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On representations of some thickness-two graphs

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Abstract

This paper considers representations of graphs as *rectangle-visibility graphs* and as *doubly linear graphs*. These are, respectively, graphs whose vertices are isothetic rectangles in the plane with adjacency determined by horizontal and vertical visibility, and graphs that can be drawn as the union of two straight-edged planar graphs. We prove that these graphs have, with n vertices, at most $6n-20$ (respectively, $6n-18$) edges, and we provide examples of these graphs with $6n-20$ edges for each $n \geqslant 8$.

Keywords: Rectangle-visibility graph; Doubly linear graph

1. Introduction

A *thickness-two graph* G is one whose edge set can be partitioned into two planar graphs, each on one copy of the vertex set of G. These graphs are of theoretical interest and arise in a multitude of applications. For example, it is an NP-complete problem to determine whether a graph has thickness two [11], and the upper bound on their chromatic number is known only to lie between 9 and 12 [6,8,14]. Thickness-two graphs arise in models for printed circuit boards [7,8] and in VLSI design and layout [21] in which all connections are either horizontal or vertical and so divide naturally into two planar layers.

We study thickness-two graphs and their representations as *rectangle-visibility graphs* and as *doubly linear graphs*; in [15] it is shown that recognizing the former graphs is an NP-complete problem. We show that the most (edge) dense thickness-two graphs have neither rectangle-visibility nor doubly linear representations, though these graph representations are ubiquitous among thickness-two graphs of lower density.

A *bar-visibility graph* [10,22] is one whose vertices can each be represented by a closed horizontal line segment in the plane, having pairwise disjoint relative interiors, with two vertices adjacent in the graph

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if and only if the corresponding segments are *vertically visible*. Two segments are considered *vertically visible* when there is a nondegenerate rectangle R such that R intersects only these two segments, and the horizontal sides of R are subsets of these two segments. (For variations on this definition, see [13,18].) Clearly, a bar-visibility graph is planar. Not all planar graphs are bar-visibility graphs since the latter are characterized as those planar graphs for which there is a planar embedding with all cut vertices on a common face [18,22].

A natural two-directional analog is that of a *rectangle-visibility graph*, a graph whose vertices can each be represented by a closed rectangle in the plane with sides parallel to the axes, having pairwise disjoint interiors, with two vertices x and y adjacent in the graph if and only if the corresponding rectangles are vertically or horizontally visible (with horizontal visibility defined analogously to vertical). Note that the bands of visibility may cross. Every planar graph is a rectangle-visibility graph [10], and it is clear that every rectangle-visibility graph has thickness at most two. Even more, a rectangle-visibility graph is the union of two bar-visibility graphs. Our main result on these graphs is that a rectangle-visibility graph with n vertices has at most $6n - 20$ edges, as distinguished from thickness-two graphs which can have as many as $6n - 12$ edges. (The latter fact follows from Euler's formula for planar graphs, which implies that a planar graph with n vertices has at most $3n - 6$ edges.) In addition, we show that for every $n > 7$ there is a rectangle-visibility graph with $6n - 20$ or fewer edges.

It is a consequence of a classical theorem of Steinitz on polyhedra (see [17]) that every planar graph G has a *linear* or *straight-line embedding* in the plane. This means that
(1) every edge is a straight line segment,
(2) no vertex lies in the interior of an edge, and
(3) edges do not cross.
If, instead of property (3), we require that
(3′) the edges of G can be partitioned into two subsets, each without crossings,
then G is called *doubly linear*. Again it is clear that doubly linear graphs have thickness at most two. We prove that a doubly linear graph with n vertices has at most $6n - 18$ edges, and for each $n > 7$ we give an example of a doubly linear graph with $6n - 20$ or fewer edges. We give examples of doubly linear graphs that are not rectangle-visibility graphs but conjecture that every rectangle-visibility graph is doubly linear.

