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Thresholds for one-parameter families of affine iterated function systems*

Andrew Vince

Department of Mathematics, University of Florida, Gainesville, FL, United States of America

E-mail: avince@ufl.edu

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Abstract

This paper examines thresholds for certain properties of the attractor of a general one-parameter affine family of iterated functions systems. As the parameter increases, the iterated function system becomes less contractive, and the attractor evolves. Thresholds are studied for the following properties: the existence of an attractor, the connectivity of the attractor, and the existence of non-empty interior of the attractor. Also discussed are transition phenomena between existence and non-existence of an attractor.

Keywords: iterated function system, attractor, fractal

Mathematics Subject Classification numbers: 28A80.

(Some figures may appear in colour only in the online journal)

1. Introduction

An iterated function system (IFS) in this paper is a finite set

$$F = \{f_1,\ f_2, \ldots, f_N\}$$

of $N \geqslant 2$ affine functions from $\mathbb{R}^d$ to $\mathbb{R}^d$. A function f is *affine* if it is of the form $f(x) = Lx + a$, where L is a *non-singular* linear map ($n \times n$ real matrix) and $a \in \mathbb{R}^d$. The linear map L will be referred to as the *linear part* of f, and a as the *translational part* of f. A special case of an IFS is a *similarity IFS* for which each function is a similarity transformation.

For the collection $\mathbb{H}$ of non-empty compact subsets of $\mathbb{R}^d$, the classical Hutchinson operator $F : \mathbb{H}(\mathbb{R}^d) \to \mathbb{H}(\mathbb{R}^d)$ is given by

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$$F(K) = \bigcup_{f \in F} f(K).$$

By abuse of language, the same notation F is used for the IFS, the set of functions in the IFS, and for the Hutchinson operator; the meaning should be clear by the context. A compact set A is the *attractor* of F if

$$A = \lim_{k\to\infty} F^k(K), \tag{1.1}$$

where F^k denotes the k-fold composition; the limit is with respect to the Hausdorff metric and is independent of the set $K \in \mathbb{H}$. Note that, if it exists, the attractor is the unique compact set A such that

$$A = F(A). \tag{1.2}$$

One-parameter affine IFS families are the topic of this paper. Let $F = \{f_1, f_2, \ldots, f_N\}$ be a set of non-singular affine transformations on $\mathbb{R}^d$ and $Q = \{q_1, q_2, \ldots, q_N\}$ a set of vectors in $\mathbb{R}^d$. We consider the general family of affine IFSs

$$F_t = \{t\, f_i(x) + q_i : \quad 1 \leqslant i \leqslant N\} \tag{1.3}$$

depending on a real parameter $t \geqslant 0$. Call such a family a **one-parameter affine family**. A one-parameter family is called a **similarity family** if each function is a similarity transformation. The parameter t is directly related to the contractivity of the IFS, the nearer the parameter t is to 0, the more contractive the functions in the IFS. Example 1.1 below shows the evolution of the attractor A_t of a particular one-parameter similarity family F_t.

The goal of this paper is to examine thresholds for certain properties of the attractor A_t of F_t. In particular, we ask whether there exist numbers t_0, t_1, t_2 such that:

- F_t has an attractor for $t < t_0$, but no attractor for $t > t_0$.
- A_t is disconnected for $t < t_1$, but connected for $t > t_1$.
- A_t has empty interior for $t < t_2$, but non-empty interior for $t > t_2$.

We also examine phenomena at the point t_0 of transition between attractor and no attractor.

An attractor with non-empty interior is important in certain IFS constructions of tilings of Euclidean space (see [6, 7]). Note that, in general, there is no direct relation between whether the attractor of an IFS is connected and whether it has non-empty interior. For example, the Koch curve is connected, but has empty interior, and the attractor in example 6.1 (figure 3) is disconnected, but has non-empty interior.

Example 1.1. As an example on $\mathbb{R}^2$, consider the one-parameter similarity family $F_t = \{f_t, g_t\}$, where

$$f_t \begin{pmatrix} x \\ y \end{pmatrix} = t\, Q \begin{pmatrix} x \\ y \end{pmatrix}, \qquad g_t \begin{pmatrix} x \\ y \end{pmatrix} = t \left(.4\, Q \begin{pmatrix} x \\ y \end{pmatrix} - \frac{0.4}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and Q is the rotation by $\pi/4$:

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Figure 1 shows the attractor of F_t for increasing values of t.

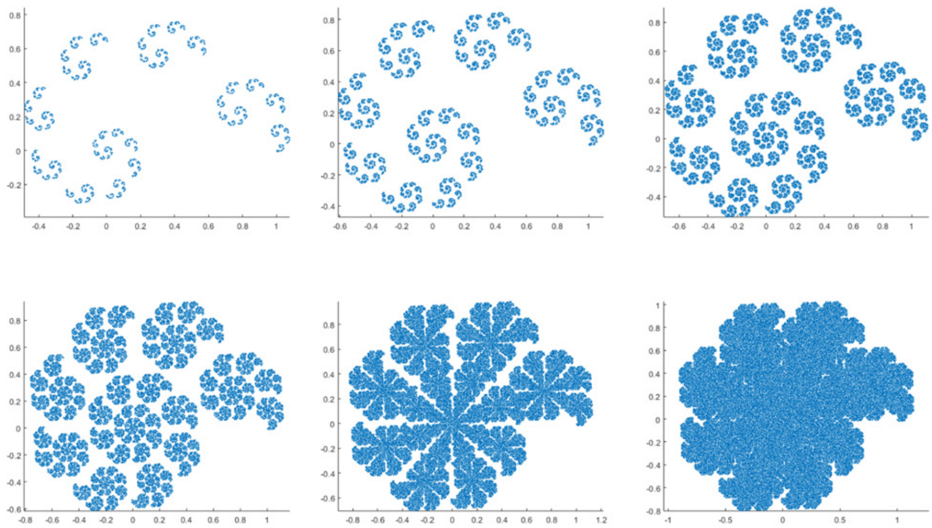

Figure 1. The attractor A_t for the one-parameter affine family F_t of example 1.1 for successive parameter values $t = 0.8, 0.85, 0.88, 0.9, 0.92, 0.94$.

2. Previous results

A particular two-dimensional one-parameter similarity family that has emerged as a topic of interest [1, 2, 4, 18, 19] is

$$\{tQ(x),\ tQ(x) + (1, 0)\},$$

where Q is a rotation about the origin. This family is often expressed in complex form, with complex parameter τ, as

$$F_\tau = \{\tau z,\ \tau z + 1\}, \tag{2.1}$$

where $\tau \in \mathbb{D} = \{z : |z| < 1\}$. It was shown already in [10] that, for the family (2.1), if $\tau \notin \mathbb{R}$ and $|\tau|$ is sufficiently close to 1, then A_τ has non-empty interior.

