Self-Similar Polygonal Tiling

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Abstract. The purpose of this paper is to give the flavor of the subject of self-similar tilings in a relatively elementary setting, and to provide a novel method for the construction of such polygonal tilings.

1. INTRODUCTION. Our goal is to lure the reader into the theory underlying the figures scattered throughout this paper. The individual polygonal tiles in each of these tilings are pairwise similar, and there are only finitely many up to congruence. Each tiling is *self-similar*. None of the tilings are periodic, yet each is *quasiperiodic*. These concepts, self-similarity and quasiperiodicity, are defined in Section 3 and are discussed throughout the paper. Each tiling is constructed by the same method from a single self-similar polygon.

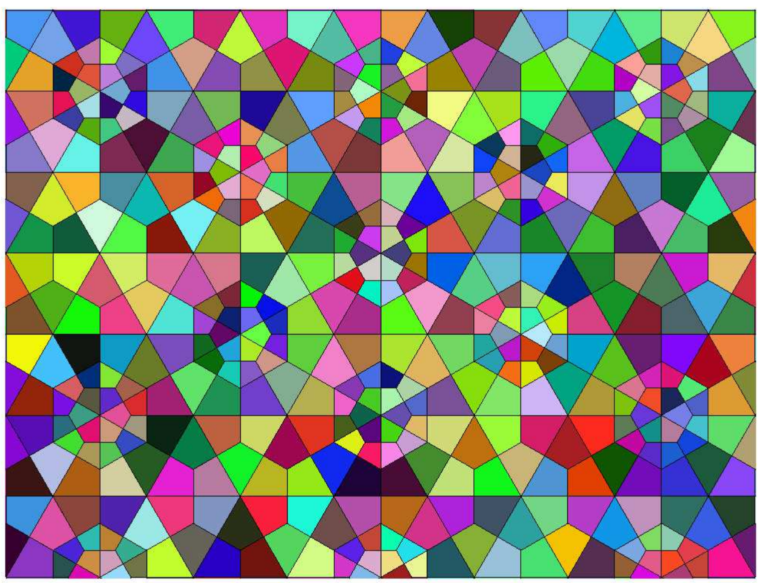


Figure 1. A self-similar polygonal tiling of order 2.

For the tiling T of the plane, a part of which is shown in Figure 1, there are two similar tile shapes, the ratio of the sides of the larger quadrilateral to the smaller quadrilateral being $\sqrt{3} : 1$. In the tiling of the entire plane, the part shown in the figure appears "everywhere," the phenomenon known as quasiperiodicity or repetitivity. The tiling is self-similar in that there exists a similarity transformation ϕ of the plane such that, for each tile $t \in T$, the "blown up" tile $\phi(t) = \{\phi(x) : x \in t\}$ is the disjoint union of the original tiles in T.

The tiling in Figure 2 is also self-similar and quasiperiodic. There are again two tile shapes, shown in dark and light grey in Figure 3. The large (dark grey) tile (L),

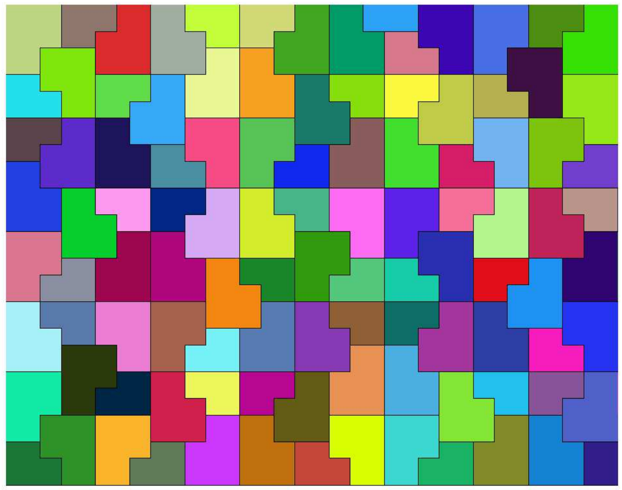


Figure 2. A golden bee tiling.

the small (light grey) tile (S), and their union, call it G, are pairwise similar polygons. The hexagon G, called the golden bee in [**15**], appears in [**9**] where it is attributed to Robert Ammann. For an entertaining piece on "The Mysterious Mr. Ammann," see the article by M. Senechal [**17**]. If $\tau = (1+\sqrt{5})/2$ is the golden ratio and $a = \sqrt{\tau}$, then the sides of L are a times as long as the sides of S, and the sides of G are a times as long as the sides of L. Except for non-isosceles right triangles, the golden bee is the only polygon that can be partitioned into a non-congruent pair of scaled copies of itself [**16**].

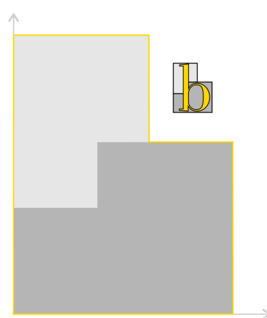


Figure 3. The golden bee is an example of a self-similar polygon. The ratio of the length of the left side to the rightmost side is the golden ratio, and also of the bottom side to the topmost side. The inset picture relates this hexagon to the letter bee.

Before veering into the general case, we give an informal description of how infinitely many golden bee tilings, like the one in Figure 2, can be obtained from the golden bee polygon in Figure 3. Figure 4 illustrates how a canonical sequence of tilings $\{Q_k\}_{k=0}^{\infty}$ can be constructed recursively by "expanding and splitting." Each tiling in the sequence uses only copies of the large (L) dark grey and small (S) light grey tiles of Figure 3. Note that Q_2 is the disjoint union of one copy of Q_0 and one copy of Q_1.