Section 2 of this paper contains basic examples of rectangle-visibility and doubly linear graphs, as well as examples that are essential to the main results of the paper. Sections 3 and 4 deal with the existence question for rectangle-visibility and doubly linear graphs, respectively, with a given number of vertices and edges, and Section 5 concludes with open questions. The concepts considered in this paper come from [7,10], and from the Workshop on Visibility Representations, McGill University Bellairs Research Institute, held in February 1993. The results of this paper (without proofs) have been announced in [9], and subsequent results on rectangle-visibility graphs appear in [2–5,15,16].

2. Examples

2.1. Complete graphs

Figs. 1 and 2 show a rectangle-visibility and a doubly linear representation, respectively, of the complete graph K_8. This is the largest complete graph so representable since K_9 has thickness three [1,19].

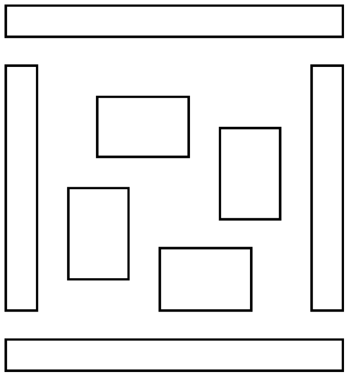

Fig. 1. A rectangle-visibility representation of K_8.

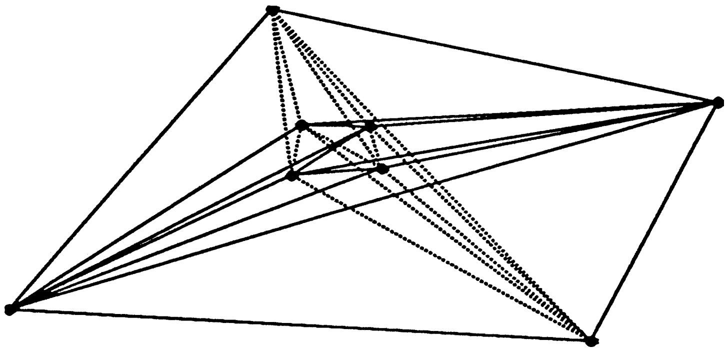

Fig. 2. A doubly linear representation of K_8.

2.2. Dense graphs

It is not hard to add another rectangle, visible to six others, to Fig. 1 (respectively, a vertex with six straight-line edges to Fig. 2) to obtain a rectangle-visibility representation (respectively, a doubly linear representation) of K_9 minus two edges; it can be arranged for these missing edges to be either mutually incident or nonincident. These graphs and K_8 have $6n - 20$ edges, $n = 9, 8$, respectively. K_9 minus one edge ($K_9 - e$) has thickness two with $6n - 19$ edges [20]. Theorem 1 will show that $K_9 - e$ is therefore not a rectangle-visibility graph, though it is the union of two bar-visibility graphs. We conjecture that $K_9 - e$ is not doubly linear.

Figs. 3 and 4 are rectangle-visibility representations that attain the upper bound $6n - 20$ of Theorem 1 on the number of edges for a given number n of vertices. They are representative of infinite families of such graphs and are used in the proof of Theorem 3 in the next section.

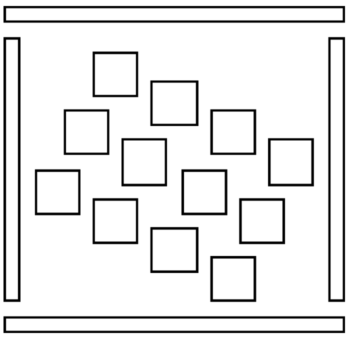

Fig. 3. A rectangular representation with $n = 16$.

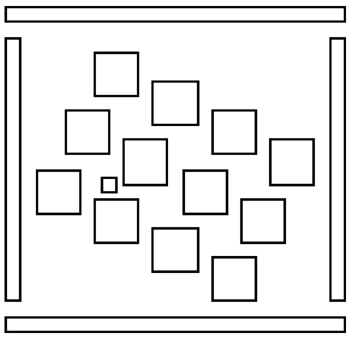

Fig. 4. A rectangular representation of $n = 17$.

