

Infinitely Many Trees Have Non-Sperner Subtree Poset

Andrew Vince · Hua Wang

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Abstract Let $C(T)$ denote the poset of subtrees of a tree T with respect to the inclusion ordering. Jacobson, Kézdy and Seif gave a single example of a tree T for which $C(T)$ is not Sperner, answering a question posed by Penrice. The authors then ask whether there exist an infinite family of trees T such that $C(T)$ is not Sperner. This paper provides such a family.

Keywords Poset · Lattice · Sperner property · Subtree

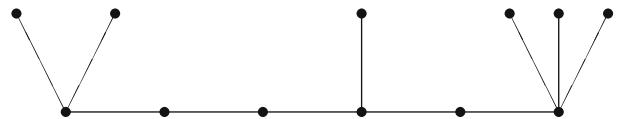
1 Introduction

A ranked partially order set (poset) is called *Sperner* if some maximum antichain consists of elements from a single rank. The set $C(T)$ of subtrees of a tree T is a poset with respect to the inclusion ordering, the rank of a subtree being the cardinality of its vertex set. Moreover, it is shown in [3] that $C(T)$ is a lattice. Such lattices appear as early as [4]. Jacobson, Kézdy and Seif [2] give a single example of a tree T for which $C(T)$ is not Sperner, answering in the negative a question posed by Penrice [5]. The authors [2] then ask whether there exists, in fact, infinitely many trees T for which $C(T)$ is not Sperner. Such an infinite family is provided in this paper. The ideas are inspired by [2]; in fact, the smallest member of our family is the example in that paper as shown in Fig. 1.

A. Vince · H. Wang (✉)
Department of Mathematics,
University of Florida, Gainesville, FL 32611, USA
e-mail: hua@math.ufl.edu

A. Vince
e-mail: vince@math.ufl.edu

Fig. 1 A non-Sperner tree on 11 vertices



Let T_m , $m \geq 3$, be the family of trees shown in Fig. 2. The path from v_1 to x has $s = s(m)$ vertices, and the path from v_2 to x has $t = t(m)$ vertices, where

$$s(m) = N_0 - 2^m + m - 2,$$

$$t(m) = N_0 - 2^m,$$

$$N_0 = \max_{0 \leq k < m} \left\{ \binom{m}{k} + \sum_{i=0}^{k+1} \binom{m}{i} \right\}. \quad (1)$$

Note that $N_0 \geq \binom{m}{m-1} + \sum_{i=0}^m \binom{m}{i} = 2^m + m$, which implies that $s(m) \geq 2m - 2$ and $t(m) \geq m$. In particular $s(m) \geq 4$ and $t(m) \geq 3$ since $m \geq 3$, and equality is achieved for the tree in Fig. 1. The proof of the following theorem appears in the next section.

Theorem 1 If $5m^2 + 12m + 8$ is not a perfect square, then $C(T_m)$ is not Sperner.

There are infinitely many m for which $5m^2 + 12m + 8$ is not a perfect square. For example, if $m \equiv 0 \pmod{3}$, then $5m^2 + 12m + 8 \equiv 2 \pmod{3}$. So $5m^2 + 12m + 8$ is not a perfect square if $m \equiv 0 \pmod{3}$ because 2 is not even a square $\pmod{3}$. A similar argument shows that $5m^2 + 12m + 8$ is not a perfect square for $m \equiv 0$ or $2 \pmod{5}$ or for $m \equiv 2, 3$ or $4 \pmod{7}$, etc. The values $m = 14$ and $m = 103$ are the first two for which $5m^2 + 12m + 8$ is a perfect square.

2 Proof of the Theorem

Let $T' := T'_m$ be the subtree of T_m rooted at vertex x and obtained by deleting vertex y from T_m . Further let $T_1 := T_1(m)$ and $T_2 := T_2(m)$ denote the two subtrees of T'

