all three fundamental domains in Fig. 6 are slightly different.

In dimensions 1 and 2, Theorem 6 is best possible in the following sense. Consider the matrix $2I$ whose unique singular value is exactly 2. According to the remarks following Corollary 1 of Theorem 3, this matrix admits no fundamental domain S such that S induces a rep-tiling of $\mathbb{Z}^n$.

7. An algebraic construction. In this section, tessellations with a ring structure are constructed, allowing for multiplication, as well as addition, of lattice points. This construction generalizes radix representation, where the base is an algebraic integer, and the hexagonal tessellation used in image processing.

To construct the lattice, consider a monic polynomial $f(x) = x^n - a_{n-1}x^{n-1} + \ldots - a_0 \in \mathbb{Z}[x]$. In the quotient ring $\Lambda_f = \mathbb{Z}[x]/(f)$, let $\alpha = x + (f)$. Then Λ_f has the structure of a free abelian group Λ_f with basis $(1, \alpha, \alpha^2, \ldots, \alpha^{n-1})$. Λ_f can be realized (in many ways) as a lattice in $\mathbb{R}^n$ by embedding the n basis elements as n linearly independent vectors in $\mathbb{R}^n$. For example, the basis vectors can be identified with the standard unit vectors along the coordinate axes of $\mathbb{R}^n$. According to Proposition 3, questions about aggregate tessellation are independent of how Λ_f is realized. Now Λ_f is the basic lattice

of our construction. Addition and multiplication of lattice points is just addition and multiplication in the ring $\Lambda_f = \mathbb{Z}[x]/(f)$.

In the special case that $f(x)$ is irreducible over $\mathbb{Z}$, then, as rings, $\Lambda_f = \mathbb{Z}[x]/(f) \cong \mathbb{Z}[\alpha]$, where α is any root of $f(x)$ in an appropropriate extension field of the rationals. For example, if $f(x) = x^2 + 1$, then the lattice Λ_f is the ring of Gaussian integers $\mathbb{Z}[i]$ with basis $(1, i)$ and can be realized as the square lattice in the complex plane. If $f(x) = x^2 + x + 1$, then the lattice Λ_f can be realized as the hexagonal lattice in the complex plane with basis $(1, -\frac{1}{2} + \sqrt{3}/2i)$. More generally, if $f(x)$ is any monic quadratic with complex roots $\alpha, \overline{\alpha}$, then $\Lambda_f = Z[\alpha] = \{a + b\alpha \,:\, a, b \in \mathbb{Z}\}$ can be considered a lattice in the complex plane. In this case, the addition and multiplication in the lattice $Z[\alpha]$ is the ordinary addition and multiplication of complex numbers.

To obtain a replicating tessellation, let β be any element of the lattice Λ_f and define the linear transformation

$$A_\beta : \Lambda \to \Lambda$$

by

$$A_\beta(x) = \beta x.$$

If S is a finite set of lattice points, then the address $s_0 s_1 \ldots s_m$ denotes the lattice point

$$\sum_{i=0}^{m} s_i \beta^i = \sum_{i=0}^{m} A_\beta^i s_i,$$

where $s_i \in S$. In other words, (A_β, S) is a rep-tiling pair for Λ_f if and only if each element of Λ_f has a unique radix representation base β with coefficients in S. Proposition 2 applies directly to this situation.

COROLLARY 6. *If every element of Λ_f has a unique base β finite address with coefficients in S, then S is a complete set of residues of Λ_f modulo $\beta\Lambda_f$ and $|S|$ is the absolute value of the constant term in the characteristic polynomial of A_β.*

Proof. $|S| = |\det(A_\beta)| = |$ is the constant term in the characteristic polynomial of $A_\beta|$. $\square$

Consider two special cases of the above construction.