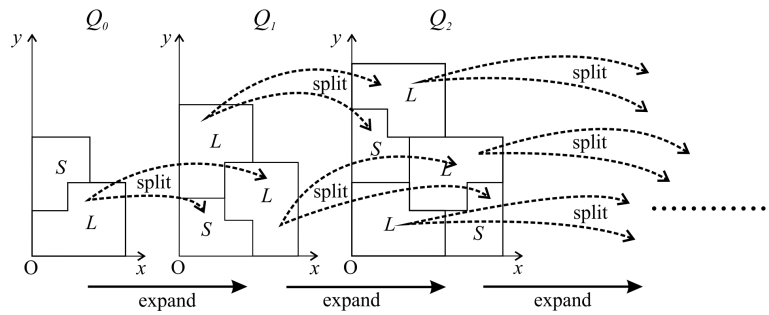


Figure 4. Inductive construction of the canonical sequence of golden tilings $\{Q_k\}$. Each tiling is obtained from its predecessor by scaling by the square root of the golden ratio, then subdividing each inflated large tile into a large (L) and small (S) tile, as in Figure 3.

Similarly, Q_{k+2} is the disjoint union of a copy of Q_k and a copy of Q_{k+1} for all $k \geq 0$. This provides a way of embedding a copy of Q_k into a copy of Q_{k+1} (call this a type 1 embedding), and another way of embedding a copy of Q_k into a copy of Q_{k+2} (call this a type 2 embedding). The first type applied successively twice yields a different tiling from the one obtained by applying the second type once.

Fix a copy of Q_0 (the first column of Figure 5). It follows from the paragraph above that, for each infinite sequence of the symbols 1 and 2, for example $21212\cdots$, one obtains a nested sequence of tilings, each tiling in the sequence congruent to Q_k for some k. The example $G(2) \subset G(21) \subset G(212) \subset G(2121) \subset G(21212) \subset \cdots$ is illustrated in the middle row in Figure 5. Similarly, the top row illustrates the construc-

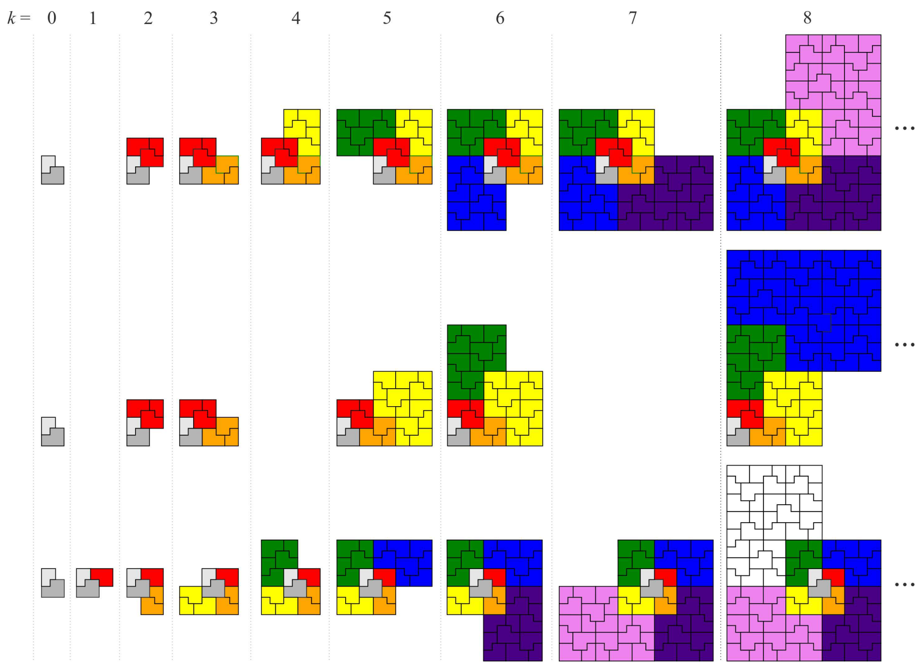


Figure 5. The construction of $G(11111111)$ (bottom), $G(21212)$ (middle) and $G(2111111)$ (top). The tilings in the kth column are congruent to Q_k. In general, the tiling $G(\theta_1\theta_2\ldots\theta_K)$ is congruent to $Q_{\theta_1+\theta_2+\cdots+\theta_K}$.

tion of $G(2111111)$ and the bottom row illustrates the construction of $G(11111111)$. The union of the tiles, for example

$$G(21212\dots) = G(2) \cup G(21) \cup G(2121) \cup G(21212) \cup \cdots ,$$

is a tiling of a region in the plane of infinite area. In this way, for each infinite sequence θ with terms in $\{1, 2\}$, there is a corresponding tiling $G(\theta)$, which is referred to as a *golden bee tiling*. This ad hoc procedure for obtaining golden bee tilings from the single golden bee G has a simple description in the general case. This is the construction in Definition 3 of Section 4.

Properties of the golden bee tilings include the following, extensions to the general case appearing in Section 6.

- $G(\theta)$ is a tiling of the plane for almost all θ. What is meant by "almost all" and for which θ the statement is true is discussed in Section 6.
- $G(\theta)$ is self-similar and quasiperiodic for infinitely many θ. Results on precisely which θ appear in Section 6.
- $G(\theta)$ is nonperiodic for all θ.
- There are uncountably many distinct golden bee tilings up to congruence.
- The ratio of large to small tiles in any a ball of radius R centered at some fixed point tends to the golden ratio $(1 + \sqrt{5})/2$ as $R \to \infty$. The general method for calculating such ratios is demonstrated after Example 3 in Section 5.

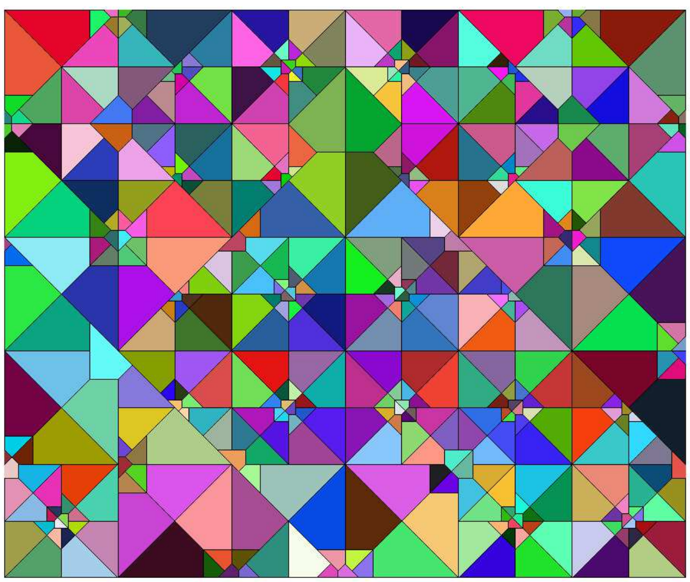


Figure 6. A self-similar polygonal tiling of order 6; see Example 4.