If A_τ denotes the attractor of F_τ, then the set

$$M := \{\tau \in \mathbb{D} : A_\tau \text{ is connected}\},$$

introduced in [3] as an analogue of the classical Mandelbrot set, is called the Mandelbrot set for F_τ. The portion of M in the first quadrant is shown in white in figure 2, this graphic due to Christoph Bandt. It is shown in [9] that M is connected and locally connected.

Call an IFS F in $\mathbb{R}^d$ *degenerate* if there exists an invariant affine subspace of dimension less than d that is common to all the functions in F; otherwise call F *non-degenerate*. Hare and Siderov [12, 13] studied the IFS $F_M := \{Mv - u, Mv + u\}$, where M is a $d \times d$ real matrix and $u \in \mathbb{R}^d$ is a vector such that $\text{span}\{M^n u : n \geqslant 0\} = \mathbb{R}^d$. They proved that if the IFS is non-degenerate and if $|\det M| \geqslant 1/2$, then the attractor A_F is connected, and if $|\det M| \geqslant 2^{-1/d}$, then A_F has non-empty interior. The latter occurs if the modulus of each eigenvalue of M is between $2^{-1/d^2}$ and 1. If the dimension $d = 2$ and the eigenvalues of M are complex, then, via a conjugacy, the IFS F_M is equivalent to the IFS F_τ of (2.1). In this case, their result states that

Figure 2. Mandelbrot set for the family (2.1), due to C Bandt.

if $1 > |\tau| > 1/2$, then A_τ is connected, and if $1 > |\tau| > 2^{-1/4} \approx 0.84$, then A_τ has non-empty interior.

In [15, 16] the authors study connectedness and disk-likeness of the attractor for a particular family of self-similar *digit tiles* (see [20]) in $\mathbb{R}^2$. The family depends on a parameter involving the digits. They provide sufficient conditions on the parameter for the attractor to be disk-like and necessary and sufficient conditions for the attractor to be connected.

3. Organization and summary of results

Concerning a threshold for the existence of an attractor of the one-parameter affine family F_t of equation (1.3), a complete answer is provided in section 4.

- There is a threshold $t_0 = 1/\rho(F)$, where $\rho(F)$ is the joint spectral radius of the linear parts of the functions in F. For all t such that $t \in [0, t_0)$, the affine IFS F_t has a unique attractor, and for all $t > t_0$ the F_t has no attractor (see corollary 4.1).

The definition of joint spectral radius is given in section 4.

At the point t_0 between the existence and non-existence of an attractor, certain transition phenomena can occur. When F_t is what we call a *bounded* family (in particular, for a bounded family there is a ball B such that $A_t \subset B$ for $t \in [0, t_0)$), *transition attractors* are defined in section 8. The *lower transition attractor* A_* is unique. The main conjecture of this paper (conjecture 8.1) is that, for a bounded family F_t, there is a unique *upper transition attractor* rather than multiple upper transition attractors. If this is the case, then the upper transition attractor is $\lim_{t \to t_0} A_t$. The conjecture is true in the one-dimensional case, and graphical evidence seems to support it in dimension two. The following results, for lower transition attractor A_* and any upper transition attractor A^*, appear in section 8.

- Although A_* and A^* may not satisfy the equation (1.1) that defines the attractor of an IFS, they do satisfy the fixed point property of equation (1.2), i.e. $F_{t_0}(A^*) = A^*$ and $F_{t_0}(A^*) = A^*$.
- It is not necessarily the case that $A_* = A^*$, but $A_* \subseteq A^*$ and $\text{conv}(A_*) = \text{conv}(A^*)$, where conv denotes the convex hull.

Some of our results hold for what we call *semi-linear* one-parameter families; some hold with the exception of *linear* or *quasi-linear* one-parameter families. The definitions and properties of these types of families appear in section 5.

Concerning a threshold for the connectivity of the attractor of a one-parameter affine family F_t, the following facts are the subject of section 6.

- If the attractor A of an affine IFS F is not connected, then A must have infinitely many components. If there are exactly two functions in F and A is not connected, then A must be totally disconnected; this, however, is not true in general for an IFS with more than two functions.
- The attractor A_t of a one-parameter affine family may be disconnected for all $t \in [0, t_0)$.
- However, for a one-parameter similarity family F_t, there exists a real number $\overline{t_1} > 0$ such that A_t is connected for all $t \in (\overline{t_1}, t_0)$ (theorem 6.2).
- The attractor A_t of any one-parameter quasi-linear family F_t is connected for all $t \in [0, t_0)$.
- However, for any affine, but not quasi-linear, one-parameter family F_t, there exists a real number $\widehat{t}_1 > 0$ such that A_t is disconnected for all $t \in [0, \widehat{t}_1)$ (theorem 6.3).
- Even if F_t is a one-parameter similarity family, there may be no single threshold t_1 such that A_t is disconnected for all $t \in (0, t_1)$ and is connected for all $t \in (t_1, t_0)$.

In section 6, however, a slightly weaker notion of connectivity, which we call *weak connectivity* (definition 6.1), is introduced that satisfies the following stronger threshold property (theorem 6.4):

- For a semi-linear, but not linear, one-parameter affine family F_t, there is a threshold $t_1 > 0$ such that A_t is weakly connected for all $t \in (t_1, t_0)$ and strongly disconnected for all $t \in [0, t_1)$.

A threshold for the appearance of non-empty interior in the attractor A_t of a one-parameter affine family F_t is the subject of section 7. The following facts are proved in that section.

- For a degenerate one-parameter affine family F_t, the attractor trivially has empty interior for all $t \in [0, t_0)$. This is even the case for some non-degenerate similarity families (example 7.1 and theorem 7.2).
- For a one-parameter affine family F_t, there is a real number $\tau > 0$ such that A_t has empty interior for all $t \in (0, \tau)$ (theorem 7.1).
- For a tame (definition 7.1) and semi-linear similarity family F_t, there is a real number $t_2 > 0$ such that A_t has empty interior for $t \in [0, t_2)$ and non-empty interior for $t \in (t_2, t_0)$. (We have no example of a semi-linear similarity IFS family that is not tame).
- Even when F_t is not tame, theorem 7.4 and corollary 7.1 provide sufficient conditions under which there exists $\tau < t_0$ such that A_t has non-empty interior for $t \in (\tau, t_0)$.