Radix representation of algebraic numbers. Let β be an algebraic integer and S a finite set of elements in $\mathbb{Z}[\beta]$. The relevant question is: Does every element of $\mathbb{Z}[\beta]$ have a unique radix representation $\sum_{i=0}^{m} s_i \beta^i$, where $s_i \in S$? If $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \mathbb{Z}[x]$ is the minimal monic polynomial for β, then $\Lambda_f = \mathbb{Z}[\beta]$ and with respect to the basis $(1, \beta, \ldots, \beta^{n-1})$

$$A_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

acts on the cubic lattice $\mathbb{Z}^n$. Now every element of $\mathbb{Z}[\beta]$ has a unique radix representation base β if and only if S induces a rep-tiling of $\mathbb{Z}^n$. By part 1 of Corollary 6 the cardinality

of a fundamental domain S is $|N(\beta)|$, where $N(\beta) = (-1)^n a_0$. $N(\beta)$ is the *norm* of β and can alternatively be defined as the product of all conjugates of β.

Gilbert [8] asks about the case where $S = \{0, 1, \ldots, N(\beta) - 1\}$. Consider the example $\beta = -1 + i$ and $S = \{0, 1\}$; then $\mathbb{Z}[\beta] = \mathbb{Z}[i]$, and this is exactly Example 2 of the Introduction concerning the Gaussian integers. Corollary 2 of Theorem 4 applies to this situation. Multiplication by the complex number β is a similarity (the composition of a $\pi/4$ rotation and a stretching by a factor of $\sqrt{2}$), and hence, to determine whether or not every element of $\mathbb{Z}[\beta]$ has a base β finite address, it is sufficient to check that each element in the ball of radius $\max\{|s| : s \in S\}/|\beta| - 1$ in the complex plane has a finite address. There are exactly 21 Gaussian integers within a ball of radius $1/\sqrt{2} - 1$. Testing with Algorithm A shows that all 21 have finite addresses (for example, $-1 = 10111$ and $-2 - i = 110010111$). Therefore, every Gaussian integer has a unique base β finite address. For β satisfying a quadratic polynomial $x^2 + cx + d$, Gilbert states [8] that every element of $\mathbb{Z}[\beta]$ has a unique radix representation with coefficients in $S = \{0, 1, \ldots, |d| - 1\}$ if and only if $d \geq 2$ and $-1 \leq c \leq d$.

Generalized balanced ternary. The following example simultaneously generalizes the balanced ternary representation of integers in the first example of the Introduction and the two-dimensional hexagonal tessellation of the third example. Let $f(x) = x^n + x^{n-1} + \cdots + 1$ and denote by $\Lambda_n = \mathbb{Z}[x]/(f)$ the corresponding n-dimensional lattice. As previously mentioned, Λ_1 and Λ_2 can be realized as the integer and hexagon lattices in dimensions 1 and 2, respectively. Let $\omega = x + (f)$ denote the image of x in the quotient ring Λ_n and note that $\omega^{n+1} = 1$ in Λ_n . Let $\beta = 2 - \omega$. With respect to the basis $(1, \omega, \omega^2, \ldots, \omega^{n-1})$

$$A_\beta = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 2 & 0 & \cdots & 0 & 1 \\ 0 & -1 & 2 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 3 \end{pmatrix}.$$

Define $S_n = \{\epsilon_0 + \epsilon_1\omega + \epsilon_2\omega^2 + \cdots + \epsilon_n\omega^n : \epsilon_i \in \{0, 1\}\}$. Note that $|S_n| = 2^{n+1} - 1$ because $1 + \omega + \cdots + \omega^n = 0$ and also $det(A_\beta) = 2^{n+1} - 1$. Therefore, S_n has the appropriate number of elements to serve as a fundamental domain for A_β. For $n = 1$, we have

$$\Lambda_1 = \mathbb{Z},$$
$$\beta = 3,$$
$$S = \{-1, 0, 1\},$$
$$A_\beta = (3),$$

which leads to the balanced ternary representation of the integers. For $n = 2$, with respect to the standard basis, we have

$$\Lambda = \text{ the hexagonal lattice},$$
$$\beta = \tfrac{5}{2} - \tfrac{\sqrt{3}}{2}i,$$

$$S = \{0, 1, \omega, \ldots, \omega^5 \ : \ \omega \text{ is a 6th root of unity}\},$$

$$A_\beta = \begin{pmatrix} \frac{5}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{5}{2} \end{pmatrix},$$

which leads to the hexagonal rep-tiling of the third example in the Introduction.

