Strongly Balanced Graphs and Random Graphs

Andrzej Ruciński*
Andrew Vince
UNIVERSITY OF FLORIDA
GAINESVILLE, FLORIDA 32611

ABSTRACT

The concept of strongly balanced graph is introduced. It is shown that there exists a strongly balanced graph with v vertices and e edges if and only if $1 \leqslant v - 1 \leqslant e \leqslant \binom{v}{2}$. This result is applied to a classic question of Erdös and Rényi: What is the probability that a random graph on n vertices contains a given graph? A rooted version of this problem is also solved.

1. INTRODUCTION

The graphs in this paper are finite, undirected, without loops and multiple edges. Throughout the paper let $v(G) = |V(G)|, e(G) = |E(G)|$,

$$d(G) = \frac{e(G)}{v(G)}$$

and

$$m(G) = \max_{H \subseteq G} d(H)$$

A graph G is *balanced* if $d(H) \leqslant d(G)$ for all subgraphs H of G, i.e., $m(G) = d(G)$. The notion of balanced graph originated in the work of Erdös and Rényi on random graphs [3]. Let $K_{n,p}$ be a random graph obtained from n isolated vertices by independent addition of each edge with probability

[†]On leave from Institute of Mathematics, Adam Mickiewicz University, Poznań, Poland.

Journal of Graph Theory, Vol. 10 (1986) 251–264
 CCC 0364-9024/86/020251-14$04.00

$p = p(n)$. They proved that if G is a balanced graph then

$$\lim_{n\to\infty} \mathrm{Prob}(K_{n,p} \supset G) = \begin{cases} 0 & \text{if } p(n)n^{1/d(G)} \to 0 \text{ as } n \to \infty \\ 1 & \text{if } p(n)n^{1/d(G)} \to \infty \text{ as } n \to \infty . \end{cases} \tag{1.1}$$

Later Bollobás generalized (1.1) to arbitrary graphs by replacing $d(G)$ with $m(G)$:

$$\lim_{n\to\infty} \mathrm{Prob}(K_{n,p} \supset G) = \begin{cases} 0 & \text{if } p(n)^{1/m(G)} \to 0 \text{ as } n \to \infty \\ 1 & \text{if } p(n)^{1/m(G)} \to \infty \text{ as } n \to \infty . \end{cases} \tag{1.2}$$

Bollobás' method involving the "grading of graph" is somewhat sophisticated and we refer to [2]. Recently, Györi, Rothschild, and Ruciński [4] proved the conjecture of Karoński and Ruciński [8]:

For any graph G there exists a balanced graph $F \supseteq G$ such that $m(F) = m(G)$. (1.3)

Results (1.1) and (1.3) easily imply (1.2). For more about random graphs see [5].

Remark. It has been found very recently [9] that both approaches to the generalization of (1.1) are unnecessarily complicated. The simplest way of proving (1.2) is by the same "second moment method" used in [3]. Erdös and Rényi did not observe this possibility and, so, introduced the notion of balanced graph. Fortunately this notion, and some variations of it, are of interest in their own right.

For a nonempty ($e(G) \neq 0$) graph G define

$$d^*(G) = \frac{e(G)}{v(G) - 1}$$

and call G *strongly balanced* if $d^*(H) \leqslant d^*(G)$ for all nonempty subgraphs H of G. It is easy to check that if G is strongly balanced then G is connected and each of its blocks B has the same value of the proportion $d^*(B)$. Obviously every strongly balanced graph is balanced, but the converse is not true. Graphs G_1 and G_2 in Figure 1 are both balanced but not strongly balanced. In particular, all trees, cycles and complete graphs are strongly balanced.

In Section 2 it is shown that for all $1 \leqslant v - 1 \leqslant e \leqslant \binom{v}{2}$, there exists a strongly balanced graph with v vertices and e edges. We use a construction which is a modification of that from [4]. This result allows us to greatly simplify the proof of (1.3). This is done in Section 3. In Section 4, a rooted version of (1.2) is proved. Moreover, some distributional results are presented in which the notion of strongly balanced graph plays a crucial role.