4. Threshold t_0 for the existence of an attractor

Concerning a threshold for the existence of an attractor of the one-parameter affine family, corollary 4.1 below, gives a complete answer. The terminology is as follows. An IFS F is *contractive* if f is a contraction for all $f \in F$ with respect to some metric that is Lipschitz equivalent to the standard Euclidean metric. An IFS F is *topologically attractive* if there exists a compact set $K \in \mathbb{H}$ such that $F(K) \subset K^{\circ}$, where K° denotes the interior of K. A *convex body* is a convex set with non-empty interior.

The *joint spectral radius* $\rho(F)$ of an IFS F is the joint spectral radius of the set of linear parts of the functions in F. The joint spectral radius of a set $\mathbb{L} = \{L_i, \quad i \in I\}$ of linear maps was introduced by Rota and Strang [17] and the generalized spectral radius by Daubechies and Lagarias [11]. Berger and Wang [8] proved that the two concepts coincide for, in particular, a finite set of maps. What follows is the definition of the joint spectral radius of $\mathbb{L} = \{L_1,\ L_2, \dots, L_N\}$. Let Ω_k be the set of all words $i_1 i_2 \cdots i_k$, of length k, where $i_j \in \{1, 2, \dots, N\} 1 \leqslant j \leqslant k$. For $\sigma = i_1 i_2 \cdots i_k \in \Omega_k$, define

$$L_\sigma := L_{i_1} \circ L_{i_2} \circ \cdots \circ L_{i_k}.$$

For a linear map L, let $\rho(L)$ denote the ordinary spectral radius, i.e., the maximum of the moduli of the eigenvalues of L. The *joint spectral radius* of $\mathbb{L}$ is

$$\hat{\rho} = \hat{\rho}(\mathbb{L}) := \limsup_{k\to\infty} \hat{\rho}_k^{1/k} \qquad \text{where} \qquad \hat{\rho}_k := \sup_{\sigma\in\Omega_k} \|L_\sigma\|,$$

which does not depend on the chosen matrix norm. The *generalized spectral radius* of $\mathbb{L}$ is

$$\rho = \rho(\mathbb{L}) := \limsup_{k\to\infty} \rho_k^{1/k} \qquad \text{where} \qquad \rho_k := \sup_{\sigma\in\Omega_k} \rho(L_\sigma).$$

Our theorem 4.1 below generalizes and extends a fundamental result of Hutchinson [14].

Theorem 4.1 ([5] **theorem** 4). *The following are equivalent for an affine IFS F.*

(*a*) *F has an attractor.*
(*b*) *F is contractive.*
(*c*) *F is topologically attractive with respect some convex body K.*
(*d*) *The joint spectral radius $\rho(F) < 1$.*

Corollary 4.1. *Given a one-parameter affine family F_t, an attractor A_t exists for $t < 1/\rho(F)$ and fails to exist for $t \geqslant 1/\rho(F)$. In other words, the threshold for the existence of the attractor is $t_0 = 1/\rho(F)$.*

Proof. If the linear parts of the affine functions in F are $\{B_1, B_2, \dots, B_N\}$, then by theorem 4.1 an attractor of F_t exists if $t\rho(F) = t\rho(B_1, B_2, \dots, B_N) = \rho(tB_1, tB_2, \dots, tB_N) = \rho(F_t) < 1$, i.e., if $t < 1/\rho(F)$. And an attractor of F_t fails to exist if $t\rho(F) = t\rho(B_1, B_2, \dots, B_N) = \rho(tB_1, tB_2, \dots, tB_N) = \rho(F_t) \geqslant 1$, i.e., if $t \geqslant 1/\rho(F)$. □

It is known that the joint spectral radius is NP-hard to compute or to approximate. Moreover, the question of whether $\rho < 1$ is an undecidable problem. However, for an IFS consisting of similarity transformations, the joint spectral radius is easily verified to be the maximum of the scaling ratios of the set of similarities in the IFS F.

Corollary 4.2. *Let $F_t = \{tf_i(x) + q_i : 1 \leqslant i \leqslant N\}$ be a one-parameter similarity family and $F = \{f_1, f_2, \dots, f_N\}$. If the scaling ratio of the similarities in F are $r_1, r_2, \dots, r_N$, the threshold for the existence of an attractor of F_t is $t_0 = 1/(\max_{1\leqslant i\leqslant N} r_i)$.*

The following theorem is relevant to results in section 8. Recall that convergence is with respect to the Hausdorff metric.

Theorem 4.2. *For the one-parameter affine family F_t, the function $t \mapsto A_t$ is continuous on the interval $(0, t_0)$.*

Proof. By theorem 4.1, all the functions in F are contractions with respect to a metric $d(\cdot, \cdot)$ that is Lipschitz equivalent to the standard Euclidean metric. So there are constants $c_1 > 0, c_2 > 0$ such that $c_1|x - y| \leqslant d(x, y) \leqslant c_2|x - y|$ for all $x, y \in \mathbb{R}^d$. Restrict our metric to the convex body K as defined in theorem 4.1. According to [4, theorem 11.1], it is sufficient to show that $d(f_t(x), f_{t'}(x)) \leqslant c|t-t'|$ for some constant c independent of $t, t', x \in K$, and $f \in F$. Let c_3 be a constant such that $|f(x)| \leqslant c_3$ for all $x \in K$ and all $f \in F$. Then

$$d(f_t(x), f_{t'}(x)) \leqslant c_2|f_t(x) - f_{t'}(x)| = c_2|(t-t')|\,|f(x)| \leqslant c_2c_3|t-t'|.$$

□

Example 4.1 (**$t \mapsto A_t$ is continuous on the interval** $(0, t_0)$ **but not uniformly continuous**). In general, the continuity guaranteed by theorem 4.2 is not uniform continuity. For example, for the one-dimensional one-parameter family $F_t = \{-tx + t + 1, -tx - t - 1\}$ the attractor is the closed interval

$$A_t = \left[-\frac{1+t}{1-t}, \frac{1+t}{1-t}\right],$$

whose length goes to infinity as t approaches $t_0 = 1$. Consequently continuity is not uniform. This question of when continuity is uniform arises in section 8.

Remark 4.1. Given an IFS family F_t, let $F'_t = t_0F_t$. The one-parameter affine family F'_t, as t varies between 0 and 1, is the same as the family F_t as t varies between 0 and t_0. Therefore there is no loss of generality in assuming that $t_0 = 1$.

