Separation Index of a Graph

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Abstract: The concepts of *separation index* of a graph and of a surface are introduced. We prove that the separation index of the sphere is 3. Also the separation index of any graph faithfully embedded in a surface of genus g is bounded by a funtion of g. J Graph Theory 41: 53–61, 2002

1. GRAPH ISOMORPHISM

This first section on the Graph Isomorphism Problem is meant only as motivation for the concept of separation index, which is defined formally in Section 2. In Section 3 we prove that the separation index of the sphere is 3. The separation index of faithfully embedded graphs is discussed in Section 4, and a few open questions are posed.

The literature on the Graph Isomorphism Problem is extensive, but we have mentioned just a few references below. The Graph Isomorphism Problem is to determine, given two graphs, whether they are isomorphic. Graph Isomorphism

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holds a special position among algorithmic questions because its complexity is problematic. It is known that, for planar graphs, there exist a linear time algorithm to solve Graph Isomorphism [6]. For graphs of bounded degree [9] or for graphs of bounded eigenvalue multiplicity [1], there exist polynomial time algorithms. But for graphs in general, it is not known whether the problem is polynomial, NP-complete, or neither.

One common approach to graph isomorphism is to devise an efficient method to partition the vertex set. Moreover, many methods for partitioning are based on probing the graph from a fixed vertex or fixed vertices. Consider the following example. Fix a vertex v in a graph G and a natural number N. Partition the vertex set V of G according to the number of walks of length N from v to each vertex. This produces a partition of V into blocks B_i consisting of those vertices with i such walks. Given two graphs G and G' and fixed vertices v and v', respectively, we obtain partitions into blocks $\{B_i\}$ and $\{B_i'\}$, respectively. For the existence of an isomorphism ϕ from G and G' taking v to v', it is necessary that ϕ take B_i onto B_i' for each i. In particular, we must have $|B_i| = |B_i'|$ for each i. If this necessary condition holds, then one can check all potential bijections to determine whether one of them is a graph isomorphism. Checking becomes more efficient the finer the partition is. (Of course, this would have to be done for each pair of fixed vertices (v, v'); but there are just $O(n^2)$ such pairs.) As an example consider the graph in Figure 1a with $N = 2$. The labels indicate the number of walks of length 2 to each of the vertices from vertex v. This provides a partition of the vertices into blocks of sizes 1, 1, 2, 2. If there existed another graph with the same parameters (there does not), then $4 = 1 \cdot 1 \cdot 2 \cdot 2$ bijections would have to be tested.

A refinement of this procedure would be to probe from several, say k, fixed pairs of vertices instead of just one pair, resulting in a set $\{\pi_1, \pi_2, \ldots, \pi_k\}$ of partitions of V. Let π be the *meet* of $\pi_1, \pi_2, \ldots, \pi_k$, the partition of V such that vertices x and y are in the same block if they are in the same block of π_i for each i. Then the efficiency of checking for isomorphism depends on how fine is the partition π. Figure 1a,b provides two partitions, but their meet does not result in a refinement of either individually. Figure 1a,c, on the other hand, provides two partitions π_1 and π_2 whose meet *separates* the set of vertices in the sense that each block of the meet has size 1.

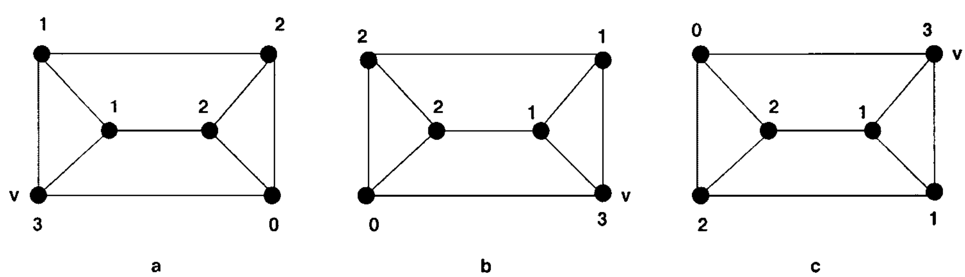

Figure 1. Probing from vertex v.