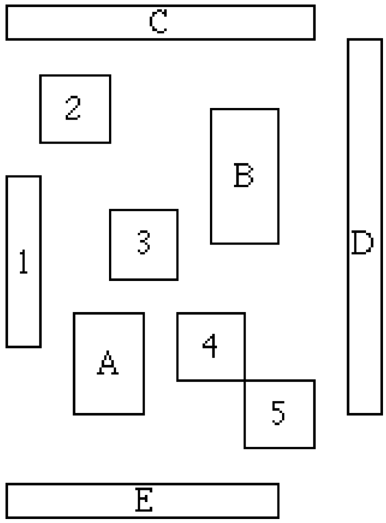

Fig. 5. A rectangle-visibility representation of $K_{5,5}$ plus four edges.

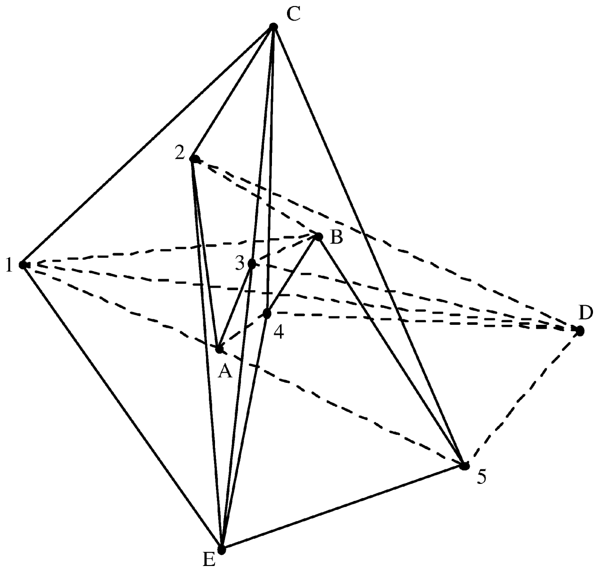

Fig. 6. A doubly linear representation of $K_{5,5}$.

2.3. Complete bipartite graphs

Fig. 5 shows a rectangular representation of $K_{5,5}$ plus four edges, and Fig. 6 shows a closely related doubly linear representation of $K_{5,5}$. Fig. 5 can be extended to a rectangular representation of $K_{5,6}$ plus edges by adding a long rectangle along the left side, and Fig. 6 can be similarly extended to a doubly linear representation of $K_{5,6}$. In [3,4] it is shown that $K_{p,q}$ with p and q at least 5 is not a rectangle-visibility graph (and that $K_{5,5}$ minus an edge and $K_{5,5}$ plus an edge are rectangle-visibility graphs). Thus $K_{5,5}$ and $K_{5,6}$ are doubly linear graphs, but not rectangle-visibility graphs. These examples point up an essential difference between the two classes of graphs: namely, that although a subgraph of a doubly linear graph is also doubly linear, the same is not true for rectangle-visibility graphs.

2.4. The join of graphs

Some infinite families of graphs having rectangle-visibility and doubly linear representations can be obtained in terms of the *join*. The *join* of two disjoint graphs G and H is the union of these two graphs together with an edge joining vertices g and h, for each vertex g of G and vertex h of H, and is denoted $G + H$. It is not difficult to obtain, for every n, a rectangle-visibility and a doubly linear representation of the join of K_4 and P_n and the join of K_4 and C_n, where P_n and C_n are, respectively, the path and the cycle on n vertices. In these examples K_4 cannot be replaced by K_5 for $n > 12$, since these graphs would contain $K_{5,13}$ which, by Euler's formula, has thickness at least 3.