5. Linear, quasi-linear, and semi-linear families

Proposition 5.1. *Let F_t be a one-parameter affine family. If $t \in [0, t_0)$, then every function in F_t has a unique fixed point, and that fixed point is contained in the attractor A_t.*

Proof. Let $f_t = tf(x) + q \in F_t$. A point x is a fixed point of f_t if and only if $(\frac{1}{t}I - L)x = a + \frac{q}{t}$, where L is the linear part of f and a is the translational part. Therefore f_t has a unique fixed point unless $1/t$ is an eigenvalue of L. If $1/t$ is an eigenvalue of L, however, then by the definition of the generalized spectral radius

$$1/t_0 = \rho(F) \geqslant 1/t,$$

which implies $t \geqslant t_0$, a contradiction.

If x is a fixed point of a function f of an IFS with attractor A, then $x = \lim_{n\to\infty} f^n(x) \in A$. □

Two functions $f, g : \mathbb{R}^d \to \mathbb{R}^d$ are *conjugate by a function h* if $g = h^{-1}fh$.

Definition 5.1. Let $F_t = \{f_{(i,t)}(x) := tf_i(x) + q_i : 1 \leqslant i \leqslant N\}$ be a one-parameter affine family.

- Call F_t **linear** if, for all $i = 1, 2, \ldots, N$ and for all $t \in [0, t_0)$, the function $f_{(i,t)}$ is conjugate to a linear function by a by a function h that is independent of i and t.
- Call F_t **quasi-linear** if, for all $i = 1, 2, \ldots, N$ and for all $t \in [0, t_0)$, the function $f_{(i,t)}$ is conjugate to a linear function by a by a function h_t that is independent of i.
- Call F_t **semi-linear** if, for all $i = 1, 2, \ldots, N$ and for all $t \in [0, t_0)$, the function $f_{(i,t)}$ is conjugate to a linear function by a function h_i that is independent of t.

It follows from the definitions that a linear family is necessarily quasi-linear and semi-linear. Converse statement do not hold. By propositions 5.2 and 5.3 below, the attractor A_t for a linear or quasi-linear family F_t is a single point for all $t \in [0, t_0)$. This is not the case for a semi-linear family. The following two lemmas are easily verified.

Lemma 5.1. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be an affine function, $q \in \mathbb{R}^d$, and $t > 0$. The following statements are equivalent for $g(x) = tf(x) + q$:*

(*a*) *q is a fixed point of g;*
(*b*) *$f(q) = 0$;*
(*c*) *$g(x) = tL(x - q) + q$, where L is the linear part of f.*

Lemma 5.2. *For an affine function $f = g^{-1}L_1 g$, where $g(x) = L_2(x) + a$ and L_1 and L_2 are linear, there is a linear function L_3 such that $f = h^{-1}L_3 h$, where $h(x) = x + a$.*

Proposition 5.2. *The following statements are equivalent for a one-parameter affine family $F_t = \{f_{(i,t)}(x) := tf_i(x) + q_i : 1 \leqslant i \leqslant N\}$.*

(*a*) *F_t is linear.*
(*b*) *There is a point q such that $q = q_i$ for all $i = 1, 2, \ldots, N$, and q is the unique fixed point of $f_{(i,t)}$ for all i and all $t \in [0, t_0)$.*

Moreover, for a linear family the attractor A_t of F_t is the single point q.

Proof. (a) $\Rightarrow$ (b) Assume that $L = h^{-1}f_{(i,t)}h$, where L is linear. By lemma 5.2, we can assume, without loss of generality, that h is a translation. By proposition 5.1, each function in F_t, $t \in [0, t_0)$, has a unique fixed point; call it $q_{(i,t)}$. The unique fixed point of the linear map $h^{-1}f_{(i,t)}h$ is the origin; therefore $h(x) = x + q_{(i,t)}$. By the definition of linear, $q_{(i,t)}$ is independent of i and t; call it q. Now $q = tf_i(q) + q_i$ for all $t \in [0, t_0)$, which implies that $f_i(q) = 0$ and hence $q = q_i$ for all i.

(b) $\Rightarrow$ (a) Let $h(x) = x + q$, and let L_i denote the linear part of f_i. Then $(h^{-1}f_{(i,t)}h)(x) = (h^{-1}f_{(i,t)})(x + q) = h^{-1}(tL_i((x + q) - q) + q) = h^{-1}(tL_i(x) + q) = tL_i(x)$.

The last statement in the proposition follows from the fact that the attractor of an IFS, all of whose functions are linear, is a single point, the origin. □

Proposition 5.3. *The following statements are equivalent for a one-parameter affine family $F_t = \{f_{(i,t)}(x) := tf_i(x) + q_i : 1 \leqslant i \leqslant N\}$.*

(*a*) *F_t is quasi-linear.*
(*b*) *There are points $q_t, t \in [0, t_0)$, such that q_t is the unique fixed point of $f_{(i,t)}$ for all $i = 1, 2, \ldots, N$.*
(*c*) *The attractor A_t of F_t is a single point for all $t \in [0, t_0)$.*

Proof. The equivalence of the first two statements, as well as (a) $\Rightarrow$ (c), can be shown exactly as was done in the proof of proposition 5.2.

(c) $\Rightarrow$ (a) Assume that F_t is not quasi-linear. Then, for some $t \in [0, t_0)$ and some $i \neq j$, the fixed points of $f_{(i,t)}$ and $f_{(j,t)}$ must be different. By proposition 5.1, the attractor A_t of F_t contains all fixed points of the functions in F_t. Therefore, the attractor A_t must contain at least two points. □

Example 5.1 (**A quasi-linear family that is not linear**). On $\mathbb{R}^2$, let

$$F_t = \left\{ f_t\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} x \\ y+1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad g_t\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} y+1 \\ -x+2y+2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

By proposition 5.2, F_t is not a linear family because $q = (1, 0)$ is not a fixed point of f_t (nor of g_t). Nevertheless, a straightforward calculation shows that $\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$ is a fixed point of both f_t and g_t for all $t \in [0, t_0) = [0, 1)$.

In fact, examples of quasi-linear families that are not linear, like the one in example 5.1, can be constructed, in general, as follows. Let W be any proper subspace of $\mathbb{R}^n$. Let $L_i, i = 1, 2, \ldots, N$, be linear maps that are equal when restricted to W, i.e., $L_i|_W = L_j|_W$ for all i, j, but $L_i \neq L_j$ for at least one pair $i \neq j$. Let w_0 be any point in W and a any point not in W. Let q be any point such that $q - a \in W$. Finally, let $f_i(x) = L_i(x - a) + w_0$ and $F_t = \{tf_i(x) + q : i = 1, 2, \ldots, N\}$. To show that, for a given t, all the functions in F_t have the same fixed point, first notice that, for all $w \in W$ and all i, we have

$$\begin{aligned} t f_i(a + w) + q &= t(L_i(a + w - a) + w_0) + q = tL_i(w) + tw_0 + a + w_1 \\ &= a + (tL_i(w) + w_2) \in a + W, \end{aligned}$$

for some $w_1 \in W$ and $w_2 = tw_0 + w_1$. Since $L_i(w) = L_j(w)$ for all i, j, also $tf_i(a + w) + q = tf_j(a + w) + q$ for all i, j and for all t. Using the notation $f_{i,t}(x) = tf_i(x) + q$, we have that the affine subspace $a + W$ is invariant under $f_{i,t}$ and that $f_{i,t}|_{a+W} = f_{j,t}|_{a+W}$ for all i, j. Therefore for $t < t_0$, all the functions $f_{i,t}$ in F_t have the same fixed point on the affine subspace $a + W$.