2. WHAT IS IN THIS PAPER. The organization of this paper is as follows. As a motivating example, we informally explored the golden bee tilings in Section 1. Section 3 contains background and definitions, in particular an explanation of exactly what is meant by a self-similar polygonal tiling. Our general construction of self-similar polygonal tilings, the subject of Section 4, is based on what we call a generating pair (Definition 2). The crux of the construction, generalizing that of the golden tilings of Section 1, is contained in Definition 3. The main result of the paper is Theorem 1, stating that our construction indeed yields self-similar polygonal tilings of the plane. Examples of self-similar polygonal tilings appear in Section 5; the question of which polygons admit self-similar tilings leads to an intriguing polygonal taxonomy. Section 6 elaborates on Theorem 1, delving into some of its subtleties. There remains much to be learned about self-similar polygonal tiling; basic problems, posed in Section 7, remain open.

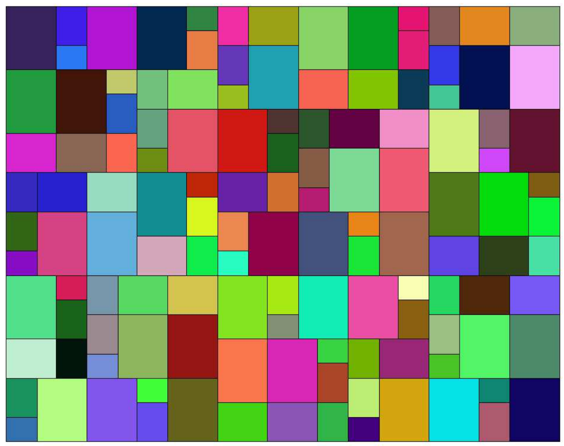


Figure 7. A tiling based on sporadic generating pair A in Figure 13; see Example 4.

3. SELF-SIMILAR POLYGONAL TILINGS. There is a cornucopia of tilings of the plane possessing some sort of regularity. The history of such tilings goes back to antiquity, and the mathematical literature is replete with papers on the subject. On the decorative side, there are, for example, the 14^{th} century mosaic tilings on the Alhambra palace in Granada, Spain, and the tilings in M. C. Escher's 20^{th} century prints. On the mathematical side, there are, for example, the tilings by regular polygons dating back at least to J. Kepler, tilings with large symmetry group as studied by Grünbaum and Shephard [**9**] and many others, and the Penrose tilings [**11**] and their relatives. Our goal in this paper is to give the flavor of the subject of self-similar tilings in a relatively elementary setting, and to provide a novel method for the construction of such polygonal tilings.

After a few basic definitions, we take a closer look at the concepts of self-similarity and quasiperiodicity. A *tiling* of the plane is a set of pairwise disjoint compact sets whose union is the plane. *Disjoint* means that a pair of tiles can intersect only at a subset of their boundaries. A set $\mathcal{P}$ of tiles is called the *prototile set* of a tiling T if, up to congruence, $\mathcal{P}$ contains exactly one copy of each tile in T. The prototile set in Figure 1 consists of the two quadrilaterals. The *order* of the tiling is the number of tiles in its prototile set. Whereas the tilings in Figures 1 and 2 have order 2, the tiling in Figure 6 has order 6. In this paper, all tilings T have finite order, and all the tiles in T have the same shape up to similarity. Moreover, all tilings in this paper are polygonal, the tiles being closed polygons with non-crossing (except at vertices) sides and positive area. Therefore, the prototile set of our tilings will consist of finitely many pairwise similar polygons.

A tiling of the plane is *periodic* if there is a translation of the plane that leaves the tiling invariant (as a whole fixed); otherwise the tiling is *nonperiodic*. Quasiperiodicity, a property less stringent than periodicity, has gained considerable attention since the 1984 Nobel Prize winning discovery of quasicrystals by Shechtman, Blech, Gratias, and Cahn [**18**]. Quasicrystals are materials intermediate between crystalline and amorphous, materials whose molecular structure is nonperiodic but nevertheless exhibits long range order as evidenced by sharp "Bragg" peaks in their diffraction pattern. Define a *patch* of a tiling T as a subset of T whose union is a topological disk. A tiling of the plane is *quasiperiodic* if, for any patch U, there is a number $R > 0$ such that any disk of radius R contains, up to congruence, a copy of U. This is what we meant by saying that every patch of the tiling appears everywhere in the tiling. If you were placed on a quasiperiodic tiling, then your local surroundings would give no clue as to where you were globally. The tilings in Figures 1, 2, and 6, although nonperiodic, are quasiperiodic.

A *similitude* f of the plane is a transformation with the property that there is a positive real number r, the *scaling ratio*, such that $|f(x) - f(y)| = r\,|x - y|$, where $|\cdot|$ is the Euclidean norm. A similitude of the plane is, according to a classification, either a stretch rotation or a stretch reflection. Self-similarity, in one form or another, has been intensely studied over the past few decades — arising in the areas of fractal geometry, Markov partitions, symbolic dynamics, radix representation, and wavelets. The tiles arising in these subjects usually have fractal boundaries. R. Kenyon [**10**], motivated by work of W. P. Thurston [**19**], proved the existence of a wide class of self-similar tilings. Explicit methods exist for the construction of certain families of self-similar tilings: digit and crystallographic tilings [**20**] and the Rauzy [**13**] and related tilings. In this paper, a tiling T is called *self-similar* if there is a similitude ϕ with scaling ratio $r(\phi) > 1$ such that, for every $t \in T$, the polygon $\phi(t)$ is the disjoint union of tiles in T. The map ϕ is called a *self-similarity* of the tiling T.

Since all tiling figures in this paper are, of necessity, just a part of the tiling, and because quasiperiodicity and self-similarity are global properties, it is not possible to say, from the figure alone, whether or not the actual tiling is quasiperiodic or self-similar.

In order to keep technicalities to a minimum, we restrict this paper to polygonal tiling.