It is also not hard to show that if G is a 2-connected planar graph or, more generally, a bar-visibility graph, then $P_2 + G$ is a rectangle-visibility graph. Moreover, if G is a planar graph, then $P_2 + G$ is doubly linear. Note that, as long as G contains a cycle, $P_2 + G$ is not planar since it contains a homeomorph of K_5.

3. Rectangle-visibility graphs

In this section we examine the number of edges possible in a rectangle-visibility graph and in a graph that does not have a rectangle-visibility representation.

Theorem 1. *A rectangle-visibility graph on $n \geqslant 5$ vertices has at most $6n - 20$ edges.*

Proof. For $5 \leqslant n \leqslant 8$, the value $6n - 20$ is at least as large as the number of edges in the complete graph K_n, and so this bound holds immediately for these values of n.

Let G be a rectangle-visibility graph with $n > 8$ vertices and rectangular representation R^*. Let R be a rectangle in R^*, and define $N(R)$ (respectively, $E(R)$, $S(R)$ and $W(R)$) to be the set of rectangles in R^* that intersect with positive area the one-way infinite band of all points "north" (respectively, "east", "south" and "west") of R.

Select R_1 to be the rectangle R with $N(R)$ empty and with the greatest y-coordinate for its bottom. Note that if R' is visible to R_1 horizontally, then $N(R')$ is empty; otherwise there is another rectangle with $N(R)$ empty and y-coordinate larger than R_1's for its bottom. Move R_1 northward until its bottom is at least two units above the top of any other rectangle; then make the height of R_1 one unit and expand it horizontally until it is as wide as the whole representation with two additional units of width to the left and to the right. The new R_1 has retained all its previous visibilities and may have gained some. Note that $S(R_1)$ is not empty in the new configuration since $n > 1$.

Next select R_2 with $S(R_2)$ empty and with the least y-coordinate of its top, $R_2 \neq R_1$. Again if R' is visible to R_2 horizontally, then $S(R')$ is also empty. Move R_2 southwards until its top is at least two units below the bottom of every other rectangle; then make R_2 one unit high and as wide as and directly below R_1. The new R_2 has retained all its previous visibilities. Note that in the new representation $|S(R_1)|$ and $|N(R_2)|$ are each at least two since they are visible to each other and $n > 2$.

Select R_3 with $W(R_3)$ empty and with the x-coordinate of its rightmost side as small as possible. Note that $R_3 \neq R_1$ and $R_3 \neq R_2$. If R_3 sees R' vertically, then $W(R')$ is empty as argued previously. Move R_3 westward until its left side is even with the left sides of R_1 and R_2. Make R_3 one unit wide and increase its height until it is one unit below the bottom of R_1 and one unit above the top of R_2. The new R_3 retains all previous visibilities. Note that $E(R_3)$ is not empty since $n > 3$. Finally, repeat this same procedure with R_4 selected to have $E(R_4)$ empty and the x-coordinate of its left side as large as possible. (See Fig. 1 for an example of the positioning of R_1, R_2, R_3 and R_4.)

Let G' be the resulting rectangle-visibility graph of these rectangles so that G is a subgraph of G'. The graph G' decomposes into two planar graphs, G'_h and G'_v, which represent, respectively, the horizontal and vertical visibilities of G'. Now count the edges of G', G'_h and G'_v. In G'_h the vertices corresponding to R_1 and R_2 have degree 0 and so, by Euler's formula,

$$\left|E\left(G'_h\right)\right| \leqslant 3(n-2) - 6 = 3n - 12.$$

In G'_v the vertices corresponding to R_3 and R_4 have degree 2 and so

$$\left|E\left(G'_v\right)\right| \leqslant 3(n-2) - 6 + 4 = 3n - 8.$$

Thus,

$$\left|E(G)\right| \leqslant \left|E\left(G'\right)\right| \leqslant 6n - 20. \quad \square$$

The following is an immediate corollary of Theorem 1.