Proposition 5.4. *The following statements are equivalent for a one-parameter family* $F_t = \{f_{(i,t)}(x) := tf_i(x) + q_i : 1 \leqslant i \leqslant N\}$.

(*a*) F_t *is semi-linear.*

(*b*) *The point* $q_i, i = 1, 2, \ldots, N$, *is the unique fixed point of* $f_{(i,t)}$ *for all* $t \in [0, t_0)$.

Moreover, q_i *is also a fixed point of* $f_{(i,t_0)}$ *for all i.*

Proof. The proof of the equivalence of statements (a) and (b) is the same as for proposition 5.2. That q_i is also a fixed point of $f_{(i,t_0)}$ follows from lemma 5.1; $f_i(q_i) = 0$ implies $f_{(i,t_0)}(q) := t_0\, f_i(q) + q_i = q_i$. □

Example 5.2 (**A semi-linear family that is not quasi-linear**). There are many such examples. For example, the one-parameter similarity family of example 1.1 is semi-linear, but not quasi-linear.

6. Threshold for connectivity of the attractor

Theorem 6.1. *The attractor of an affine IFS consisting of two functions cannot have a finite number, greater than one, of connected components.*

Proof. Let K be a connected compact set such that $F(K) \subset K^o$ as guaranteed by theorem 4.1. If $F^n(K)$ is connected for all n, then so is A_F, being the intersection of compact connected sets. Otherwise there is a first m such that $F^m(K)$ is not connected. By substituting $F^m(K)$ for K, it can be assumed without loss of generality that $m = 1$. Let f_1 and f_2 be the two functions in the IFS. Since $F(K)$ is not connected, $f_1(K)$ and $f_2(K)$ are disjoint. This implies that $f_1 \circ f_1(K), f_1 \circ f_2(K), f_2 \circ f_1(K), f_2 \circ f_2(K)$ all lie in four distinct components of $F^{(2)}(K)$, with the first two lying in $f_1(K)$ and the last two in $f_2(K)$. Continuing in this way, the number of components of $F^n(K)$ goes to infinity with n. Therefore A_F has infinitely many components. □

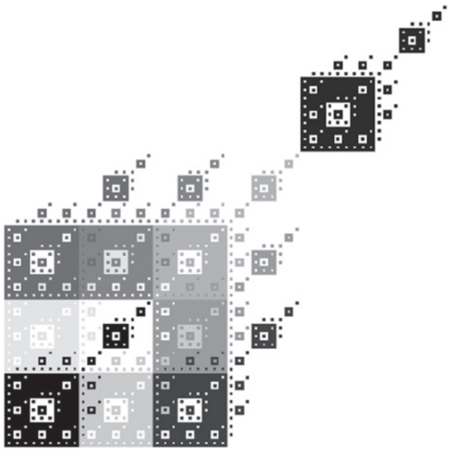

Figure 3. An attractor with infinitely many connected components that is not totally disconnected.

Remark 6.1. In general, it is possible for an IFS to have a finite number, greater than one, of connected components. For example, the one-dimensional IFS whose functions are the following has attractor $[0, 15/32] \cup [17/32, 1]$.

$$f_1(x) = \frac{3}{8}x \qquad f_2(x) = \frac{3}{8}x + \frac{3}{32} \qquad f_3(x) = \frac{3}{8}x + \frac{17}{32} \qquad f_4(x) = \frac{3}{8}x + \frac{5}{8}$$

Example 6.1 (**An attractor with infinitely many connected components that is not totally disconnected**). Such an example is shown in figure 3. This is the attractor of an affine IFS consisting of nine functions indicated by the different grey shades. On the other hand, the attractor A of any contractive IFS consisting of exactly two functions is know to be either connected or totally disconnected [4].

Example 6.2 (**A one-parameter affine family F_t such that A_t is disconnected for all $t \in [0, t_0)$**). Consider the one-parameter affine family

$$F_t = \left\{ f_t\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},\ g_t\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \left(1 - \frac{t}{10}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right.$$

By corollary 4.1, if $t \geqslant 1$, then F_t has no attractor. And if $t < 1$, then the attractor of F_t is disconnected. This can been seen because, if K is the unit square with vertices $(0,0), (1,0), (0,1), (1,1)$, then for all $t \in [0,1)$, we have $F_t(K) \subset K, (0,0) \in f_t(K), (1,1) \in g_t(K)$. However, $f_t(K) \cap g_t(K) = \emptyset$.

Example 6.3 (**A one-parameter similarity family with no single threshold for connectivity of the attractor**). An examination of the Mandelbrot set for the family given in (2.1), in particular the region of the 'ram's horn' about a third of the way up from the horizontal in figure 2, shows that, even for the simple one-parameter family of the form (2.1), there can exist $0 < t < t' < t'' < 1$ such that A_t and $A_{t''}$ are connected, but $A_{t'}$ is not connected. Thus there can, in general, exist no single threshold for connectivity for a one-parameter affine family, even a family consisting of two similarities.

Because of example 6.2, it is assumed in theorem 6.2 below that the functions in the IFS are similarities. Because of example 6.3, we prove the existence of two thresholds τ and τ' such that, for t greater then τ, the attractor A_t of a one-parameter similarity family is connected, and for t less than τ', the attractor A_t of an affine, but not linear, family is disconnected. A slightly weaker notion of connectivity, called *weak connectivity*, is then introduced, for which a single threshold is proved; see theorem 6.4.