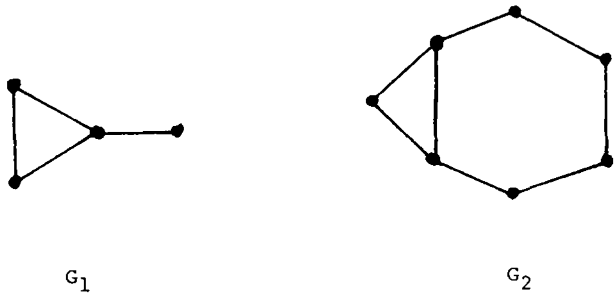


FIG. 1. Balanced, but not strongly balanced, graphs.

2. STRONGLY BALANCED GRAPHS

Since a strongly balanced graph G must be connected, $v(G) - 1 \leqslant e(G)$ must hold. Conversely let v and e be any integers such that $1 \leqslant v - 1 \leqslant e \leqslant \binom{v}{2}$. Below we construct a strongly balanced graph $B(v,e)$ with v vertices and e edges.

Throughout this section set $n = v - 1$ and express $e = kn + r$ where k and r are integers and $0 \leqslant r < n$. Note that $n \geqslant 2k - 1$ and $n \geqslant 2k$ if $r > 0$. Denote by $C(n,k)$ a graph with vertex set $\{1,2,\ldots,n\}$ and edge set $\{\{x,y\}: |x - y| \leqslant k - 1, \text{ modulo } n\}, k \geqslant 2$. Denote by $D(n,k)$ the graph obtained from $C(n,k)$ by joining a new vertex $n + 1$ to all vertices of $C(n,k)$. Let $R(n,r)$ be the following set of points almost equidistributed around $C(n,k)$:

$$R(n,r) = \{x : \lfloor (x - 1)r/n \rfloor < \lfloor xr/n \rfloor, 1 \leqslant x \leqslant n, x \text{ an integer}\}$$

Note that $|R(n,r)| = r$, and for any n' consecutive modulo n integers, less than $n'r/n + 1$ of them belong to $R(n,r)$. Five cases are now considered separately.

Case 1. $k = 1$ and $r = 0$. Let $B(v,e)$ be any tree with v vertices and e edges.

Case 2. $k = 1$ and $r = 1$. Let $B(v,e)$ be a v-cycle.

Case 3. $k = 1$ and $r > 1$. Let $B(v,e)$ be the graph obtained from the cycle $C(n,2)$ by joining a new vertex $n + 1$ to all vertices from $R(n,r)$.

Case 4. $k > 1$ and $n \leqslant 2k$. Then $e \geqslant \binom{n}{2}$. Define $B(v,e)$ to be a graph obtained from K_n by adding one vertex and $e - \binom{n}{2}$ edges.

Case 5. $k > 1$ and $n \geqslant 2k + 1$. Let $B(v,e)$ be the graph obtained from $D(n,k)$ by adding the r edges $\{x, x + k\}, x \in R(n,r)$. Note that the condition $n \geqslant 2k + 1$ assures that these r edges are distinct.

In each of the five cases the graph $B(v,e)$ has v vertices and e edges. Examples of Cases 3 and 5 are shown in Figure 2. In the proof of the main theorem

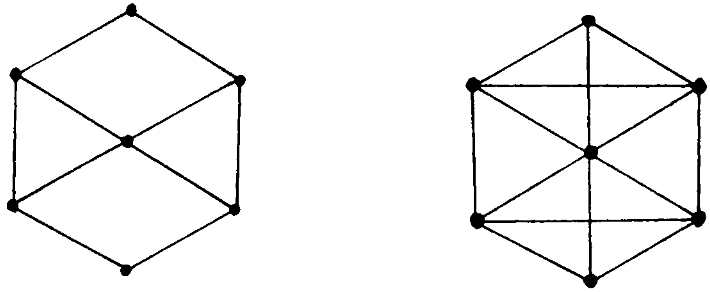


FIG. 2. Strongly balanced graphs.