Corollary 2. *Let G' with $n \geqslant 5$ vertices be a subgraph of a rectangle-visibility graph G. Then G' has at most $6n - 20$ edges.*

See [3,4] for similar proofs that a bipartite rectangle-visibility graph and a bipartite subgraph of a rectangle-visibility graph with n vertices has at most $4n - 12$ edges. Bipartite rectangle-visibility examples with at most $4n - 16$ edges are given also for each $n > 7$, and for each $n \geqslant 16$ bipartite graphs with n vertices and $4n - 12$ edges are known that are subgraphs of rectangle-visibility graphs [12].

Next we show that the bound of Theorem 1 is best possible for all $n \geqslant 8$. (For $n \leqslant 8$, as noted in Section 2, the complete graphs give the best possible bound.) Figs. 3 and 4 show rectangular representations with $6n - 20$ edges and $n = 16$ and 17 vertices, respectively, and the next result shows that this pattern holds for all $n \geqslant 8$.

Theorem 3. *There is a* (*connected*) *rectangle-visibility graph with n vertices and $6n - 20$ edges for each $n \geqslant 8$.*

Proof. First, for each n of the form $n = ks + 4$ with $k, s > 1$, we describe a representation with (mainly) squares, 3 units by 3 units. Then we show how to vary this for $n = p + 4$ where p is a prime.

We use i and j as coordinates of the squares. Let

$$\mathrm{LL} = \{iu + jv \mid 0 \leqslant i \leqslant k - 1,\ 0 \leqslant j \leqslant s - 1\},$$

where u and v are the vectors $(4, -2)$ and $(2, 4)$, respectively. LL is the set of lower-left corners of squares in the construction. Let S be the set of 3×3 squares

$$\big\{(x, y) - (x + 3, y + 3) \mid (x, y) \text{ in LL}\big\}.$$

(See Fig. 3 for the case of $k = 4$, $s = 3$.)

In addition, put a tall rectangle to the left of the squares and to the right of the squares, stretching slightly above them. Then above all the squares and rectangles place a long, horizontal rectangle, stretching from the left to the right of the configuration below. Similarly place a long rectangle below the whole configuration.

The four rectangles just placed around the outside form a K_4, having 6 edges. The remaining edges are between squares or between one rectangle and a square and fall into two sets, horizontal and vertical.

First we count the edges in the vertical set by examining the rectangle or square at the bottom of each such edge.

(a) The long rectangle at the bottom sees all squares with $j = 0$, $j = 1$, or $i = k - 1$. There are $2k + s - 2$ such squares.
(b) No other rectangle sees a square from below.
(c) Squares with $j = s - 1$ see only the top rectangle from below. There are k such squares.
(d) Squares with $i = 0$ and $j \neq s - 1$ see the top rectangle and one other square. There are $s - 1$ such squares.
(e) All other squares see three objects from below. There are $(k - 1)(s - 1)$ such squares.

Totaling (a)–(e) we get

$$e = 2k + s - 2 + 0 + k \cdot 1 + (s - 1) \cdot 2 + (k - 1)(s - 1) \cdot 3 = 3ks - 1 \text{ edges.}$$

A similar count shows the number of horizontal edges also equals $3ks - 1$. Thus the total number of edges is $2(3ks - 1) + 6 = 6ks + 4 = 6n - 20$ since $n = ks + 4$, demonstrating the theorem when $n - 4$ is composite.

Suppose now that $n = p + 4$ for some prime p. Perform the previous construction with $n - 1 = p - 1 + 4$ rectangles and $6(n - 1) - 20 = 6n - 26$ edges. Then add a unit square with lower-left hand corner at $(4.5, 1.5)$. This added square sees 6 other objects, four vertically and two horizontally, and blocks no previous visibility. (See Fig. 4 for the case of $n = 13 + 4$.) Thus we have n rectangles with $6n - 20$ visibility edges. □

Rectangle-visibility graphs with fewer edges are also possible, as given in the next result. Simple graphs with 6 vertices and 16 edges or with 7 vertices and 22 edges do not exist, thus the exceptions. The result follows by taking essentially the same rectangular arrangements as in the proof of Theorem 3 and adding additional rectangles in fairly obvious ways.