Theorem 6.2. *Given a one-parameter similarity family F_t, there exists a real number τ with $\tau \in [0, t_0)$ such that the attractor A_t is connected for all $t \in (\tau, t_0)$.*

Proof. By remark 4.1, there is no loss of generality in assuming that $t_0 = 1$. For a similarity f, let s_f denote the scaling ratio of f. Let g be a function in F with the maximum scaling ratio, which is 1 since $t_0 = 1$, and let h be a function in F with the minimum scaling ratio. Let $\tau = 1/(s_g + s_h) < 1$. Then for t such that $1 > t > \tau$ we have

$$s_{g_t} + s_{h_t} = t(s_g + s_h) > \tau(s_g + s_h) = 1.$$

For t such that $1 > t > \tau$, we claim that F_t has a connected attractor. Assume, by way of contradiction, that this is not the case. This implies that either (a) F_t has no attractor or (b) F_t has an attractor A_t, but A_t is not connected. Case 1 is not possible by corollary 4.1 since $\rho(F_t) = t\rho(F) = t < 1$. Concerning case 2, and referring to theorem 4.1, let K be a compact convex set containing the attractor A_t such that $F_t(K) \subset K$. Letting $K_j := F_t^j(K)$, this implies that $K_{j+1} \subset K_j$ for all j. Since $A_t = \lim_{j\to\infty} K_j$ and the intersection of a nested sequence of compact connected sets is connected, there must exist an integer j such that K_j is connected, but $K_{j+1} = \bigcup_{f\in F} f_t(K_j)$ is not. Therefore, with functions g and h as defined above, there must exist an $f \in F$ such that $g_t(K_j) \cap f_t(K_j) = \emptyset$. Now, with vol denoting volume,

$$\text{vol}\, g_t(K_j) + \text{vol}\, f_t(K_j) = s_{g_t}\, \text{vol}\, K_{j+1} + s_{f_t}\, \text{vol}\, K_{j+1} \geqslant (s_{g_t} + s_{h_t})\, \text{vol}\, K_{j+1} > \text{vol}\, K_{j+1}.$$

But this is impossible since $g_t(K_j) \cap h_t(K_j) = \emptyset$ and $K_{j+1} \subset K_j$. □

Example 6.4 **(An affine family F_t for which the attractor A_t is connected for all $t \in [0, t_0)$).** Any quasi-linear one-parameter family is such an example (see example 5.1 and the general construction following it). For a quasi-linear family the attractor A_t is, by proposition 5.3, a single point for all $t \in [0, t_0)$, hence connected.

Theorem 6.3. *If $F_t = \{tf_i(x) + q_i : 1 \leqslant i \leqslant N\}$ is a one-parameter affine, but not quasi-linear, family, then there is a real number $\tau > 0$ such that A_t is disconnected for all $t \in [0, \tau)$.*

Proof. First assume that not all the q_i are equal. If B is a ball containing the origin and all the q_i in its interior, and if t is sufficiently small, say $t < \alpha_1$, then $(f_i(B) + q_i) \cap (f_j(B) + q_j) = \emptyset$ for all i, j such that $q_i \neq q_j$ and $f_i(B) + q_i \subset B$ for all i. Therefore, for $t < \alpha_1$, we have $F_t(B) \subset B$ and $F_t(B)$ is disconnected. This is sufficient to insure that A_t is disconnected.

From the paragraph above, we may assume that all the q_i are equal, that the one-parameter family has the form $F_t = \{tf_i(x) + q : 1 \leqslant i \leqslant N\}$. Let B be a ball containing q in its interior. If t is sufficiently small, say $t < \alpha_2 \leqslant \alpha_1$, then $tf_i(B) + q_i \subset B$ for all i; hence, for $t < \alpha_2$, we have $F_t(B) \subset B$. Since $A_t = \bigcap_{n\geqslant 0} F_t^{(n)}(B)$, to prove that A_t is disconnected, it is sufficient to show that $F_t^{(n)}(B)$ is disconnected for some integer n. Assume, by way of contradiction, that $X_n := F_t^{(n)}(B)$ is connected for all n. For any compact set $X \subset \mathbb{R}^d$, denote its diameter by $D(X)$, i.e., the largest distance between any two points of X. For each i, there is a constant c_i such that, for any $X \subset \mathbb{R}^d$,

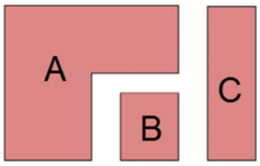

Figure 4. The sets $A \cup B$ and C are the weak components.

it is the case that $D(f_i(X)) \leqslant c_i D(X)$. Therefore there is a constant c (depending on N) such that, if $\bigcup_{i=1}^N f_i(X)$ is connected, then $D(\bigcup_{i=1}^N f_i(X)) \leqslant c\, D(X)$, and therefore $D(\bigcup_{i=1}^N t\, f_i(X)) \leqslant ct\, D(X)$. Under our assumption that X_n is connected for all n, we now have $D(X_{n+1}) \leqslant ctD(X_n)$, which implies that $D(X_n) \leqslant (ct)^n D(B)$. For $t < \alpha_3 := \min\{c, \alpha_2\}$, this implies that the diameter of $A_t = \lim_{n\to\infty} X_n$ is 0; hence A_t is a single point, say x_t. Thus x_t is the fixed point of the function $tf_i(x) + q$ for all i, which contradicts that F_t is not quasi-linearity. □

Definition 6.1. A hyperplane H in $\mathbb{R}^d$ *separates* a compact set S if $S \cap H = \emptyset$ but S has non-empty intersection with both half-spaces determined by H. A compact subset S of $\mathbb{R}^d$ is *strongly disconnected* if there exists a hyperplane that separates S. A compact set S is *weakly connected* if there is no such hyperplane. For a compact set S, call a maximal weakly connected subset of S a *weak component* of S, and call the convex hull of a weak component a *convex component* of S. Figure 4 shows the weak components of a set.

Proposition 6.1. *The set of weak components of a compact set S form a partition of S.*

Proof. Assume, by way of contradiction, that C_1 and C_2 are weak components of S with non-empty intersection. By the definition of a weak component, $C_1 \cup C_2$ is not weakly connected and hence there is a hyperplane H that separates $C_1 \cup C_2$. Since C_1 and C_2 are weakly connected, C_1 lies in one of the two halfspaces determines by H as does C_2. If C_1 and C_2 are contained in the same half-space, then H does not separate $C_1 \cup C_2$, a contradiction. If C_1 and C_2 are contained different half-spaces, then C_1 and C_2 have empty intersection, again a contradiction. □

The proofs of the following lemmas are straightforward. For a compact set C denote its convex hull by conv$C$.

Lemma 6.1. *Given an affine function f, a compact set C can be separated by a hyperplane if and only if f(C) can be separated by a hyperplane.*

Lemma 6.2. *If K is a compact set, $\{C_i\}$ a finite collection of compact sets, and f is an affine function, then*

(*a*) $f(\text{conv}\, K) \subseteq \text{conv}\, f(K)$
(*b*) $\bigcup_i \text{conv}\, C_i \subseteq \text{conv}\, (\bigcup_i C_i)$.