we make use of a deficit function f for a graph G; for any subgraph H of G let

$$f(H) = d^*(G)(v(H) - 1) - e(H).$$

Note that $f(G) = 0$ and that G is strongly balanced if and only if

$$f(H) \geq 0 \tag{2.1}$$

for all subgraphs H of G. It is easy to check that

$$f(H_1 \cup H_2) = f(H_1) + f(H_2) - f(H_1 \cap H_2)$$

for all subgraphs H_1 and H_2 of G. If H_1 and H_2 have at most one vertex in common then $f(H_1 \cup H_2) \geq f(H_1) + f(H_2)$. In this case if $H_1 \cup H_2$ violates condition (2.1) so must either H_1 or H_2. This means that we can restrict ourselves to 2-connected subgraphs when checking whether a graph is strongly balanced.

Theorem 1. There exists a strongly balanced graph with v vertices and e edges if and only if $1 \leq v - 1 \leq e \leq \binom{v}{2}$.

Proof. It is sufficient to show that each of the five graphs $B(v, e)$ is strongly balanced. This is trivial in Cases 1 and 2. Otherwise let H be a 2-connected subgraph of $B = B(v, e)$.

In Case 3 if $n + 1 \notin V(H)$ then $H = C(n, 2)$ and $f(H) = [(n + r)/n] \cdot (n - 1) - n > 0$. On the other hand let $n + 1 \in V(H)$ and $n' = v(H) - 1$. Then H contains less than $n'r/n + 1$ edges incident to vertex $n + 1$. Hence

$$\begin{aligned} f(H) &= (1 + r/n)(v(H) - 1) - e(H) \\ &> (1 + r/n)n' - (n' - 1 + n'r/n + 1) = 0. \end{aligned}$$

In Case 4 assume the worst possibility, that H is a complete proper subgraph of B. Then

$$d^*(H) = \binom{v(H)}{2} \Big/ (v(H) - 1) = v(H)/2 \leqslant (v - 1)/2 \leqslant d^*(B).$$

In Case 5 it may be assumed that $n + 1 \in V(H)$; otherwise it is easy to check that $d^*(H + (n + 1)) > d^*(H)$. Let $B_s = B(v, ns + r)$ and let f_s be the corresponding deficit function: $f_s(H) = d^*(B_s)(v(H) - 1) - e(H)$. It is easy to check that $f_s(H) \geqslant 0$ for all subgraphs H of B_s implies $f_{s+1}(H) \geqslant 0$ for all subgraphs H of $B_{s+1}, s = 2, 3, \ldots$. Let B_s' denote the graph which looks like B_s but the r additional edges are now replaced by loops at the vertices of $R(n, r)$; let H' be the subgraph corresponding to H and let f_s' be the corresponding deficit function. It is again easy to check that $f_s'(H') \geqslant 0$ implies $f_s(H) \geqslant 0$. Hence it is sufficient to prove that $f_2'(H') \geqslant 0$ for all 2-connected subgraphs of B'. Since there are less than $n'r/n + 1$ loops in H', we have

$$\begin{aligned} f_2'(H') &= (2 + r/n)(v(H') - 1) - e(H') \\ &> (2 + r/n)n' - (n' - 1 + n' + n'r/n + 1) = 0, \end{aligned}$$

where $n' = v(H') - 1$. ∎

Problem. The proof of Theorem 1, in fact, can be slightly strengthened so that the constructed strongly balanced graphs with the exception of trees are *strict* in the following sense: For all proper subgraphs H of G there is strict inequality $d^*(H) < d^*(G)$. The problem arises of finding a graph G with given number of vertices and edges which maximizes

$$d^*(G) - \max_{H \not\subseteq G} d^*(H).$$

Even more natural is the similar question for balanced graphs, since strictly balanced graphs play an important role in random graph theory (see [2, 6, 7]).

3. AN APPLICATION

The following theorem appeared in [4] in conjunction with the random graph problem discussed in the introduction. Using Theorem 1 results in a significant simplification of the proof.

