COMBINATORICA 5 (3) (1985) 257—260

A SHELLABLE POSET THAT IS NOT LEXICOGRAPHICALLY SHELLABLE

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Received 6 April 1983

Revised 19 October 1984

It is known that a lexicographically shellable poset is shellable, and it has been asked whether the two concepts are equivalent. We provide a counterexample, a shellable graded poset that is not lexicographically shellable.

1. Shellability

A pure d-dimensional simplicial complex Γ is *shellable* if its facets can be ordered $F_1, F_2, \dots, F_N$ such that $F_k \cap \left(\bigcup_{i=1}^{k-1} F_i\right)$ is a pure $(d-1)$-dimensional complex for $k=2, 3, \dots, N$. In particular, this intersection is homeomorphic to either a $(d-1)$-ball or a $(d-1)$-sphere. Important examples of shellable complexes are the boundary complexes of simplicial polytopes [8]. Moreover, a shellable manifold must be a ball or sphere, and a shellable complex must have the homotopy type of a wedge of spheres. Shellability has proved an important tool in polyhedral theory [11], in the topology of Coxeter complexes and buildings [3] and in the algebra of Cohen—Macauley rings [5, 12].

Recently, shellability has been investigated in the context of partially ordered sets [1, 2, 5, 6, 7, 10, 15]. Let P be a finite graded poset. The *order complex* ΔP is the abstract simplicial complex whose simplexes are the chains of P. A finite graded poset P is *shellable* if ΔP is shellable. A method, involving a labeling of the edges of the Hasse diagram of the poset, can be used to show that a poset is shellable.

When this is possible, the poset is said to be lexicographically shellable (EL-shellable). Precise definitions appear in [2]. Examples of EL-shellable posets include distributive lattices, semimodular lattices and supersolvable lattices. In certain situations it is useful to consider a related labeling of the chains of the poset. If such a labeling exists, we say that the poset is chain lexicographically shellable (CL-shellable) [7]. An example is Bruhat order in a Coxeter group [6]. The following result is also proved in [6].

Theorem 1. *EL-shellable $\Rightarrow$ CL-shellable $\Rightarrow$ shellable.*

AMS subject classification (1980): 05 C 65, 06 A 10.

It has been asked in [5] whether the three concepts coincide. In the next section of this paper we provide an example of a shellable poset that is not EL or CL-shellable. Our technique involves finding a non-shellable complex whose barycentric subdivision is shellable. This idea was suggested to J. Walker who independently implemented it by finding an example smaller than ours [16].

2. A shellable, non lexicographically shellable, poset

Given a simplicial complex Δ its poset P of faces is obtained by ordering the faces of Δ by inclusion and adjoining a top element $\hat{1}$ and a bottom element $\hat{0}$. Note that the order complex $\Delta(P-\{0, 1\})$ is the barycentric subdivision of Δ. The following theorem, in the case where Δ is a simplicial complex (or more generally polyhedral complex), is proved in [7]. Here the dual poset is obtained by reversing the ordering.

Theorem 2. *The dual of the poset of faces of complex Δ is CL-shellable if and only if Δ is shellable.*

In what follows Theorem 2 is applied to a finite cell complex whose cells are simplexes, but where the intersection of two faces can possibly consist of the union of more than one face of each simplex. For such a *pseudosimplicial complex* the definition of shellability, as well as the proof of Theorem 2, carries over without alteration. In fact, Björner [4] proves a theorem analogous to Theorem 2 for CW complexes.

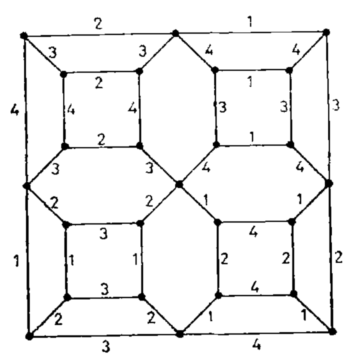

Fig. 1

Let G be the edge colored graph in Figure 1, the color set being $I=\{1, 2, 3, 4\}$. For $J\subseteq I$ let S_J be the set of connected components of the subgraph of G induced by lines colored in J. Note that $S_\emptyset$ is the set of points of G. Let $P(G)$ be the poset consisting of all pairs (H, J) where $J\subseteq I$ and $H\in S_J$, together with a bottom element $\hat{0}$. The elements of $P(G)$ are ordered as follows: $(H, J)<(H', J')$ if $H\subseteq H'$ and $J\subset J'$ and $\hat{0}<(H, J)$ for all J and H. Note that $P(G)$ also has a top element $\hat{1}=(G, I)$.

Theorem 3. *The poset $P(G)$ is shellable but not CL-shellable.*

Proof. To the graph G is associated a pseudosimplicial complex as follows. For each point v of G let Δv be a 3-simplex whose vertices are colored 1, 2, 3 and 4. A subsimplex $s \in \Delta v$ is said to be of type J if the set of colors on the vertices of s is J. Let X be the disjoint union of $\{\Delta v | v \in G\}$. Identify two simplexes $s \in \Delta v$ and $s' \in \Delta v'$ if they are the same type J and if v and v' are connected by a path in G with colors in $I-J$. If $\sim$ denotes this identification let $\Delta G = X/\sim$. By [13, Theorem 2] $P(G)$ is the dual of the poset of faces of ΔG, and by [13, Theorems 7, 10] ΔG is not shellable. Therefore Theorem 2 implies that $P(G)$ is not CL-shellable. The order complex $\Delta' = \Delta(P(G) - \{\hat{0}, \hat{1}\})$ is the barycentric subdivision of ΔG. Now there is an edge colored graph G' such that $\Delta' = \Delta G'$ and G' is easily produced from G. Also [13, Theorem 7] gives a simple criterion for checking the shellability of Δ' in terms of G'. A short computer calculation then substantiates that Δ' is shellable. Hence $\Delta P(G)$ is shellable, i.e. $P(G)$ is shellable. ∎

Let a poset be called *dual CL-shellable* if its dual is CL-shellable. Since CL-shellability and dual CL-shellability each imply shellability, it is natural to ask whether there exists shellable graded posets which are neither CL-shellable nor dual CL-shellable. If P is a graded poset which is shellable but not CL-shellable then we form the ordinal sum Q, of P and its dual. Now Q is a shellable graded poset since ordinal sums of shellable graded posets are shellable and graded (see [2]). But Q is neither CL-shellable nor dual CL-shellable since P and its dual are intervals of Q and any interval of a CL-shellable (dual CL-shellable) poset is CL-shellable (dual CL-shellable). Hence there exists shellable graded posets which are neither CL-shellable nor dual CL-shellable.

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