Lemma 6.3. *If K is compact set and $\widehat{K}$ denotes the union of the convex components of K, then*

(*a*) $\widehat{\widehat{K}} = \widehat{K}$,
(*b*) $\widehat{K}$ *is strongly disconnected if and only if K is strongly disconnected.*

Theorem 6.4. *Let F_t be a semi-linear, but not linear, affine one-parameter family. There is a number $\tau \in (0, t_0]$ such that the attractor A_t is weakly connected for $t \in (\tau, t_0)$ and strongly disconnected for $t \in (0, \tau)$. Moreover, $\tau < t_0$ if the functions in F_t are similarities.*

Proof. Let $F_t = \{tf_i(x) + q_i : 1 \leqslant i \leqslant N\}$. Theorem 4.1 guarantees that, for $t \in [0, t_0)$, there is a convex body K (depending on t) such that $F_t(K) \subset K^o$. That F_t is semi-linear implies, by proposition 5.4, that the fixed point of $tf_i(x) + q_i$ is q_i for all $t \in [0, t_0)$. That F_t is not linear implies that not all the q_i are equal. Let H be a hyperplane that separates the set $\{q_i : i = 1, 2, \ldots, N\}$ of fixed points. If t is sufficiently small, then the set $\{tf_i(K) + q_i : i = 1, 2, \ldots, N\}$ is also separated by H. Since $K \supset F_t(K) \supset F_t^2(K) \supset \cdots$, the attractor A_t of F_t is also separated by H. Therefore A_t is strongly disconnected for t sufficiently small. Let τ be the supremum of those t such that A_t is strongly disconnected.

Since A_t is weakly connected for $t \in (\tau, t_0)$, it only remains to show that A_t is strongly disconnected for $t \in (0, \tau)$. For this it is sufficient to show the following: if $A_{t'}$ is strongly disconnected and $t < t'$, then A_t is also strongly disconnected. Let K be a convex body such that $F_{t'}(K) \subset K^o$. Without loss of generality we may assume that K contains the origin. If not, then consider the conjugate semi-linear affine one-parameter family $\widehat{F}_t$ defined as follows. Let p be a point on the interior of K and h the translation that takes p to 0. Using the notation $f_{i,t}(x) := tf_i(x) + q_i$, let $\widehat{F}_t = \{h f_{i,t} h^{-1} : 1 \leqslant i \leqslant N\} = \{t\, f_i(x - p) + (q_i + p) : 1 \leqslant i \leqslant N\}$. It is easily checked that, with $0 \in \widehat{K} := h(K)$, we have $F_{t'}(\widehat{K}) \subset \widehat{K}^o$, and the relevant connectivity properties of the attractors are the same for F_t and $\widehat{F}_t$.

Now let $K_n = F_{t'}^n(K)$ and let $C_n = F_t^n(K)$. Thus, with respect to the Hausdorff metric, $K_n \to A_{t'}$ and $C_n \to A_t$. Note that the fixed points $q_i, i = 1, 2, \ldots, N$, all lie in K. Because K is convex, if L is any linear map, then $tL(K) \subset t'L(K)$ if $t < t'$. It then follows from the definition of semi-linear that $tf_i(K) + q_i \subset t'f_i(K) + q_i$. Since $C_0 = K$, we have $C_1 = F_t(C_0) = F_t(K) = \bigcup_{i=1}^N t\, f_i(K) + q_i \subset \bigcup_{i=1}^N t'\, f_i(K) + q_i = F_{t'}(K) = K_1 \subset K = C_0$, which implies inductively that $C_0 \supset C_1 \supset C_2 \supset C_3 \supset \cdots$. Since $A_{t'}$ is assumed strongly disconnected, there is a least natural number n such that K_n is strongly disconnected. We claim that C_n is also strongly disconnected, which would imply that A_t is strongly disconnected, thus proving the theorem.

Denote by $\widehat{K_n}$ and $\widehat{C_n}$ the union of the convex components of K_n and C_n respectively. To simplify notation, let $f_{i,t}(x) := tf_i(x) + q_i$. To prove the claim, first note that $\widehat{C}_j$ and $\widehat{K}_j$ are connected for $j = 1, 2, \ldots, n-1$. For these values of j, by lemma 6.1, $f_{i,t'}(K_j)$ and $f_{i,t}(C_j)$ are weakly connected for all i. We have previously in this proof shown that $C_1 \subset K_1$, hence $\widehat{C}_1 \subset \widehat{K}_1$. Proceeding inductively, if $\widehat{C}_j \subseteq \widehat{K}_j$, then

$$C_{j+1} \subseteq \bigcup_{i=1}^N f_{i,t}(\widehat{C}_j) \subseteq \bigcup_{i=1}^N f_{i,t}(\widehat{K}_j) \subseteq \bigcup_{i=1}^N f_{i,t'}(\widehat{K}_j) \subseteq \widehat{K_{j+1}}.$$

The last inclusion above is true by lemma 6.2 because $f_{i,t'}(K_j) \subseteq K_{j+1}$ implies that $f_{i,t'}(\widehat{K}_j) = \widehat{f_{i,t'}(K_j)} \subset \widehat{K_{j+1}}$ for all i. The second to last inclusion is true because K_j contains the fixed points of all i and, using statement (c) of lemma 5.1, as in the paragraph above, $f_{i,t}(K_j) \subseteq f_{i,t'}(K_j)$ for all i. From $C_{j+1} \subseteq \widehat{K_{j+1}}$ it follows from lemma 6.3 that $\widehat{C_{j+1}} \subseteq \widehat{\widehat{K_{j+1}}} = \widehat{K_{j+1}}$. Applying this inclusion to the case $j = n-1$ gives $\widehat{C_n} \subseteq \widehat{K_n}$. Since K_n is strongly disconnected, so is $\widehat{K_n}$ by lemma 6.3. But because $\widehat{C_{n-1}}$ and $\widehat{K_{n-1}}$ are both connected and $\widehat{C_{n-1}} \subseteq \widehat{K_{n-1}}$, each connected component of $\widehat{K_n}$ contains a connected component of $\widehat{C_n}$. Therefore $\widehat{K_n}$ strongly disconnected implies that $\widehat{C_n}$ is also strongly disconnected, which is the result required.