Best possible is when the set of partitions separate V. In this case only one bijection of the vertex sets of G and G' would be required to test for isomorphism of G and G'. However, if $Aut(G)$ denotes the automorphism group of G, then *any* method that probes from a fixed vertex v cannot distinguish between vertices in the same orbit of the stabilizer subgroup $Aut(G)_v$ of v. (This is the case in Figure 1a,b for the vertices labeled 1 and for those labeled 2.) No method of probing from vertex v can give a partition finer than the orbit partition of the stabilizer of v. This observation leads to the problem of determining the least number of vertices such that the meet of the orbit partitions of the respective stabilizers separates the set of vertices.

2. SEPARATION INDEX

In this section, the notions of separation index of a graph and separation index of a surface are formally defined. Let $V = \{1, 2, \ldots, n\}$. The set of partitions of V form a lattice, where the partial order is $\pi < \sigma$ if π is a finer partition than σ. For example, for $n = 6$ the partition $(12)(34)(56)$ is finer than $(1234)(56)$. The meet in the lattice will be denoted by $\wedge$, so that x and y are in the same block of $\pi \wedge \sigma$, if and only if, x and y are in the same block of π and in the same block of σ. The 0 and 1 denote, respectively, the partition into singletons and the partition with a single block. A set $\{\pi_1, \pi_2, \ldots, \pi_k\}$ of partitions is said to *separate* V if $\pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_k = 0$. In other words, for any two elements $u, v \in V$, there exists a j such that u and v lie in different blocks of π_j.

Consider a finite group Γ acting on V. Let $\Gamma_v \leq \Gamma$ denote the stabilizer of $v \in V$. Moreover, let $\pi(v)$ denote the *orbit partition* of Γ_v. In other words, the blocks of $\pi(v)$ are the orbits of Γ_v acting on V. Elements $v_1, v_2, \ldots, v_k$ of V are said to *separate* V if the set $\{\pi(v_1), \pi(v_2), \ldots, \pi(v_k)\}$ of orbit partitions separates V. We define the *separation number* of Γ, denoted $sep(\Gamma)$, as the minimum number k of elements of V that suffice to separate V:

$$sep(\Gamma) = \min\{k \mid k \text{ elements separate } V\}.$$

If, for example, Γ_v is trivial for all v, then $sep(\Gamma) = 1$. At the other extreme, if Γ consists of the full symmetric group of permutations of V, then $sep(\Gamma) = n - 1$.

Let G be a connected graph on n vertices, without multiple edges or loops. Define the *separation index* of G as the separation index of its automorphism group $Aut(G)$ acting on the vertex set:

$$sep(G) = sep(Aut(G)).$$

If, for example, K_n is the complete graph, then $sep(K_n) = n - 1$. More generally, if G is a complete r-partite graph, then $sep(G) = n - r$. If C_n is a cycle, then $sep(C_n) = 2$. Clearly $sep(G) = 1$ if and only if the stabilizer of some vertex acts trivially.

Let S be a closed surface (compact and without boundary). The *separation index* of S, denoted $sep(S)$, is defined as the minimum number k such that k vertices suffice to separate any 3-connected graph embedded in S:

$$sep(S) = \min\{k \mid sep(G) \leq k \text{ for any 3-connected graph embedded in } S\}.$$

The condition that G be 3-connected is needed. Consider, for example, the two families of graphs: $A_n = K_1 \vee \overline{K_n}$ and $B_n = \overline{K_2} \vee \overline{K_n}$, where $n \geq 2$ and the complement of K_n consists of n isolated vertices. The *join* $G \vee H$ denotes the graph obtained from the disjoint union of graphs G and H by adding the edges $\{uv | u \in V(G), v \in V(H)\}$. Then A_n is 1-connected, but not 2-connected, and B_n is 2-connected but not 3-connected. Both A_n and B_n are planar for all n, but the separation indices, $sep(A_n) = n - 1$ and $sep(B_n) = n$, are arbitrarily large.

3. SEPARATION INDEX OF THE SPHERE

It can be shown that a nontrivial automorphism of a 3-connected planar graph can have at most 2 fixed points. This means that the pointwise stabilizer $Aut(G)_{\{a,b,c\}}$ of 3 distinct vertices a, b, c must be trivial. Notice, however, that this is not as strong a statement as Theorem 3.1 below. If $\pi(abc)$ denotes the orbit partition of $Aut(G)_{\{a,b,c\}}$, then $\pi(abc) \leq \pi(a) \wedge \pi(b) \wedge \pi(c)$ with equality often not the case. Consider, for example, the two antipodal points on the 3-cube and any distinct third point.