Corollary 4. *With the exception of the cases* $(n, m) = (6, 16)$ *and* $(7, 22)$, *the following holds for all* $n \geqslant 4$:

(a) *for each* m *with* $0 \leqslant m \leqslant 6n - 20$, *there is a rectangle-visibility graph with* n *vertices and* m *edges,*
(b) *for each* m *with* $n - 1 \leqslant m \leqslant 6n - 20$, *there is a connected rectangle-visibility graph with* n *vertices and* m *edges.*

Families of graphs with $n' \geqslant 9$ vertices and $m' \geqslant 35$ edges that are not representable by rectangles can also be found. For any $0 \leqslant m \leqslant 6n - 20$ take a rectangle-visibility graph G with n vertices and m edges, as guaranteed by Corollary 4, and form the disjoint union of G with $K_9 - e$ to obtain a graph with $m + 35$ edges and $n + 9$ vertices. By Corollary 2, the new graph is not a rectangle-visibility graph since $K_9 - e$ has more than $6n - 20$, $n = 9$, edges. Connected graphs G together with $K_9 - e$ plus an adjoining edge similarly give connected examples.

4. Doubly linear graphs

The results in this section parallel those of Section 3. Our upper bound on the number of edges for a graph with a doubly linear representation is $6n - 18$. We give, for each $n \geqslant 8$, an example of a doubly linear graph with $6n - 20$ edges, two short of the upper bound. An embedding is called a *near-triangulation* of the plane if all faces, except possibly for the infinite face, are bounded by three edges.

Lemma 5. *Let* G *be a maximal linear near-triangulation of the plane with vertex set* V, *and let* L *be a line through a vertex* v *of* V. *Let* V_1 *and* V_2 *be the subsets of* V *on either side of* L *and* V_3 *all vertices on* L *so that* $V = V_1 \cup V_2 \cup V_3$. *If* V_1 *is nonempty, then there is an edge of* G *from* v *to some member of* V_1.

Proof. Without loss of generality, not all vertices lie on L; otherwise V_1 and V_2 are both empty. Let P the boundary of the external, infinite face of G; by maximality, P is a convex polygon. Successive pairs of neighbors of v form triangles around v and so form angles less than π at v, except for one pair when v lies on P. Thus if v does not lie on P, v has a neighbor in V_1 and in V_2. If v lies on P and if L is a support line of P (i.e., all vertices of G lie on or on one side of L), then all neighbors of v lie in $V_1 \cup V_3$ by maximality, and at least one neighbor lies off L and in V_1. If L is not a support line, then v has neighbors in both V_1 and V_2, for example, its two neighbors on P. $\square$

We shall use Lemma 5 when V_1 is a singleton $\{u\}$. We then call the edge uv a *forced edge* of the triangulation.

Theorem 6. *If G is a doubly linear graph with $n \geqslant 4$ vertices, then G has at most $6n-18$ edges.*

Proof. The bound is best possible for $n=4$, but for $n=5$, 6, 7 and 8, the complete graphs show that G has at most $6n-20$ edges. Thus we may assume $n>8$.

Let G be drawn as a doubly linear graph in the plane and two-color the edges of G so that the two sets, R and B, each form a straight-line planar embedding. Assume G is maximal, i.e., no edge can be added without destroying double-linearity.

Add as many edges to R as possible, retaining straight-line planarity, giving R'. Similarly, add edges to B giving B'. Each edge added to R must be in B, and each edge added to B must be in R by the maximality assumption. Thus, the number d of duplicate edges (edges in both R' and B') equals the number of edges that have been added. Furthermore, if h is the number of edges on the convex hull of the embedded G, then the number of edges in R' or B' is $3n-6-(h-3)$, so the total number of edges in G is $2(3n-6-h+3)-d=6n-12-2h+6-d$. Set $s=2h+d-6$ so that G is s edges short of being the union of two complete planar triangulations.