Definition 1. A **self-similar polygonal tiling** is a tiling of the plane by pairwise similar polygons that is (1) self-similar, (2) quasiperiodic, and (3) of finite order.

Immediate consequences of the above definition are the following.

- Self-similar polygonal tilings are hierarchical. Using the notation ϕ^k for the k-fold composition, if T is a self-similar tiling with self-similarity ϕ, then $T, \phi(T), \phi^2(T), \dots$ is a sequence of nested self-similar tilings, each at a larger scale than the previous.

- If p is any polygon in the prototile set of a self-similar polygonal tiling, then there exist similitudes $f_1, f_2, \dots, f_N$, $N \geq 2$, each with scaling ratio less than 1, such that

$$p = \bigsqcup_{n=1}^{N} f_n(p), \tag{1}$$

where $\bigsqcup$ denotes a pairwise disjoint union. In the fractal literature, $F = \{f_1, f_2, \dots, f_N\}$ is an example of an iterated function system and p is an example of the attractor of the iterated function system [**2**].

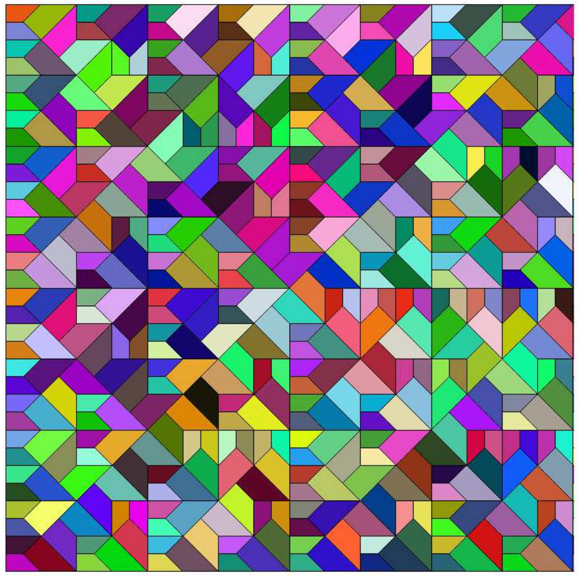


Figure 8. Tiling based on sporadic generating pairs C in Figure 13; see Example 4.

4. THE CONSTRUCTION. Our construction of self-similar polygonal tilings is contained in Definition 3. It relies on what we call a generating pair, whose Definition 2 is clearly motivated by the equation (1) that must hold for any self-similar polygon tiling.

Definition 2. Let p be a polygon and $F = \{f_1, f_2, \dots, f_N\}$, $N \geq 2$, a set of similitudes with respective scaling ratios $r_1, r_2, \dots, r_N$. If there there exists a real number $0 < s < 1$ and positive integers $a_1, a_2, \dots, a_N$ such that

$$p = \bigsqcup_{f \in F} f(p), \qquad \text{and} \qquad r_n = s^{a_n}, \tag{2}$$

for $n = 1, 2, \dots, N$, then (p, F) will be called a **generating pair**. The second equation is essential for insuring that the constructed tilings have finite order; see Question 3 and Proposition 1 of Section 6.

Example 1 (Generating pair for the golden bee). Let $s = 1/\sqrt{\tau}$, where τ is the golden ratio. For the golden bee, the generating pair is $(G, \{f_1, f_2\})$, where

$$f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} s \\ 0 \end{pmatrix}, \qquad f_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s^2 & 0 \\ 0 & -s^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The respective scaling ratios are $r_1 = s$, $r_2 = s^2$ and $(a_1, a_2) = (1, 2)$.

From the equations (2), since the area of p is equal to the sum of the areas of $f(p)$, for $f \in F$, we must have

$$\sum_{n=1}^{N} s^{2a_n} = 1. \tag{3}$$

Note that, for any set $\{a_1, a_2, \dots, a_N\}$ of positive integers, equation (3) has a unique positive solution s. We will, without loss of generality, always assume that $g := \gcd(a_1, \dots, a_N) = 1$; otherwise s can be replaced by s^g.

Let $[N] = \{1, 2, 3, \dots, N\}$. Denote by $[N]^*$ the set of all finite strings over the alphabet $[N]$ and by $[N]^\omega$ the set of all infinite strings over the alphabet $[N]$. For $\sigma \in [N]^*$, the length of σ is denoted $|\sigma|$. The following simplifying notation is useful. For $\sigma = \sigma_1\, \sigma_2\, \dots\, \sigma_k \in [N]^*$ let

$$e(\sigma) := \sum_{i=1}^{|\sigma|} a_{\sigma_i} \qquad\qquad e^-(\sigma) := \sum_{i=1}^{|\sigma|-1} a_{\sigma_i}$$

$$\begin{aligned} f_\sigma &:= f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k} \\ f_{-\sigma} &:= f_{\sigma_1}^{-1} \circ f_{\sigma_2}^{-1} \circ \cdots \circ f_{\sigma_k}^{-1}. \end{aligned}$$

For $\theta \in [N]^\omega$ let

$$\theta|k := \theta_1\, \theta_2\, \dots \theta_k.$$

From a single generating pair (p, F), a potentially infinite number of self-similar polygonal tilings will be constructed. There are three steps in the construction. All tiling figures in this paper are based on the construction in Definition 3.

Definition 3. Let the generating pair (p, F) and $\theta \in [N]^\omega$ be fixed.

(1) For each positive integer k and each $\sigma \in [N]^*$, construct a single tile $t(\theta, k, \sigma)$ that is similar to p:

$$t(\theta, k, \sigma) := (f_{-(\theta\,|\,k)} \circ f_\sigma)(p).$$

(2) Form a patch $T(\theta, k)$ consisting of all tiles $t(\theta, k, \sigma)$ for which σ satisfies a certain property:

$$T(\theta, k) := \left\{ t(\theta, k, \sigma) \,:\, e(\sigma) > e(\theta|k) \geq e^-(\sigma) \right\}.$$

(3) The final tiling $T(\theta)$, depending only on θ, is the nested union of the patches $T(\theta, k)$:

$$T(\theta) := T(p, F, \theta) := \bigcup_{k \geq 1} T(\theta, k).$$

The tiling $T(\theta)$ is called a θ-**tiling generated by the pair** (p, F). Each set $t(\theta, k, \sigma) \in T(\theta)$ is a tile of $T(\theta)$.