Theorem 2. For any graph G, there exists a balanced graph F containing G as a subgraph with $m(F) = m(G)$.

Proof. For any graph H define the deficit as

$$\varepsilon(H) = mv(H) - e(H),$$

where $m = m(G)$. It is easily verified that

$$\varepsilon(H_1 \cup H_2) = \varepsilon(H_1) + \varepsilon(H_2) - \varepsilon(H_1 \cap H_2)$$

for any graphs H_1, H_2. Note that for

$$\overline{H} = \bigcup_{\substack{H \subseteq G \\ \varepsilon(H)=0}} H \text{ we have } \varepsilon(\overline{H}) = 0 .$$

If $v(\overline{H}) = v(G)$ we are done. If not, a graph $G' \supset G$ will be constructed such that $m(G') = m(G)$ and

$$v(G') - v(\overline{H}') < v(G) - v(\overline{H}), \qquad \text{where } \overline{H}' = \bigcup_{\substack{H \subseteq G' \\ \varepsilon(H)=0}} H .$$

The result then follows by induction. Let

$$\varepsilon^* = \min_{\substack{H \subseteq G \\ H \not\subseteq \overline{H}}} \varepsilon(H) \quad \text{and} \quad \varepsilon(G^*) = \varepsilon^* .$$

Note that $m \geqslant \varepsilon^* > 0$.

The first inequality comes from considering a subgraph H obtained from $\overline{H}$ by adding one isolated vertex. For this H we have $\varepsilon(H) = m$. Without loss of generality we may assume $G^* \supset \overline{H}$; choose $x \in V(G^*) - V(\overline{H})$. Let v and e be any positive integers such that $v - 1 \leqslant e \leqslant \binom{v}{2}$ and

$$mv - e = m - \varepsilon^*. \tag{3.1}$$

Since the right-hand side of Eq. (3.1) is rational, the existence of such a solution is assured by the elementary theory of linear Diophantine equations. By Theorem 1 there is a strongly balanced graph B with v vertices and e edges. Now let G' be the graph obtained by adjoining B to G at vertex x, i.e., $V(B) \cap V(G) = \{x\}$. Equation (3.1) is equivalent to

$$\varepsilon(B \cup G^*) = \varepsilon(B) + \varepsilon(G^*) - \varepsilon(B \cap G^*) = 0 .$$

Thus $\overline{H}' \supseteq B \cup G^*$ and

$$v(G') - v(\overline{H}') \leqslant v(B \cup G) - v(B \cup G^*) = v(G) - v(G^*) < v(G) - v(\overline{H}) .$$

It only remains to show that $m(G') = m(G)$, equivalently $\varepsilon(H_0) \geq 0$ for any subgraph H_0 of G'. Let $B_0 = H_0 \cap B$ and $G_0 = H_0 \cap G$. Without loss of generality we may assume $B_0 \neq \emptyset$; let v_0, e_0 be the number of vertices and edges of B_0. Because B is strongly balanced,

$$\frac{e_0}{v_0 - 1} \leq \frac{e}{v - 1},$$

which, in turn, implies that

$$\frac{e - e_0}{v - v_0} \geq \frac{e}{v - 1} > m;$$

the last inequality follows from (3.1). Thus

$$\varepsilon(B_0) > \varepsilon(B).$$

If $G_0 = \emptyset$, then $\varepsilon(H_0) = \varepsilon(B_0) > \varepsilon(B) \geq 0$; the last inequality follows from (3.1) because $m \geq \varepsilon^*$. If $G_0 \neq \emptyset$, then by the definition of G^*, $\varepsilon(G_0) \geq \varepsilon(G^*)$. Therefore

$$\varepsilon(H_0) = \varepsilon(B_0) + \varepsilon(G_0) - \varepsilon(B_0 \cap G_0) > \varepsilon(B) + \varepsilon(G^*) - m = 0. \quad \blacksquare$$

A comparison of the proof in [4] with that above reveals that the constructed graph F in each case requires the same number of new vertices. The problem from [4] of determining the minimum such F is still open.