To show G has at most $6n-18$ edges, it suffices to show that $s \geqslant 6$. We examine cases based on the number of edges h on the convex hull. Note that both R' and B' contain all edges of the convex hull so that $d \geqslant h$.

Case 1. $h \geqslant 4$. Since $d \geqslant h$, $s \geqslant 3h-6 \geqslant 6$.

But we can do better for $n>8$. For $h \geqslant 5$, we have $s \geqslant 3h-6 \geqslant 9$. For $h=4$, there must be some vertex v inside the convex hull $abcd$. (Assume that $abcd$ is the clockwise order of the exterior vertices.) Let e be the intersection point of the two diagonals ac and bd of $abcd$, and define four closed triangles abe, bce, cde and dae that include all points within and on the boundary of each triangle. Since $n>5$, there is an additional vertex in some triangle, say in abe, distinct from a and b.

Start with the line collinear with cb, and rotate it counter-clockwise about the point c until it hits another vertex $x \neq a$. Using Lemma 5 with $v=x$ and $V_1=\{b\}$, we have that bx is a forced edge. Similarly, starting with da and rotating clockwise about d, we find a point $y \neq b$ so that ay is a forced edge. Note that x and y may be the same vertex, but this is not important. Note also this argument is valid when two edges of the convex hull, say bc and cd, are collinear.

Each forced edge must be in both R' and in B', as must the convex hull. Thus $d \geqslant h+2$ and $s \geqslant 2h+h+2-6=8$.

Case 2. $h=3$. We will find three forced edges as in Case 1. Let abc be the convex hull, with vertices specified in clockwise order. Triangle abc contains interior vertices. Starting with the line collinear with ba, rotate it counterclockwise about b until it hits some point $x \neq c$. By Lemma 5, ax is a forced edge. Starting with ac, rotate the line counterclockwise about a until it hits some point $y \neq b$; by Lemma 5, cy is a forced edge. Finally, do the same line rotation on cb about c so that a forced edge bz is established. Note that x, y and z need not be distinct. Each forced edge lies in both R' and B', as must the convex hull edges. Thus $d \geqslant h+3$ and $s \geqslant 2h+h+3-6=6$. Thus in both cases $s \geqslant 6$ and so the number of edges of G is at most $6n-12-s \leqslant 6n-18$. $\square$

Note that for $n > 4$, the edge-bound of $6n - 20$ is established in the proof above except in the case when the convex hull consists of 3 vertices. Except for K_4, we have no example of a doubly linear graph with more than $6n - 20$ edges.

The examples of the next result are closely related to those of Theorem 3.

Theorem 7. *There is a* (*connected*) *doubly linear graph with* n *vertices and* $6n - 20$ *edges for each* $n \geqslant 8$.

Proof. For $n = 8$, K_8 is doubly linear as seen in Fig. 2. For $n = 9$, K_9 minus two edges can be seen to be doubly linear by suitably adding a vertex in the center of Fig. 2, and so we assume $n > 9$. As in the proof of Theorem 3, first we consider $n = ks + 4$ with $k, s > 1$, and then $n = p + 4$ where p is a prime.

Suppose that $n = ks + 4$ with $k, s > 1$, $s \leqslant k$, and let $q \geqslant 2$ be an integer such that $k \leqslant qs$. Let T be the set of grid points in the rectangle

$$Q = \{(x, y) \mid 0 \leqslant x \leqslant k - 1,\ 0 \leqslant y \leqslant s - 1\},$$

and let

$$a = (-4k, -s), \qquad b = (4k, 3s), \qquad c = (8k, 3s - 10qs), \qquad d = (-k, 3s + 13qs).$$

The vertices a, b, c and d can be joined by straight edges to form a K_4 with no edge intersecting the rectangle Q; note that the edge ab passes above Q, the edge ac passes below Q since $q > 1$, and b lies inside the triangle acd.