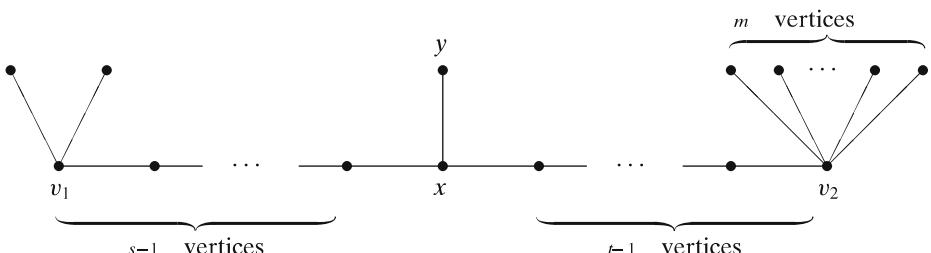


Fig. 2 The tree T_m

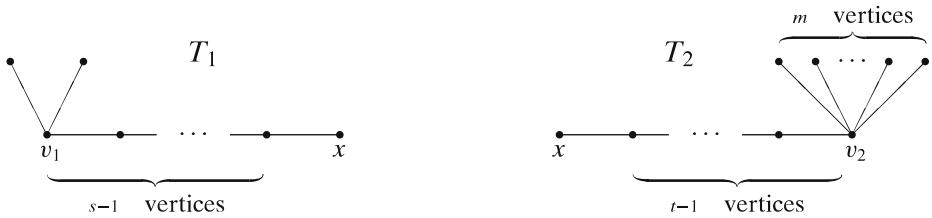


Fig. 3 The rooted trees T_1 and T_2

rooted at x that appear in Fig. 3. Note that T_1 has $s + 2$ vertices, T_2 has $t + m$ vertices, and $t + m = s + 2$ by the formulas 1.

For any tree T rooted at vertex x , let $C(T, x)$ denote the poset of all subtrees of T containing x . (Moreover, as pointed out by a referee, $C(T, x)$ is a locally distributive lattice [1].) Let $r'(i)$ denote the number of subtrees in $C(T', x)$ of rank i , and let $r_1(i)$ and $r_2(i)$ denote the number of subtrees in $C(T_1, x)$ and $C(T_2, x)$, respectively, of rank i . Then it is easy to see that

$$r'(i) = \sum_{j=1}^i r_1(j) \cdot r_2(i-j+1) \quad (2)$$

where

$$r_1(i) = \begin{cases} 2 & \text{if } i = s + 1 \\ 1 & \text{if } 1 \leq i \leq s + 2, i \neq s + 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$r_2(i) = \begin{cases} 1 & \text{if } 1 \leq i < t \\ \binom{m}{i-t} & \text{if } t \leq i \leq t + m \\ 0 & \text{otherwise.} \end{cases}$$

Our proof of Theorem 1 depends on the fact, proved below, that $r'(i)$ attains its maximum on exactly two values a and b of i and that a and b differ by at least 2.

Lemma 1 *With notation as in the introduction, let $0 \leq k_0 < m$ be such that $\binom{m}{k_0} + \sum_{i=0}^{k_0+1} \binom{m}{i} = N_0$ and let $a = s + 2$ and $b = 2s + 2 - k_0$. If $5m^2 + 12m + 8$ is not a perfect square, then $b \geq a + 2$ and*

$$\begin{aligned} r'(a) &= r'(b) = N_0 \\ r'(i) &< N_0 \quad \text{if } i \neq a, b. \end{aligned}$$

Proof Concerning the first statement, it follows from Eq. 1 and the fact that $s + 2 = t + m$ that $b - a = s - k_0 \geq s - m + 1 = t - 1 \geq 2$. Concerning the second statement, formula 2 implies

$$\begin{aligned} r'(a) &= r'(t + m) = \sum_{i=1}^{t+m} r_1(i) \cdot r_2(t + m - i + 1) \\ &= \sum_{i=1}^{m+1} r_1(i) \cdot r_2(t + m - i + 1) + \sum_{i=m+2}^{t+m} r_1(i) \cdot r_2(t + m - i + 1) \\ &= \sum_{i=1}^{m+1} \binom{m}{m-i+1} + t = \sum_{i=0}^m \binom{m}{i} + t = 2^m + t = N_0 \end{aligned}$$

and

$$\begin{aligned} r'(b) &= r'(2t + 2m - 2 - k_0) = \sum_{i=t+m-1-k_0}^{t+m} r_1(i) \cdot r_2(2t + 2m - 1 - k_0 - i) \\ &= \sum_{i=0}^{k_0+1} r_1(i + t + m - 1 - k_0) \binom{m}{i} = \binom{m}{k_0} + \sum_{i=0}^{k_0+1} \binom{m}{i} = N_0. \end{aligned}$$

Letting $\Delta(i) := r'(i + 1) - r'(i)$ it follows from Eq. 2 that

$$\Delta(i) = r_2(i - s + 1) - r_2(i - s) - r_2(i - s - 1) + r_2(i + 1). \quad (3)$$

It is then easy to check that

$$\begin{aligned} \Delta(i) &> 0 & \text{if } 1 \leq i < a \\ \Delta(i) &< 0 & \text{if } a \leq i < s + t \\ \Delta(i) &> 0 & \text{if } i = s + t. \end{aligned}$$

In addition we claim that

$$\begin{aligned} \Delta(i) &> 0 & \text{if } s + t < i < b \\ \Delta(i) &< 0 & \text{if } b \leq i, \end{aligned}$$

which would complete the proof of the lemma.