4. ROOTED SUBGRAPHS OF A RANDOM GRAPH

The concept of strongly balanced graph is central to the proof of Theorem 2 which, in turn, immediately implies Bollabás' result (1.2) on random graphs. In this chapter we give a rooted version of (1.2) as well as some distributional results, in which a modification of the notion of strongly balanced graph plays a crucial role.

Recall that by random graph $K_{n,p}$ we mean a graph obtained from the complete graph K_n on vertex set $\{1, \ldots, n\}$ by an independent deletion of each edge with probability $1 - p, p = p(n)$.

Let G be a graph. We call a subgraph G_0 of K_n a *rooted copy* of G if $1 \in V(G_0)$ and $G_0 \simeq G$. Denote by $X_n(G)$ the random variable which counts all the rooted copies of G in $K_{n,p}$. Our main result concerns the probability that a random graph contains at least one rooted copy of G. In order to prove this theorem we need a lemma for which we introduce the following terminology.

We call a subgraph G_0 of K_n an *x-rooted copy* of G if $1 \in V(G_0)$ and there is an isomorphism between G_0 and G which maps 1 to x. Denote by $X_n(G,x)$ the number of all x-rooted copies of G in $K_{n,p}$. The lemma will be proved by the so-called "second moment method" used in proving (1.1) in [3]. Recall that for every graph $G, e(G) \geqslant 1$,

$$d^*(G) = \frac{e(G)}{v(G) - 1}.$$

Lemma. Let G be an arbitrary graph with v vertices and e edges, $e \geqslant 1$, and let $x \in V(G)$. If we denote

$$m_x^* = m_x^*(G) = \max_{\substack{x \in H \subset G \\ e(H) \geqslant 1}} d^*(H)$$

then

$$\lim_{n\to\infty} \operatorname{Prob}(X_n(G,x) > 0) = \begin{cases} 0, & \text{if } pn^{1/m_x^*} \to 0 \text{ as } n \to \infty \\ 1, & \text{if } pn^{1/m_x^*} \to \infty \text{ as } n \to \infty. \end{cases} \tag{4.1}$$

Proof. Assume $pn^{1/m_x^*} \to 0$ as $n \to \infty$. Let H be a subgraph of G with $x \in V(H)$ and $d^*(H) = m_x^*$. Number all x-rooted copies of H in K_n, say, $H_1, H_2, \ldots$, and put

$$Y_i = \begin{cases} 1, & \text{if } H_i \subset K_{n,p} \\ 0, & \text{otherwise} \end{cases}$$

$i = 1, 2, \ldots$. Then $X_n(H,x) = \Sigma_i Y_i, EY_i = p^{e(H)}, i = 1, 2, \ldots$, and

$$\operatorname{Prob}(X_n(G,x) > 0) \leqslant \operatorname{Prob}(X_n(H,x) > 0) \leqslant EX_n(H,x)$$

$$= \sum_i EY_i = 0(n^{v(H)-1} p^{e(H)}) = 0(pn^{1/m_x^*}).$$

Thus the first statement of (4.1) is proved. Let $X = X_n(G,x)$. To prove the other one we make use of the following consequence of Tchebysheff's inequality:

$$P(X = 0) \leqslant \frac{\operatorname{Var} X}{(EX)^2} = \frac{E(X(X - 1))}{(EX)^2} + \frac{1}{EX} - 1.$$

