A SHELLABLE POSET THAT IS NOT LEXICOGRAPHICALLY SHELLABLE

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It is known that a lexicographically shellable poset is shellable, and it has been asked whether the two concepts are equivalent. We provide a counterexample, a shellable graded poset that is not lexicographically shellable.

1. Shellability

A pure d-dimensional simplicial complex $\Gamma$ is shellable if its facets can be ordered $F_1, F_2, ..., F_N$ such that $F_k \cap \left( \bigcup_{i=1}^{k-1} F_i \right)$ is a pure $(d-1)$-dimensional complex for $k=2, 3, ..., N$. In particular, this intersection is homeomorphic to either a $(d-1)$-ball or a $(d-1)$-sphere. Important examples of shellable complexes are the boundary complexes of simplicial polytopes [8]. Moreover, a shellable manifold must be a ball or sphere, and a shellable complex must have the homotopy type of a wedge of spheres. Shellability has proved an important tool in polyhedral theory [11], in the topology of Coxeter complexes and buildings [3] and in the algebra of Cohen–Macauley rings [5, 12].

Recently, shellability has been investigated in the context of partially ordered sets [1, 2, 5, 6, 7, 10, 15]. Let $P$ be a finite graded poset. The order complex $\Delta P$ is the abstract simplicial complex whose simplexes are the chains of $P$. A finite graded poset $P$ is shellable if $\Delta P$ is shellable. A method, involving a labeling of the edges of the Hasse diagram of the poset, can be used to show that a poset is shellable.

When this is possible, the poset is said to be lexicographically shellable (EL-shellable). Precise definitions appear in [2]. Examples of EL-shellable posets include distributive lattices, semimodular lattices and supersolvable lattices. In certain situations it is useful to consider a related labeling of the chains of the poset. If such a labeling exists, we say that the poset is chain lexicographically shellable (CL-shellable) [7]. An example is Bruhat order in a Coxeter group [6]. The following result is also proved in [6].

Theorem 1. EL-shellable $\Rightarrow$ CL-shellable $\Rightarrow$ shellable.

It has been asked in [5] whether the three concepts coincide. In the next section of this paper we provide an example of a shellable poset that is not EL or CL-shellable. Our technique involves finding a non-shellable complex whose barycentric subdivision is shellable. This idea was suggested to J. Walker who independently implemented it by finding an example smaller than ours [16].

2. A shellable, non lexicographically shellable, poset

Given a simplicial complex $Δ$ its poset $P$ of faces is obtained by ordering the faces of $Δ$ by inclusion and adjoining a top element $\hat{1}$ and a bottom element $\hat{0}$. Note that the order complex $Δ\left(P - \{0, \hat{1}\}\right)$ is the barycentric subdivision of $Δ$. The following theorem, in the case where $Δ$ is a simplicial complex (or more generally polyhedral complex), is proved in [7]. Here the dual poset is obtained by reversing the ordering.

**Theorem 2.** The dual of the poset of faces of complex $Δ$ is CL-shellable if and only if $Δ$ is shellable.

In what follows Theorem 2 is applied to a finite cell complex whose cells are simplexes, but where the intersection of two faces can possibly consist of the union of more than one face of each simplex. For such a pseudosimplicial complex the definition of shellability, as well as the proof of Theorem 2, carries over without alteration. In fact, Björner [4] proves a theorem analogous to Theorem 2 for CW complexes.

Let $G$ be the edge colored graph in Figure 1, the color set being $I = \{1, 2, 3, 4\}$. For $J \subseteq I$ let $S_J$ be the set of connected components of the subgraph of $G$ induced by lines colored in $J$. Note that $S_\emptyset$ is the set of points of $G$. Let $P(G)$ be the poset consisting of all pairs $(H, J)$ where $J \subseteq I$ and $H \in S_J$, together with a bottom element $\hat{0}$. The elements of $P(G)$ are ordered as follows: $(H, J) < (H', J')$ if $H \subseteq H'$ and $J \subseteq J'$ and $\hat{0} < (H, J)$ for all $J$ and $H$. Note that $P(G)$ also has a top element $\hat{1} = (G, I)$.

**Theorem 3.** The poset $P(G)$ is shellable but not CL-shellable.
Proof. To the graph $G$ is associated a pseudosimplicial complex as follows. For each point $v$ of $G$ let $Av$ be a 3-simplex whose vertices are colored 1, 2, 3 and 4. A subsimplex $s \subseteq Av$ is said to be of type $J$ if the set of colors on the vertices of $s$ is $J$. Let $X$ be the disjoint union of $\{Av | v \in G\}$. Identify two simplexes $s \subseteq Av$ and $s' \subseteq Av'$ if they are the same type $J$ and if $v$ and $v'$ are connected by a path in $G$ with colors in $I - J$. If $\sim$ denotes this identification let $\Delta G = X / \sim$. By [13, Theorem 2] $P(G)$ is the dual of the poset of faces of $\Delta G$, and by [13, Theorems 7, 10] $\Delta G$ is not shellable. Therefore Theorem 2 implies that $P(G)$ is not CL-shellable. The order complex $A' = \Delta (P(G) - \{0, 1\})$ is the barycentric subdivision of $\Delta G$. Now there is an edge colored graph $G'$ such that $A' = \Delta G'$ and $G'$ is easily produced from $G$. Also [13, Theorem 7] gives a simple criterion for checking the shellability of $A'$ in terms of $G'$. A short computer calculation then substantiates that $A'$ is shellable. Hence $\Delta P(G)$ is shellable, i.e. $P(G)$ is shellable.

Let a poset be called dual CL-shellable if its dual is CL-shellable. Since CL-shellability and dual CL-shellability each imply shellability, it is natural to ask whether there exist shellable graded posets which are neither CL-shellable nor dual CL-shellable. If $P$ is a graded poset which is shellable but not CL-shellable then we form the ordinal sum $Q$, of $P$ and its dual. Now $Q$ is a shellable graded poset since ordinal sums of shellable graded posets are shellable and graded (see [2]). But $Q$ is not CL-shellable nor dual CL-shellable since $P$ and its dual have intervals of $Q$ and any interval of a CL-shellable (dual CL-shellable) poset is CL-shellable (dual CL-shellable). Hence there exist shellable graded posets which are neither CL-shellable nor dual CL-shellable.

References


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