Theorem 3.1. *The separation index of the sphere is 3.*

Proof. The separation index is at least 3 because two points do not suffice to separate K_4.

To show that 3 vertices suffice, let G be a 3-connected planar graph. By a theorem of Whitney [17] (or see [16]), G has a unique embedding in the 2-sphere. This also implies that each automorphism of G induces an automorphism of the embedded graph M. By *automorphism* of M is meant a graph automorphism that preserves faces, an automorphism of M considered as a map on the sphere. Let B be the barycentric subdivision of M with each vertex of B colored 0, 1, or 2 according to the dimension of the face it represents, 0 for a vertex, 1 for an edge, and 2 for a face. The colors on the three vertices of any triangle in B are 0, 1, and 2. Now, let G' be the edge-colored dual graph of B constructed as follows. Inside each triangle of B there corresponds a vertex of G'. Two vertices of G' are joined by an edge colored i, if the corresponding two triangles in B share an edge whose two colors do not include i. Hence each vertex in G' has degree 3 with incident edges colored 0, 1, and 2. Let $Aut(B)$ and $Aut(G')$ denote the color preserving automorphism groups of B and G', respectively. Each automorphism of M corresponds to a color preserving automorphism of B, which in turn corresponds to a color preserving automorphism of G'. Therefore $Aut(G') \cong Aut(B) \cong$

$Aut(M) \cong Aut(G)$. These are not just group isomorphisms. Each automorphism in $Aut(G)$ induces a unique automorphism in $Aut(M)$, in $Aut(B)$ and in $Aut(G')$.

Consider the lexicographic order on all words in the alphabet $\{0, 1, 2\}$ where $0 < 1 < 2$. For example, $21 < 012$ and $01 < 02$. For vertices $u, v \in G'$ let $d(u, v)$ denote the minimum word whose corresponding edge path in G' is from u to v. Fix a vertex $v_0 \in G'$ and let $\{v_0, v_1, \ldots, v_k\}$ be the orbit of v_0 under $Aut(G')$. Define

$$N_i = \{v \in G' \colon d(v_i, v) < d(v_j, v)\ \forall j \neq i\}.$$

If $\langle U \rangle$ denotes the graph induced by vertex set U, note that

(1) $\{N_i : i = 0, 1, \ldots, k\}$ forms a partition of the vertex set of G'.
(2) $\langle N_i \rangle$ is connected for each i.
(3) $\langle N_i \rangle \cong \langle N_j \rangle$ for all i,j, where the graph isomorphism is color preserving.

The second statement is an easy exercise. The last statement follows from that fact that $Aut(G')$ acts transitively on the set $\{v_0, v_1, \ldots, v_k\}$ of vertices.

A new map M' on the sphere, related to the original map M, will now be constructed as follows. Start with the triangulation B. To each set N_i of vertices of G' there corresponds a set B_i of triangles in B. The union of the (closed) triangles in any such B_i is connected and simply connected, simply connected because the 2-sphere cannot be partitioned into a finite number of homeomorphic, connected, nonsimply connected regions. From each B_i remove any edge common to two triangles corresponding to two vertices in N_i. Then remove any isolated vertices. If we do this for all i, a map M' on the sphere is obtained. It has no vertices of degree 1. There may be vertices of degree 2, which will be mentioned later.

Before proceeding, a degenerate case must be considered. The map M' will have one face (no edges and no vertices) exactly when $Aut(G')$ is trivial. In this case $Aut(G)$ is also trivial and hence $sep(G) = 1$, and the theorem is proved.

Each element of $Aut(B)$ acts on the map M'. Hence $Aut(G)$ also acts on M' as a group of map automorphisms. Since $Aut(G')$ acts simply transitively on $\{v_0, v_1, \ldots, v_k\}$, the group $Aut(G)$ acts simply transitively on the set of faces of M'. The classification of maps on the sphere that are face transitive is elementary, combinatorial, and well known [3]. They are the duals of the semi-regular polyhedra (the duals of the Platonic and Archimedean solids, and the duals of the prisms and antiprisms) and two degenerate cases that will be considered separately at the end of the proof. The degenerate cases aside, each semi-regular polyhedron P can be realized in Euclidean 3-space so that each automorphism of P as a map on the sphere is induced by a symmetry of P. A *symmetry* of P is an isometry of 3-space preserving P. Since a polyhedron and its dual have the same automorphism and symmetry groups, respectively, each automorphism of the dual of P as a map on the sphere is induced by a symmetry of the dual of P.