As was done for $X_n(H,x)$ above, we can express $X_n(G,x)$ as a sum of 0–1 random variables, say, $X_n(G,x) = \Sigma_i Z_i$, with $EZ_i = p^e, i = 1, 2, \ldots$. Hence

$$EX = \sum_i EZ_i = \binom{n-1}{v-1} c(G,x) p^e > c(pn^{1/d^*(G)})^e$$
$$\geq c(pn^{1/m_x^*})^e \to \infty \text{ as } pn^{1/m_x^*} \to \infty\,, \tag{4.2}$$

where $c(G,x)$ and c are appropriate constants, i.e., both do not depend on n. Let us split

$$E(X(X-1)) = \sum_i \sum_{j \neq i} P(Z_i = Z_j = 1) = E' + E'',$$

where E' is the sum taken over all ordered pairs of (i, j) which correspond to the pairs, say, (G_i, G_j), of edge-disjoint x-rooted copies of G in K_n. It is easy to check that

$$E' = \sum_{s=0}^{v-1} \binom{n-1}{v-1} \binom{v-1}{s} \binom{n-v}{v-1-s} c(G,x) c_s p^{2e} \leq (EX)^2,$$

since $\sum_{s=0}^{v-1} \binom{v-1}{s} \binom{n-v}{v-1-s} = \binom{n-1}{v-1}$ and $c_s \leq c(G,x), s = 0, \ldots, v-1$, where $s+1$ is the number of common vertices in a pair of edge-disjoint x-rooted copies of G in K_n. Thus

$$P(X = 0) \leq \frac{E''}{(EX)^2} + \frac{1}{EX}\,. \tag{4.3}$$

Let G_i and G_j be two x-rooted copies of G in K_n with at least one edge in common. Then, by the definition of m_x^*,

$$e(G_i \cap G_j) \leq m_x^*(v(G_i \cap G_j) - 1)\,,$$

and

$$E'' = 0\left(\sum_{s=1}^{v-1} n^{2(v-1)-s} p^{2e-m_x^* s}\right) \tag{4.4}$$
$$= 0((EX)^2 (pn^{1/m_x^*})^{-1}) = o((EX)^2)$$

provided $pn^{1/m_x^*} \to \infty$ as $n \to \infty$. This, together with (4.2) and (4.3), completes the proof. ∎

Remark. The above proof can be easily adapted to the case of ordinary (not rooted) subgraphs, giving a "new" elementary proof of (1.2).

Theorem 3. Let G be an arbitrary graph with v vertices and e edges, $e \geq 1$. If we denote

$$m^* = m^*(G) = \min_{x \in V(G)} m_x^*(G)$$

then

$$\lim_{n\to\infty} \operatorname{Prob}(X_n(G) > 0) = \begin{cases} 0, & \text{if } pn^{1/m^*} \to 0, \quad \text{as } n \to \infty \\ 1, & \text{if } pn^{1/m^*} \to \infty, \quad \text{as } n \to \infty . \end{cases}$$

Proof. For every $x \in V(G)$ let H_x stand for a subgraph of G with $x \in V(H_x)$ and $m_x^* = d^*(H)$. Then, if $pn^{1/m^*} \to 0$,

$$\operatorname{Prob}(X_n(G) > 0) \leq \sum_{x \in V(G)} \operatorname{Prob}(X_n(H_x, x) > 0) = o(1)$$

by the lemma and the definition of m^*. On the other hand, applying the lemma for G and $x \in V(G)$ such that $m_x^* = m^*$, we have

$$\operatorname{Prob}(X_n(G) > 0) \geq \operatorname{Prob}(X_n(G, x) > 0) = 1 - o(1),$$

provided $pn^{1/m^*} \to \infty$. ∎

In the remainder of the paper we consider the asymptotic distribution of $X_n(G)$ as $n \to \infty$. As usual (see [6, 7]), we may expect Poisson distribution when $p = p(n)$ is of the same order of magnitude as n^{-1/m^*}, and a "normal phase" in the case $pn^{1/m^*} \to \infty$ but not too fast.