The patches $T(\theta, k)$ are nested because every tile in $T(\theta, k)$ is a tile in $T(\theta, k+1)$:

$$f_{-(\theta|k)} \circ f_\sigma(p) = f_{-(\theta|k)} \circ (f_{\theta_{k+1}})^{-1} \circ f_{\theta_{k+1}} \circ f_\sigma(p) = f_{-(\theta|k+1)} \circ f_{\theta_{k+1}\sigma}(p).$$

Figure 9 illustrates the tree-like structure underlying the construction of $T(\theta, 2)$, where the generating pair is, for example, the golden bee of Figure 3 and $\theta = 1\,2\cdots$. The eight tiles in $T(\theta, 2)$ are represented by the leaves of the tree, three of these tiles (in black) are small and five (in red) are large (larger by a factor of $\sqrt{\tau}$). The numbers on the edges are $a_1 = 1$ and $a_2 = 2$. Each sequence σ of edge labels from the root to a leaf satisfies $e(\sigma) > e(\theta|2) = 3 \geq e^-(\sigma)$. The numbers on the leaves are $e(\sigma)$ for the corresponding sequence.

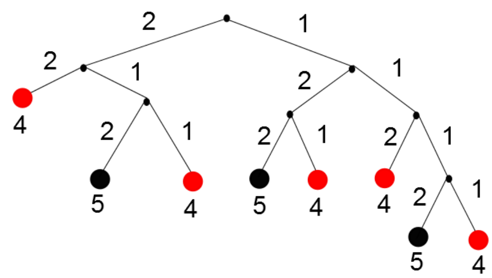


Figure 9. The tree structure underlying the set $T(\theta, 2)$ for the golden bee.

An issue is that $T(\theta)$, for some $\theta \in [N]^\omega$, may cover just a "wedge" — a closed subset of the plane between two rays, for example a quadrant of the plane. That this almost never occurs is reflected in Theorem 1 and is discussed in detail in Question 2 of Section 6.

Our method, encapsulated in Definition 3, assumes a generating pair (p, F) but does not find one. Examples appear in Section 5, and questions concerning their existence appear in Section 7.

Theorem 1. *For any generating pair, there exist infinitely many $\theta \in [N]^\omega$ for which $T(\theta)$ is a self-similar polygonal tiling.*

5. EXAMPLES. Viewed geometrically, equation (1) states that polygon p is a disjoint union of the smaller similar polygons $f_1(p), f_2(p), \ldots, f_N(p)$. Tilings of polygons by smaller polygons has a long history. For example, in a 1940 paper, Brooks, Smith, Stone, and Tutte [**5**] investigated the problem of tiling a rectangle with squares of different sizes. In 1978 Duijvestijn [**6**], by computer, showed that the smallest possible number of squares in a tiling of a larger square by smaller squares of different sizes is 21. In general, the term for a tiling of a polygon p by pairwise non-congruent smaller similar copies of p is a *perfect* tiling. A tiling of a polygon p by smaller similar copies of p, all congruent, is called a *rep-tiling* and p is called a *rep-tile*. The term was coined by S. W. Golomb [**7**]; also see [**8**]. For a generating pair, the smaller similar copies of p need not be pairwise congruent nor pairwise non-congruent. The term for a polygon p that is the disjoint union of smaller similar polygons seems to be *irreptile*; see [**14, 15, 16**]. For (p, F) to be a generating pair, the polygon p must be an irreptile that satisfies the ratio condition in equation (2).

Examples 2 and 3 below provide two infinite families of generating pairs. Example 4 provides a few of what we call sporadic examples. Self-similar polygonal tilings of order 1 are fairly common because there are many known rep-tiles. Therefore self-similar polygonal tilings of higher order, not being prevalent, are illustrated in this section.

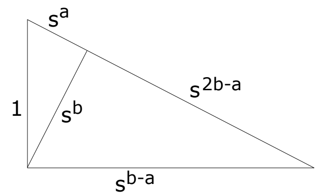


Figure 10. A right triangle in the family $p(a, b)$; see Example 2.

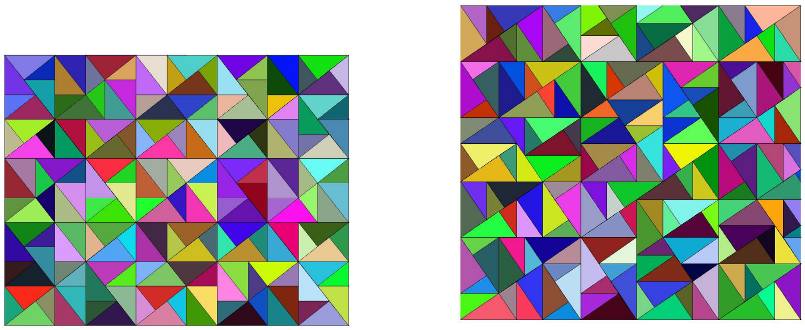


Figure 11. Two tilings by right triangles, based on $p(2, 1)$ and $p(3, 1)$, respectively; see Example 2.

Example 2 (Right Triangles). Consider a right triangle decomposed into two smaller similar triangles; see Figure 10. For every distinct pair of positive integers a, b, let s^2 be the unique positive solution of $x^a + x^b = 1$; this is equation (3). The triangle $p(a, b)$ in Figure 10 is an irreptile for which the scaling ratios, as given in (2), are:

$$r_1 = s^a, \qquad\qquad r_2 = s^b.$$

If we denote the corresponding set of similitudes by $F(a, b) = \{f_1, f_2\}$, then $(p(a, b), F(a, b))$, where $a > b \geq 1$, is an infinite family of generating pairs. Figure 11 illustrates two corresponding self-similar tilings by right triangles, one of order 2, the other of order 3.