So far the argument has been combinatorial. We now take into consideration the Euclidean metric. We consider M' realized as a polyhedron in Euclidean 3-space as described in the previous paragraph. (To be accurate, M' is realized as such a polyhedron with possible additional vertices of degree 2 on some edges. Recall that M' may have such vertices of degree 2.) The group $Aut(M')$, and hence the group $Aut(G)$, act as groups of symmetries of M'. This action can be extended to an action of $Aut(G)$ on B as follows. First fill in just one face of M' by appropriately triangulating it to correspond to B. Then extend this triangulation to all faces of M' by letting the group $Aut(G)$, regarded as a symmetry group which acts simply transitively on the faces of M', act on the first face. Since $Aut(G)$ acts as a group of symmetries on this realization of B, so $Aut(G)$ also acts as a group of symmetries on the induced realization of M (by removing the extra vertices and lines in the barycentric subdivision B). Consequently $Aut(G)$ acts as a group of symmetries on a realization of G.

Every finite group of isometries is a subgroup of the orthogonal group $O(3)$. Hence $Aut(G)$ acts as a finite subgroup of $O(3)$ on G. Let O denote the fixed point of $O(3)$. In the polyhedral realization of M', each face contains at least one point (not O) corresponding to a vertex of G. This is sufficient to guarantee that, in the realization of G, there exists three distinct vertices a, b, c such that the corresponding vectors $\mathbf{a} = \vec{O}a$, $\mathbf{b} = \vec{O}b$ and $\mathbf{c} = \vec{O}c$ do not lie in a plane. We now show by contradiction that a, b, c separate the vertices of G. Assume the contrary, that two distinct vertices u, v of G are simultaneously in the same orbit of each of the three stabilizers $Aut(G)_a$, $Aut(G)_b$, $Aut(G)_c$. Since u and v lie in the same orbit of $Aut(G)_a$, then u and v both lie on a circle in a plane orthogonal to vector $\mathbf{a}$. Likewise, u and v lie in a plane orthogonal to $\mathbf{b}$ and in a plane orthogonal to $\mathbf{c}$. Since $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ do not themselves lie in a plane, the three planes orthogonal to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, intersect in at the most a single point, contradicting that both u, v lie in this intersection.

The proof is now complete except for the two degenerate cases mentioned earlier. The first case is when M' has exactly two faces, each bounded by the same, say m-gon. Then $Aut(G)$ consists of exactly two elements, the identity and an involution g. If G has a vertex whose stabilizer is trivial, then $sep(G) = 1$. Otherwise, the involution stabilizes every vertex of G, which would imply that g is the identity, which is a contradiction.

The other degenerate case occurs when M' is the dual of the case just described; M' consists of two points, say a and b, joined by m (multiple) edges. Then $Aut(G)$, acting simply transitively on the faces of M', must be a cyclic group of order m. In its action on the vertices of G, a generator of $Aut(G)$ can be expressed as the disjoint product of cycles, each of length m, and possibly one cycle of length 2 (reversing a and b). In this case $sep(G) \leq 2$. ■

The proof of Theorem 3.1 could be simplified by using a theorem that states the following: each finite group of homeomorphisms of the 2-sphere is topologically equivalent to a finite subgroup of the orthogonal group $O(3)$. An immediate corollary of this and Whitney's Theorem is that any 3-connected graph G can be embedded in the unit sphere such that the automorphism group of G is a subgroup of the orthogonal group $O(3)$ acting on the unit sphere in 3-space. This is essentially what is shown as part of our proof of Theorem 3.1 The topological theorem appears in an article by Babai et al. [2] (see also the book of Gross and Tucker [4]). Babai et al. [2] state that they later learned that the result was found in 1919 by Kerékjártó [8], but mentioned the need for a proof "in modern language and accuracy." Their proof is somewhat long, so our short proof may be of independent interest.

The proof of Theorem 3.1 could also be shortened using a generalization of Steinitz' Theorem due to Mani [10]. Steinitz' Theorem states that, to any finite, 3-connected, planar graph G, there exists a 3-dimensional convex polyhedron P such that G is isomorphic to the 1-skeleton of P. Mani [10] further showed that such a P exists with the property that every automorphism of G is induced by a symmetry of P. All known proofs of Steinitz' Theorem are fairly difficult; see [18] for a version a little shorter than [14] or [5]. The proof of Theorem 3.1 in this study is self contained in that it makes no reference to Steinitz' result.