Assume first

$$pn^{1/m^*} \to c > 0 \qquad \text{as } n \to \infty .$$

It can be deduced similarly to (4.2) that

$$EX_n(G) > c(pn^{1/d^*(G)})^e \geq c(pn^{1/m^*})^e. \tag{4.5}$$

Since any asymptotic Poisson distribution requires

$$\lim_{n\to\infty} EX_n(G) = \lambda, \qquad \lambda > 0,$$

we arrive, via (4.5) at the necessary condition $d^*(G) = m^*(G)$, which means there is a vertex $x \in V(G)$ such that for all subgraphs H of G containing $x, d^*(H) \leq d^*(G)$. We call a graph G with the above property *locally strongly balanced.*

Remark. Every strongly balanced graph is locally strongly balanced. There is no such relationship between balanced and locally strongly balanced graphs. The graph in Figure 3(a) is balanced but not locally strongly balanced, whereas the graph in Figure 3(b), conversely, is locally strongly balanced but not balanced. The graph in Figure 3(c) has both properties.

Note also that in proving Theorem 2 it was not necessary to require B strongly balanced; it would have been enough to assume that the auxiliary graph B is both balanced and locally strongly balanced.

Although the locally strongly balanced property is necessary, it is not sufficient for asymptotic Poisson distribution. The property must be strengthened as follows. A graph G has *property* P if for every vertex $x \in V(G)$ for which $d^*(H) \leq d^*(G)$ for all subgraphs H of G containing x, we have strict inequality $d^*(H) < d^*(G)$ for all proper subgraphs H of G containing x. A graph G is called *locally strictly balanced* if G is locally strongly balanced and has property P. A locally strictly balanced graph is the analog of a strictly balanced graph [5, 6] for the unrooted case. Observe that trees are not locally strictly balanced but cycles and complete graphs are. Define

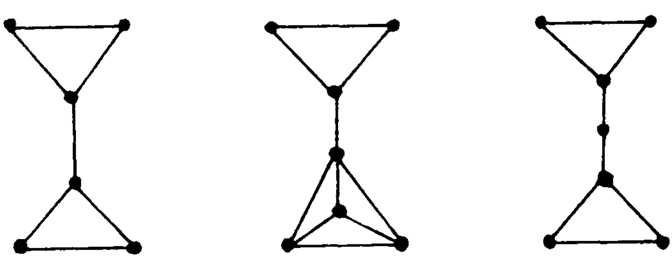
$$\eta = d^*(G) - \max_{x \in S} \max_{\substack{x \in H \subsetneq G \\ e(H) \geq 1}} d^*(H) \tag{4.6}$$

as a measure of the extent of locally strict balance, where

$$S = \{x \in V(G) : m_x^*(G) = d^*(G)\}.$$

Theorem 4. Let G be a locally strictly balanced graph, let $0_1, 0_2, \ldots$ be the orbits of the vertex set of G under the action of its automorphism group, and let $x_i \in 0_i, i = 1, 2, \ldots$ Define the set $T = \{i : m_{x_i}^*(G) = d^*(G)\}$. If

$$\lim_{n \to \infty} pn^{1/d^*} = c > 0$$



(a) (b) (c)

FIG. 3

then

$$\lim_{n\to\infty} \operatorname{Prob}(X_n(G) = k) = \lambda^k \exp\{-\lambda\}/k!, \qquad k = 0, 1, \ldots, \tag{4.7}$$

where $\lambda = c^e \sum_{i\in T} [a(G, x_i)]^{-1}$ and $a(G,x)$ is the number of automorphisms of G which fix x.

Proof. We prove (4.7) for $\overline{X}_n = \sum_{i\in T} X_n(G, x_i)$. The result will then follow by the fact that

$$\lim_{n\to\infty} \operatorname{Prob}(X_n(G) - \overline{X}_n > 0) = 0$$

by the lemma. According to a result from [1] (see also [6])

$$\sup_{A\subset Z^+} |\operatorname{Prob}(\overline{X}_n \in A) - \operatorname{Prob}(Y_n \in A)| \leq 2p^e + 4E''/E\overline{X}_n, \tag{4.8}$$

where Y_n has Poisson distribution with $EY_n = E\overline{X}_n$ and E'' is the expectation of the number of all ordered pairs of x_i-rooted copies of G in $K_{n,p}$, $i \in T$, with at least one edge in common. Then, as in (4.4)

$$E'' = 0\left(\sum_{s=1}^{v-1} n^{2(v-1)-s} p^{2e-(d^*-\eta)s}\right) = 0(p^\eta) = o(1), \quad \text{since } \eta > 0.$$

To complete the proof note that $E\overline{X}_n \to \lambda$ as $n \to \infty$. ∎

The assumption of locally strict balance in Theorem 4 is not only sufficient, but also necessary. In fact, if G is locally strongly balanced but not locally strictly balanced, then there is a vertex $x \in V(G)$ and a proper subgraph H of G such that

(i) $x \in V(H)$
(ii) $d^*(H) = m_x^*(H) = d^*(G) = d^*$.