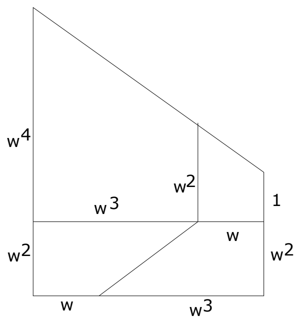
Example 3 (Trapezoids). Consider a trapezoid decomposed into four smaller similar trapezoids as in Figure 12. The length w has the form $w = s^{(b-a)/2}$, where $a > b \geq 1$ are any two positive integers of the same parity, and s is the unique solution of $x^a + x^b = 1$, coming from equation (3): $(x^a + x^b)^2 = x^{2a} + x^{a+b} + x^{a+b} + x^{2b} = 1$. The trapezoid, denoted $q(a, b)$, is an irreptile with scaling ratios:

$$r_1 = s^a, \qquad\qquad r_2 = r_3 = s^{(a+b)/2}, \qquad\qquad r_4 = s^b.$$

The tiling on the right in Figure 12 is a self-similar polygonal tiling based on the case $a = 3, b = 1$.

As mentioned in Section 1, the ratio of large to small tiles in any self-similar golden tiling is, in the limit, the golden ratio. In general, given a generating pair (p, F), let $\{a_1, a_2, \ldots, a_N\}$ and s be as in Definition 2. Further, let $M = \max\{a_i : i \in [N]\}$ and let D_R be a disk of radius R centered at the origin. For $i = 1, 2, \ldots, M$, let

$$d_i = \lim_{R\to\infty} \frac{\text{the number of tiles congruent to } s^{i-1}(p) \text{ in } D_R}{\text{the total number of tiles in } D_R}.$$



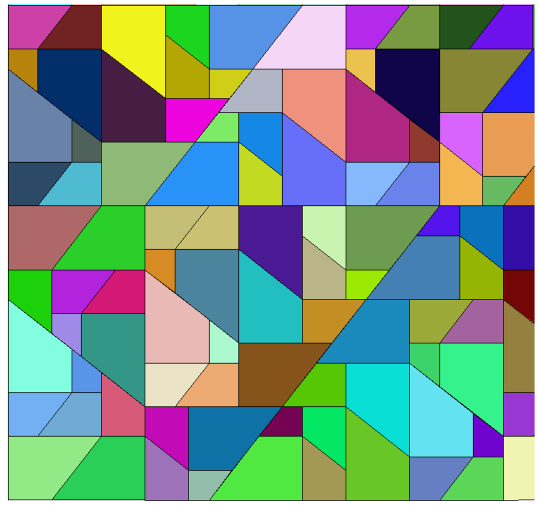


Figure 12. A trapezoid irreptile and a tiling of order 3 based on $q(3,1)$; see Example 3.

For any golden bee tilings, the proportion of large tiles is $d_1 = 1/\tau \approx 0.6180$ and the proportion of small tiles is $d_2 = 1 - 1/\tau \approx 0.3820$, where τ is the golden ratio. For the trapezoid tiling $Q(3,1)$ of Example 3 in Figure 12, the proportions are, $d_1 \approx 0.3826,\ d_2 \approx 0.4392,\ d_3 \approx 0.1781$, the proportion for the largest of the three tiles being d_1, the proportion for the smallest being d_3. These numbers are derived as follows. Let

$$C = \begin{pmatrix} c_1 & 1 & 0 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & & & \\ c_{M-1} & 0 & 0 & 0 & \cdots & 1 \\ c_M & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where c_i, $i = 1, 2, \dots, M$, is the number of functions in F with scaling ratio s^i. Letting $\mathbf{c} = (c_1, c_2, \dots, c_M)$ and $\mathbf{d} = (d_1, d_2, \dots, d_M)$, it can be shown that,

$$\mathbf{d} = \frac{C^n * \mathbf{c}}{\mathbf{j} * (C^n * \mathbf{c})},$$

where $\mathbf{j}$ is the all ones vector. This holds for any θ-tiling by the generating pair (p, F), independent of θ.

Example 4 (Sporadic generating pairs). The irreptiles in Figure 13 do not belong to an infinite family. For that reason we call them and the associated generating pairs *sporadic*. For polygons A and B, the constant $w = \sqrt{\tau}$, where τ is the golden ratio; in

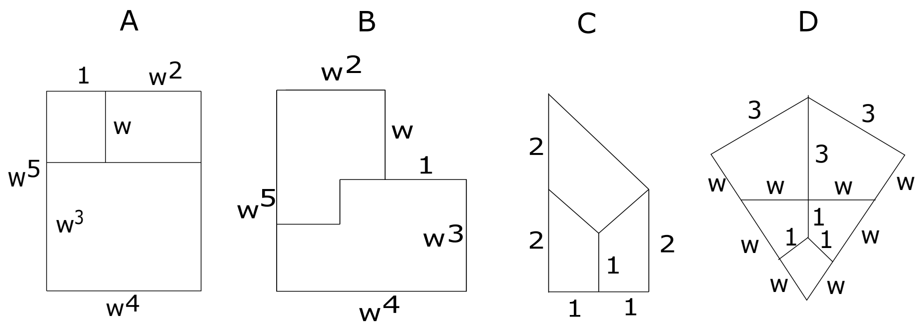


Figure 13. Sporadic irreptiles; see Example 4.

figure D, the constant is $w = \sqrt{3}$. The scaling ratios are:

$$\begin{aligned}
A:&\quad r_1 = \frac{1}{\sqrt{\tau}},\ r_2 = \frac{1}{\tau\sqrt{\tau}},\ r_3 = \frac{1}{\tau^2}\\
B:&\quad r_1 = \frac{1}{\sqrt{\tau}},\ r_2 = \frac{1}{\tau}\\
C:&\quad r_1 = \sqrt{2}/2,\ r_2 = r_3 = 1/2\\
D:&\quad r_1 = r_2 = \sqrt{3}/3,\ r_3 = r_4 = r_5 = 1/3.
\end{aligned}$$

The golden bee in figure B was discussed in Section 1. Figures 1, 2, 7, and 8 illustrate self-similar polygonal tilings based on D, B, A, and C, respectively. Other self-similar polygonal tilings based on sporadic generating pairs appear in Figures 6, 16 and 18. And there are many more sporadic generating pairs.