To prove the claim, let $i > s + t$. Since T_2 has $t + m = s + 2 < s + t$ vertices, $r_2(i + 1) = 0$. Equation 3 then implies that $\Delta(i) > 0$ if and only if

$$\begin{aligned} \binom{m}{i-t-s+1} &= r_2(i - s + 1) > r_2(i - s) + r_2(i - s - 1) \\ &= \binom{m}{i-t-s} + \binom{m}{i-t-s-1}. \end{aligned}$$

Letting $k = m - i + (s + t)$, the above inequality becomes

$$\binom{m}{k-1} > \binom{m}{k} + \binom{m}{k+1},$$

and an easy calculation using the quadratic formula shows that this is equivalent to $k > \alpha$ where

$$\alpha = \frac{1}{2} \left(-(m+2) + \sqrt{5m^2 + 12m + 8} \right).$$

Note that α is not an integer because $5m^2 + 12m + 8$ is not a perfect square. This implies that $\Delta(i) < 0$ if and only if $k < \alpha$. For $i > s+t$ the rank function of T' as given in formula 2, after some simplification, is

$$r'(k) = \binom{m}{k} + \sum_{j=0}^{k+1} \binom{m}{j}.$$

By the comments above, this function achieves a unique maximum at $k = k_0 = \lfloor \alpha \rfloor$. Equivalently, this maximum is at $i = m+s+t-k_0 = 2s+2-k_0 = b$. \square

Proof of Theorem 1 Let $T = T_m$. It is first shown that $C(T, x)$ is not Sperner. A maximum antichain \mathcal{A} is provided whose elements are not all from a single rank. With vertex y as shown in Fig. 2, integers a and b as in Lemma 1, and $|S|$ denoting the number of vertices in a subtree S , let

$$\mathcal{A} = \{ S \in C(T, x) : (|S| = a+1 \text{ and } y \in S) \text{ or } (|S| = b \text{ and } y \notin S) \}.$$

By Lemma 1 we know that $b > a+1$, so \mathcal{A} is an antichain in $C(T, x)$, and the elements of \mathcal{A} do not all have the same rank. Moreover $|\mathcal{A}| = r'(a) + r'(b) = 2N_0$. On the other hand, letting $r_x(i)$ denote the number of elements of rank i in $C(T, x)$, we have by Lemma 1 that $r_x(i) = r'(i) + r'(i-1) < 2N_0$. Hence $C(T, x)$ is not Sperner.

To prove that $C(T)$ is not Sperner, it will be shown that the number of elements at any given rank in $C(T)$ is less than the number of elements in the antichain \mathcal{A} . Let $r(i)$ denote the number of (unrooted) subtrees of $C(T)$ with i vertices. It is now sufficient to show that $r(i) < |\mathcal{A}| = 2N_0$ for all i . If $i \geq s+2 = t+m$, then any subtree of T with i vertices must contain x . Hence in this case $r(i) = r_x(i) < 2N_0$. It is easy to check that $r(1) < 2N_0$.

For $1 < i < s+2$, let $r_+(i)$ and $r_-(i)$ denote the number of subtrees with i vertices in $C(T)$ that do and do not, respectively, contain vertex y . Then

$$r_-(i) = s+t-i+2 + \sum_{j=0}^{i-1} \binom{m}{j}$$

and

$$r_+(i) = \begin{cases} i-1 & \text{if } i \leq t \\ t-1 + \sum_{j=0}^{i-t-1} \binom{m}{j} & \text{if } i > t. \end{cases}$$

Then $r(i) = r_-(i) + r_+(i)$ is maximized either when $i = s + 1$ or when $i = t$. Noting that $s \geq m$, $t + m = s + 2$ and $N_0 = 2^m + t$ from formula 1, we have

$$r(s+1) = 2t + \sum_{j=0}^m \binom{m}{j} + \sum_{j=0}^{m-2} \binom{m}{j} = 2^{m+1} + 2t - m - 1 < 2(2^m + t) = 2N_0.$$

and

$$r(t) = s + t + 1 + \sum_{j=0}^{t-1} \binom{m}{j} \leq 2^m + s + t + 1 = 2N_0 - 2^m + m - 1 < 2N_0. \quad \square$$

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