Assume $pn^{1/d^*} \to c > 0$, so $EX_n(G) \to \lambda > 0$. Then

$$E\{X_n(G)[X_n(G) - 1]\} \geq E' + E''$$

where E' is the expected number of pairs of vertex disjoint rooted copies of G in $K_{n,p}$, whereas E'' is the expected number of pairs of x-rooted copies of G in $K_{n,p}$ whose intersection is exactly H. Then $E' \to \lambda^2$ and

$$E'' \geq \binom{n}{v-1}\binom{n-v}{n-v(H)} p^{2e-e(H)} \to c_0 > 0 \quad \text{as} \quad n \to \infty.$$

Thus $\liminf_{n\to\infty} E\{X_n(G)[X_n(G) - 1]\} > \lambda^2$ and $X_n(G)$ cannot converge in distribution to Poisson distribution.

For Y_n with Poisson distribution if $EY_n \to \infty$ then, by the Central Limit Theorem, the sequence $(Y_n - EY_n)/\sqrt{EY_n}$ has asymptotically standard normal distribution, and so does $(X_n(G) - EX_n(G))/\sqrt{EX_n(G)}$, as long as the right-hand side of (4.8) tends to zero. We conclude the paper with a result establishing the "normal phase" of $X_n(G)$.

Theorem 5. Let G be a locally strictly balanced graph and let η be as in (4.6). If

$$\lim_{n\to\infty} pn^{1/d^*} = \infty$$

but

$$\lim_{n\to\infty} pn^{(v-3)/[d^*(v-3)+\eta(v-1)]} = 0$$

then for every $x \in (-\infty, \infty)$

$$\lim_{n\to\infty} \left|\text{Prob}\{(X_n(G) - EX_n(G))/\sqrt{EX_n(G)} < x\} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt\right| = 0\,. \quad \blacksquare$$

Remark. The greater η, the longer the "normal phase" that can be established. This corresponds to the problem from Section 2. For instance, if G is a cycle on v vertices or $G = K_v$ then $\eta = 1/(v-1)$, which is the trivial upper bound for η (taking H as G with an edge deleted).

References

[1] A. D. Barbour, Poisson convergence and random graphs. *Math. Proc. Cambr. Phil. Soc.* 92 (1982) 349–359.

[2] B. Bollobás, Threshold functions for small subgraphs. *Math. Proc. Cambr. Phil. Soc.* 90 (1981) 187–206.

[3] P. Erdös, A. Rényi, On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci.* 5 (1960) 17–61.

[4] E. Györi, B. Rothschild, A. Ruciński, Every balanced graph is contained in a sparsest possible balanced graph. *Math. Proc. Cambr. Phil. Soc.* 98 (1985) 397–401.

[5] M. Karoński, A review of random graphs. *J. Graph Theory* 6 (1982) 349–389.

[6] M. Karoński, *Balanced Subgraphs of Large Random Graphs*. A. M. Univ. Press, Poznań (1984).

[7] M. Karoński, A. Ruciński, On the number of strictly balanced subgraphs of a random graph. *Lecture Notes in Math. 1018,* Springer-Verlag (1983) 79–83.
[8] M. Karoński, A. Ruciński, Problem 4, *Proc. Third Czech. Symp. on Graph Th.*, Prague, 1982, p. 350.
[9] A. Ruciński and A. Vince, Balanced graphs and the problem of subgraphs of random graphs, *Congressus Numerantum* 49 (1985) 181–190.