Example 5 (Reducible generating pairs). Given a generating pair (p, F), there is a trivial way to obtain infinitely many related generating pairs. Replace a function $f \in F$ (or several functions) by the set of functions $\{f \circ f_n : n = 1, 2, \dots, N\}$. An example is depicted geometrically in Figure 14, where one rectangle in the subdivision of p (right figure) is replaced by three smaller similar rectangles (left figure). More generally, call a generating pair (p, F) *reducible* if there is a proper subset S of the tiles in p such that $\bigcup\{t : t \in S\}$ is similar to p; otherwise call (p, F) *irreducible*. A θ-tiling is called *(ir)reducible* if its generating pair is (ir)reducible.

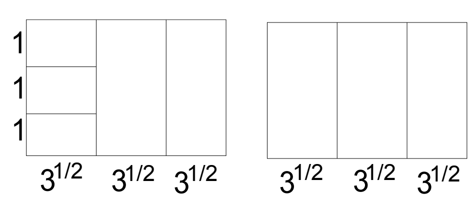


Figure 14. A reducible irreptile.

Example 6 (An irreptile that does not induce a generating pair). Most irreducible irreptiles seem not to induce a generating pair; they fail to satisfy the ratio condition in equation (2). The equilateral triangle in Figure 15 is subdivided into 6 smaller similar equilateral triangles. The two scaling ratios are $1/3$ and $2/3$. However, the existence of a real number s and integers a, b such that $s^a = 1/3$ and $s^b = 2/3$ would imply that $\log 3/\log 2$ is rational.

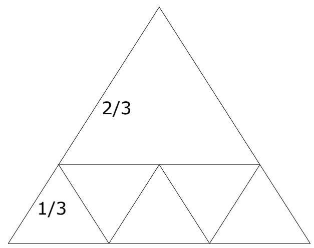


Figure 15. An irreptile that does not induce a generating pair; see Example 6.

6. THE FINE POINTS. For a given generating pair (p, F), the following questions concerning Theorem 1 are addressed in this section. Proofs of all statements not proved here appear in [**4**]. A string $\theta \in [N]^\omega$ is *periodic* if it is a concatenation of the form $\overline{\alpha} := \alpha\,\alpha\,\alpha \cdots$, where α is a finite string. A string $\theta \in [N]^\omega$ is *eventually periodic* if it is a concatenation of the form $\beta\overline{\alpha}$, where α and β are finite strings.

Questions

For which $\theta \in [N]^\omega$

1. are distinct tiles in $T(\theta)$ pairwise disjoint?
2. does $T(\theta)$ fill the plane?
3. is $T(\theta)$ of finite order, and what is the order?
4. is $T(\theta)$ self-similar?
5. is $T(\theta)$ quasiperiodic?

Answers are summarized as follows.

Answer 1. Distinct tiles in $T(\theta)$ are pairwise disjoint for all $\theta \in [N]^\omega$.

Answer 2. It is a tiling of the whole plane for "almost all" $\theta \in [N] \in [N]^\omega$ in the following three senses.

First, there are infinitely many eventually periodic strings θ for which $T(\theta)$ tiles the entire plane.

Second, $T(\theta)$ fills the plane for all disjunctive θ. An infinite string θ is *disjunctive* if every finite string is a consecutive substring of θ. An example is the binary *Champernowne sequence*

$$0\,1\,00\,01\,10\,11\,000\,001 \cdots,$$

formed by concatenating all finite binary strings in lexicographic order. There are infinitely many disjunctive sequences in $[N]^\omega$ if $N \geq 2$. See [**3**] for a discussion and proofs of this and the next statement.

Third, define a word $\theta \in [N]^\omega$ to be a *random word* if there is a $p > 0$ such that each θ_k, for $k = 1, 2, \ldots$, is selected at random from $\{1, 2, \ldots, N\}$, where the probability that $\theta_k = n$, for $n \in [N]$, is greater than or equal to p, independent of the preceeding outcomes. If $\theta \in [N]^\omega$ is a random word, then, with probability 1, the tiling $T(\theta)$ covers $\mathbb{R}^2$.

Answer 3. The tiling $T(\theta)$ is of finite order for all $\theta \in [N]^\omega$. The order, i.e., the number of tiles in the prototile set, is equal to $M = \max\{a_i : i \in [N]\}$. See Proposition 1 below.

Answer 4. If $\theta = \overline{\alpha}$, $\alpha \in [N]^*$ is periodic, then $T(\theta)$ is self-similar with self-similarity map $\phi = f_{-\alpha}$.

To better understand this answer, note that the set $[N]^*$ is a semigroup, where the operation is concatenation. Let $\mathbb{T}$ denote the set of all θ-tilings for the pair (p, F). There is a natural semigroup action

$$\widehat{\alpha} : \mathbb{T} \to \mathbb{T}$$

of $[N]^*$ on $\mathbb{T}$ defined by:

$$\widehat{\alpha}(T(\theta)) = T(\alpha\,\theta)$$

for $\alpha \in [N]^*$ and $T(\theta) \in \mathbb{T}$. If $\theta = \overline{\alpha}$ is periodic, then clearly $\widehat{\alpha}(T(\theta)) = T(\theta)$. It can then be shown that any such fixed point $T(\theta)$ of $\widehat{\alpha}$ is self-similar.

More generally, if θ is eventually periodic of the form $\theta = \beta\overline{\alpha}$ where $e(\beta) = e(\alpha)$, then $T(\theta)$ is self-similar. This statement follows from the one above for the following reason. Call two tilings *congruent* if one can be obtained from the other by a Euclidean motion, i.e., by a translation, rotation, reflection or glide. Under the given assumptions, tilings $T(\overline{\alpha})$ and $T(\beta\overline{\alpha})$ can be shown to be congruent. It is also not hard to show that if $T(\theta)$ is self-similar and $T(\theta')$ is congruent to $T(\theta)$, then then $T(\theta')$ is also self-similar.

Answer 5. The tiling $T(\theta)$ is quasiperiodic for all θ such that $T(\theta)$ tiles $\mathbb{R}^2$.

In light of these answers, there are infinitely many θ that meet the requirements for $T(\theta)$ to be a self-similar polygonal tiling, thus verifying Theorem 1.