COROLLARY 7. *The generalized balanced ternary pair* (A_β, S_n) *yields a rep-tiling of* Λ_n.

The proof of Corollary 7 will follow from some properties of the addition and multiplication of addresses in the generalized balanced ternary. An element $s = \epsilon_0 + \epsilon_1\omega + \epsilon_2\omega^2 + \cdots + \epsilon_n\omega^n \in S_n$ can be encoded by a corresponding binary string $b_s = \epsilon_n\epsilon_{n-1}\ldots\epsilon_0$. Note that, as in ordinary integer notation, the order of the digits is reversed. In the generalized balanced ternary, addition and multiplication can be carried out by simple and fast bit string routines. Define three operations on such binary strings as follows. First, $b \oplus b'$ is circular base 2 addition; a carry from the ith column goes to the $(i+1)$st column mod$(n+1)$. Note that the column numbers increase to the left. This first operation is equivalent to ordinary addition mod$(2^{n+1} - 1)$. For example, $1011 \oplus 1110 = 1010$. Second, $b \boxplus b'$ is base 2 addition with no carries. For example, $1011 \boxplus 1010 = 0001$. Third, $T(s)$ is the shift one position to the right mod$(n+1)$. For example, $T(1011) = 1101$. Using the facts that $\omega^{n+1} = 1$ and $2 = \omega + \beta$ it can be routinely checked that if $s, s' \in S_n$, then in Λ_n we have

$$s + s' = s_0 + s_1\beta,$$

where

$$b_{s_0} = b_s \oplus b_{s'},$$

$$b_{s_1} = T[b_s \boxplus b_{s'} \boxplus (b_s \oplus b_{s'})].$$

(The latter expression for b_{s_1} yields a 1 or 0 at those positions where a carry in $b_s \oplus b_{s'}$ has or has not, respectively, occurred.) Addition of addresses is accomplished by using the carry rule above (sum $= s_0$; carry $= s_1$). Addition corresponds to vector addition in $\mathbb{R}^n$. Multiplication also uses the rule for addition and $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$. For example with $n = 2$, let $x = (110) + (010)\beta$ and $y = (101) + (110)\beta$. Then $x + y = (100) + (001)\beta + (110)\beta^2$ and $xy = (010) + (100)\beta + (001)\beta^2$. (We have used the fact that $111 = 000$.)

Proof of Corollary 7. Note (1) S contains a basis $1, \omega, \ldots, \omega^{n-1}$ for the lattice Λ_n. Also, the comments above concerning bit string operations imply that and (2) $S \pm S \subseteq S + \beta S$. The corollary then follows immediately from Theorem 4. $\square$

Remark. The eigenvalues of A_β for the generalized balanced ternary are $\{2 - \omega \ : \ \omega$ is an (n+1)st root of unity, $\omega \neq 1\}$. Therefore, as $n \to \infty$, the minimum modulus of an eigenvalue tends to 1, but (A_β, S_n) is a rep-tiling pair for all n. This gives the example mentioned at the end of §3.

Appealing geometric properties of the generalized balanced ternary tessellation can be obtained by embedding the generator vectors $1, \omega, \ldots, \omega^n$ for the lattice Λ_n at the points $b_0, \ldots b_n$ that generate the dual root lattice A_n^* as descibed in the previous section. Then the Voronoi regions in dimensions 2 and 3, as previously mentioned, are regular hexagons and truncated octahedra, respectively.

Acknowledgments. The author thanks both referees for their valuable suggestions, in particular, for drawing our attention to recent references and for the proof of the last part of Proposition 4.

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