We form two sets R and B of edges, each a linear triangulation. If R and B share e edges, then we can remove these edges from R to get R' such that R' and B share no edge and have a total of $6n - 12 - e$ edges.

Let R have the edges

- $\{t, t + (1, 0)\}$, $\{t, t + (1, 1)\}$, $\{t, t + (2, 1)\}$ for all t in T (and when the second vertex is an element of T),
- $\{a, (x, y)\}$ for $x = 0$ or $x = 1$ or $y = 0$,
- $\{b, (x, y)\}$ for $x = k - 1$ or $x = k - 2$ or $y = s - 1$,
- the edges of the K_4 formed by a, b, c and d, and
- $\{c, (k - 1, 0)\}$.

Thus, a is connected to the left two columns of vertices and to the bottom vertices of Q, and b is connected to those on the top and in the two rightmost columns. (It is a routine slope calculation that a and b can be joined to these vertices by nonintersecting straight edges.)

Let B have the edges

- $\{t, t + (0, 1)\}$, $\{t, t + (-1, 1)\}$, $\{t, t + (-1, 2)\}$ for all t in T (and when the second vertex is in T),
- $\{c, (x, y)\}$ for $x = k - 1$ or $y = 0$ or $y = 1$,
- $\{d, (x, y)\}$ for $x = 0$ or $y = s - 1$ or $y = s - 2$,
- the edges of the K_4 except for the edge ab,
- $\{a, (0, 0)\}$ and $\{b, (k - 1, s - 1)\}$.

Thus, c is connected to the bottom and to the right of the rectangle by straight edges (since $q \geqslant 2$), and d is connected to the left and to the top vertices.

Both R and B are triangulations. R and B share 5 edges of the K_4 $abcd$ and the three edges $\{a, (0, 0)\}$, $\{b, (k - 1, s - 1)\}$ and $\{c, (k - 1, 0)\}$, for a total of $e = 8$ shared edges. Thus R' and B share no edge for a total of $6n - 12 - e = 6n - 20$ edges.

For the case of $n = p + 4$ with p a prime, the same trick as in Theorem 3 works here: construct the doubly linear representation of a graph with $n - 1 = p - 1 + 4$ vertices as described above. Then add a vertex with coordinates $(0.5, 0.375)$ and in each of the triangulations R and B join it to the three vertices of the triangle within which it lies. Thus one vertex and six edges are added to the graph as required to achieve the $6n - 20$ edge bound for this case also. □

Since a subgraph of a doubly linear graph is doubly linear, we can construct a doubly linear graph with n vertices and with any desired number of edges less than $6n - 20$. To construct families of non-doubly linear graphs one can begin with a specific graph that is not doubly linear and form the union with a doubly linear graph with n vertices and any desired number of edges, at most $6n - 20$. For example, one can begin with K_9, which is not doubly linear since it has thickness three. Or one can begin with $K_{12} - F$, the complete graph on 12 vertices minus a one-factor. Let G_1 be the graph of the icosahedron, and let G_2 be the graph on the same set of vertices with vertices adjacent if they are at distance two in G_1. In fact, G_1 and G_2 are isomorphic, and their union is $K_{12} - F$, showing the latter to have thickness two. However, $K_{12} - F$ has 12 vertices and $60 = 6n - 12$ edges, and so by Theorem 6 is not doubly linear.

5. Open questions

Our upper bound on the number of edges in a doubly linear graph on n vertices is $6n - 18$, and we gave examples of doubly linear graphs with $6n - 20$ edges. Whether the upper bound is tight remains open.

Question 1. For $n > 4$, does there exist a doubly linear graph on vertices with either $6n - 18$ or $6n - 19$ edges?

The second question concerns the relationship between rectangle-visibility and doubly linear representations. An example was given in Section 2 of a doubly linear graph that is not a rectangle-visibility graph.

Question 2. Is there a rectangle-visibility graph that is not doubly linear?

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