Proposition 1. *For any generating pair (p, F) and $\theta \in [N]^\omega$, the prototile set of $T(p, F, \theta)$ consists of $M = \max\{a_i : i \in [N]\}$ tiles similar to p.*

Proof. Since the inverse of a similarity and any composition of similarities is a similarity, each tile $t(\theta, k, \sigma)$ is similar to p. Therefore, all tiles in $T(\theta)$ are similar to p.

To show that there are at most finitely many tiles up to congruence, consider the tiles in $T(\theta, k)$ for any $k \geq 1$. The scaling ratio of a tile in $T(\theta, k)$ is of the form

$$r(f_{-(\theta\,|\,k)} \circ f_\sigma) = s^a, \qquad \text{where} \qquad a = e(\sigma) - e(\theta|k).$$

Let $\sigma = \sigma_1\,\sigma_2 \cdots \sigma_K$. The restriction $e(\sigma) > e(\theta|k) \geq e^-(\sigma) = e(\sigma) - a_{\sigma_K}$ immediately implies that

$$a_{\sigma_K} \geq a > 0.$$

Therefore $a \in \{1, 2, \ldots, M\}$, verifying that there are at most M tiles up to congruence.

To show that a can take any value in $\{1, 2, \ldots, M-1\}$, recall that we can assume that $\gcd(a_1, \ldots, a_N) = 1$ and, by an elementary result from number theory, every sufficiently large positive number is a sum of terms $a_1, a_2, \ldots, a_N$. Therefore, if k is sufficiently large, then $a = e(\sigma) - e(\theta|k)$ can take any positive integer value subject to the restriction $e(\sigma) > e(\theta|k) \geq e^-(\sigma)$ or equivalently, subject to the restriction $aleqa_{\sigma_K}$. But for k sufficiently large, $\sigma_K \in [N]$ can be chosen arbitrarily, so that a_{σ_K} can be chosen to be M. ∎

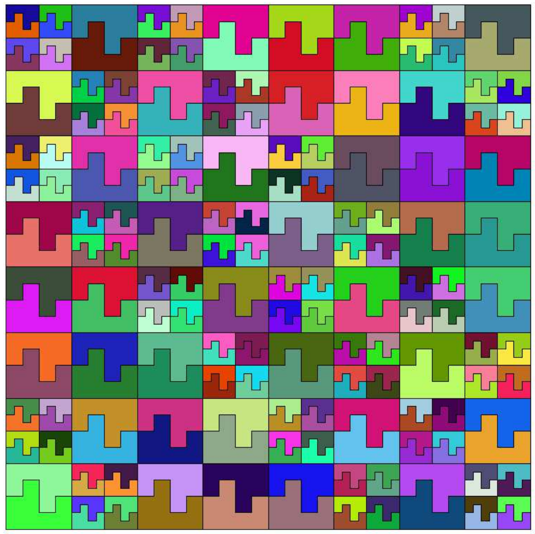


Figure 16. A self-similar polygonal tiling of order 2 based on a sporadic generating pair; see Example 4.

7. OPEN PROBLEMS.

Some basic questions remain open.

Question 1. Can every self-similar polygonal tiling be obtained by the generating pair method of Definition 3?

Question 2. In Section 5, Examples 2 and 3, two infinite families of irreducible generating pairs are given. Are there additional infinite families of irreducible generating pairs?

Question 3. Several sporadic irreducible generating pairs are given in Section 5. Are there at most finitely many sporadic irreducible generating pairs? If not, given N, are there at most finitely many irreducible generating pairs (p, F) for which $|F| = N$?

Question 4. The pinwheel tilings of C. Radin [**12**] are based on the subdivision of a right triangle with side lengths $1, 2, \sqrt{5}$ due to J. Conway; see Figure 17. These tilings are order 1 tilings in the terminology of this paper. Do there exist higher order analogs? In other words, does there exist an irreducible (in the sense of Example 5) self-similar polygonal tiling of order at least 2 for which the tiles appear in infinitely many rotational orientations?

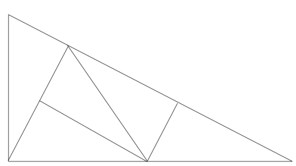


Figure 17. Pinwheel rep-tile.

Given a generating pair (p, F), let $\mathbb{T}(p, F)$ denote the set of its θ-tilings of the plane. Let $\pi : [N]^\omega \to \mathbb{T}(p, F)$ denote the map defined by $\theta \mapsto T(\theta)$. As stated in

Section 1, there are infinitely many self-similar golden bee tilings up to congruence, none of which is periodic. On the other hand, the image of $[N]^\omega$ under π need not always be infinite. There may be many strings θ for which their images $T(\theta)$ are pairwise congruent tilings. This is the case, for example, when p is a square whose images under four functions in F subdivide p into four smaller squares. In this case, $T(\theta)$ is, for all θ, the standard tiling of the plane by squares. Moreover, all such square θ-tilings are periodic.

Question 5. Let (p, F) be an irreducible generating pair, where p is not a triangle or a parallelogram. Is it the case that there exist infinitely many θ-tilings up to congruence, none of which is periodic?

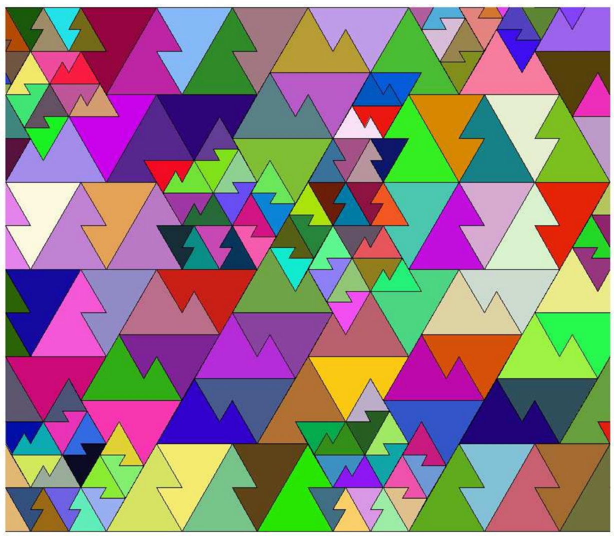


Figure 18. Another sporadic tiling.

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