4. OPEN PROBLEMS

Some questions that suggest themelves:

Question 1. Does every surface have a finite separation index?

Question 2. What are the separation indices of the torus? and of the projective plane?

Let S_g denote the orientable surface of genus g. The *genus* $\gamma(G)$ of a graph G is the minimum g such that G embeds in S_g. Ringel [13] determined the genus of the complete bipartite graphs, in particular $\gamma(K_{3,n}) = \lceil \frac{n-2}{4} \rceil$. Since $sep(K_{3,n}) = n + 1$ we have

$$sep(S_g) \geq 4g + 3. \tag{1}$$

In particular, the separation index of the torus is at least 7.

Question 3. Does equality always hold in equation 1?

Some of the questions posed above become more tractible when restricted to Whitney embeddings of graphs, a notion we now define. Whitney's Theorem [17] (mentioned in the proof of Theorem 3.1) states that every automorphism of a 3-connected planar graph maps face boundaries of the embedded graph to face boundaries. An embedding of a graph G in a surface such that every automorphism of G maps face boundaries to face boundaries is called a *Whitney embedding* by Hutchinson [7] and a *faithful embedding* by Negami [12]. There is a nice sufficient condition for an embedding of a graph to be Whitney, conjectured by Hutchinson [7] and proved by Thomassen [15], that uses the notion of edge-width. The *edge-width* of an embedded graph G is the length of the smallest noncontractible cycle in G. A *large-edge-width embedding* (abbreviated *LEW-embedding*) of a graph in a surface is one whose edge-width is larger than the maximum length of a face boundary. (See [11] for more in LEW-embedding.)

Theorem 4.1 (Thomassen [15]). *A LEW-embedding of a 3-connected graph is a Whitney embedding.*

The answer to Question 1 can be answered in the affirmative when restricted to Whitney embeddings on surfaces of genus greater than 1. The result is likely not best possible, but it does show finiteness.

Theorem 4.2. *If G is Whitney embedded on a surface of genus $g > 1$, then $sep(G) \leq 168(g-1)$.*

The proof of Theorem 4.2 relies on the following lemma. Here S_n denotes the symmetric group acting on the set $[n] = \{1, 2, \ldots, n\}$.

Lemma. *If Γ is a subgroup of S_n, then $sep(\Gamma) \leq |\Gamma|$.*

Proof. If, for some $k \in [n]$, the stabilizer Γ_k consists of just the identity e, then $\{k\}$ separates $[n]$ and $sep(\Gamma) = 1$. So, for each $k \in [n]$, let $g_k \in \Gamma_k$, $g_k \neq e$. Define an equivalence relation on $[n]$ by $i \equiv j$ if $g_i = g_j$. There are clearly at most $|\Gamma|$ equivalence classes. These equivalence classes partition $[n]$. Let $A = \{a_1, i_2, \ldots, a_m\}$, $m \leq |\Gamma|$ be a set consisting of exactly one element from each equivalence class. It now suffices to show that A separates $[n]$. Let s, t be distinct elements of $[n]$. Then $s \equiv a_i$ for some i; in other words $g_s = g_{a_i}$. So g_{a_i} leaves s fixed. Hence s and t lie in different orbits of Γ_{a_i}. ■

Proof of Theorem 4.2. Considering $Aut(G)$ as a group of permutations of the vertices of G, the lemma above implies $sep(G) \leq |Aut(G)|$. According to Hurwitz's Theorem (see [4, pg. 296]), the order of any finite group acting as homeomorphisms on the surface of genus $g > 1$ has order at most $168(g-1)$. Since the embedding of G is a Whitney embedding, each automorphism of G induces a homeomorphism of the surface. Hence $|Aut(G)| \leq 168(g-1)$. ■

In the case of embeddings on a not necessarily orientable surface with Euler characteristic $\chi < 0$, a similar proof shows that $sep(G) \leq -84\chi$.

The last question concerns the validity of a statement much stronger than Theorem 4.2. Call G a *Whitney graph* if G has a Whitney embedding.

Question 4. The separation index of any Whitney graph is at most 3.

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