That $\tau < t_0$ for a similarity family follows from theorem 6.2, which states that there is a $\tau' < t_0$ such that A_t is connected, and hence weakly connected, for $t \in (\tau', t_0)$. Therefore $\tau < t_0$. □

7. Threshold for the appearance of an attractor with non-empty interior

Lemma 7.1. *Let F be an affine IFS and $G \subset F$. If F has an attractor, then so does G, and $A_G \subseteq A_F$. In particular, if A_G has non-empty interior, then A_F also has non-empty interior.*

Proof. If F has an attractor, then by part (c) of theorem 4.1 G also has an attractor. Moreover, since $G^n(A_F) \subseteq F^n(A_F) \subseteq A_F$ for all n, we have $A_G = \lim_{n\to\infty} G^n(A_F) \subseteq A_F$. □

Theorem 7.1. *If $F_t = \{tf_i(x) + q_i : 1 \leqslant i \leqslant N\}$ is a one-parameter affine family, then there is a real number $\tau > 0$ such that A_t has empty interior for $t \in (0, \tau)$.*

Proof. By lemma 7.1 it suffices to prove the theorem when F_t consists of two functions $f_t(x) = tf(x) + p$ and $g_t = tg(x) + q$. Let $\mu(B)$ denote the Lebesgue measure of a compact set B. Then $\mu(f_t(B)) = t^d \det(L_f)\mu(B)$, where L_f is the linear part of f; similarly for g_t. Since $A_t = f_t(A_t) \cup g_t(A_t)$, we have $\mu(A_t) \leqslant \mu(f_t(A_t)) + \mu(g_t(A_t)) = t^d(\det(L_f) + \det(L_g))\mu(A_t)$. If A_t has non-empty interior, then $\mu(A_t) > 0$, which implies that $t^d(\det(L_f) + \det(L_g)) > 1$. This is not possible if t is sufficiently small. □

It is not hard to construct affine families F_t for which there is a proper affine subspace $W \subsetneq \mathbb{R}^d$ that is invariant under F_t for all $t \in [0, t_0)$. For such a family the attractor satisfies $A_t \subset W$, which implies, trivially, that the interior of A_t is empty. Call a family F_t with such an invariant proper affine subspace **degenerate**, otherwise **non-degenerate**.

Example 7.1 (**An non-degenerate one-parameter affine family F_t such that A_t has empty interior for all $t \in [0, t_0)$**). Consider the one-parameter similarity IFS family

$$F_t = \left\{ t f_1 \begin{pmatrix} x \\ y \end{pmatrix},\ t f_2 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \qquad \text{where}$$
$$f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad f_2 \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} x \\ y \end{pmatrix} - \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.1}$$

and $0 < \alpha < 1/3$. Note that the threshold t_0 for the existence of an attractor is 1 in this example.

Theorem 7.2. *The attractor A_t of the family F_t of equation (7.1) has empty interior for all $t \in [0, t_0)$.*

Proof. It is easily verified that $F_t(B_t) \subset B_t$, where B_t is the compact set consisting of four solid quadrilaterals depicted in figure 5. Denote by g_1 and g_2 the two functions in F_t. Note that $g_1(B_t) \subset B_t$ and $g_2(B_t)$, shown in red, is contained in the disk of radius $t\alpha$ centred at $(1 - t\alpha, 0)$.

By way of contradiction, assume that A_t has non-empty interior. Then there is a ball contained in the interior, and hence there is an arc of a circle centred at the origin contained in A_t. Let γ be the arc of greatest length contained in A_t. Since g_1 and g_2 are similarities, there must exist two circular arcs γ_1 and γ_2 in A_t such that $\gamma \subseteq g_1(\gamma_1) \cup g_2(\gamma_2)$. Since g_1 and g_2 are contractions, the radii r_1, r_2 of the circles corresponding to γ_1 and γ_2 must be greater than the radius r of the circle corresponding to γ. Either $\gamma \subseteq g_1(\gamma_1)$ or $\gamma_2 \neq \emptyset$. If $\gamma_2 \neq \emptyset$, then (a) γ is contained in the rightmost quadrilateral of B_t and (b) by the paragraph above $r < r_2 \leqslant t\alpha$, and (a) for γ to be in the range of g_2 it must be the case that $r > 1 - 2t\alpha$. But $t\alpha > 1 - 2t\alpha$ implies that $\alpha > 1/3$, a contradiction. Therefore $\gamma \subseteq g_1(\gamma_1)$. In this case γ is the image of an arc in A_t of length greater than that of γ, a contradiction. □

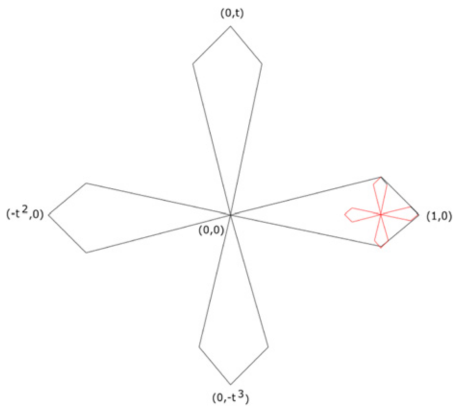

Figure 5. The set B_t in the proof of theorem 7.2.

Lemma 7.2. *If the attractor A of an affine IFS has non-empty interior, then A is the closure of its interior.*

Proof. By definition, each function in the IFS F has non-singular linear part. Therefore each function in F takes the interior of A into the interior of A. If a point x lies in the closure of the interior of A, call it $\overline{A}$, then $f(x)$ must lie in $\overline{A}$. Therefore the Hutchinson operator F takes $\overline{A}$ into $\overline{A}$, which implies that $A = \lim_{n\to\infty} F^n(\overline{A}) \subseteq \overline{A} \subseteq A$. □

Definition 7.1. By a *cone* in $\mathbb{R}^d$ in this paper, we mean a right circular non-degenerate (not a line segment) cone. For a compact set K with non-empty interior, define a boundary point x to be a *cusp* of K if there is no cone with apex at x contained in K. For an affine IFS F that has an attractor A, theorem 4.1 implies that each functions in F has a unique fixed point that is contained in A. Let Z denote the set of fixed points of the functions in F. Call F **tame** if not all points in Z are cusps of A. In particular, if there is even one point of Z that lies in the interior of A, then F is tame. Call a one-parameter affine family F_t **tame** if F_t is tame for all t such that the attractor A_t has non-empty interior.

Remark 7.1. We have no example of an IFS that has an attractor with non-empty interior and is not tame.

Theorem 7.3. *Let F_t be a tame, semi-linear one-parameter similarity family. Then there is a real number $t_2 > 0$ such that A_t has empty interior for $t \in [0, t_2)$ and non-empty interior for $t \in (t_2, t_0)$.*

Proof. If A_t has empty interior for all $t \in [0, t_0)$ (which is possible by example 7.1 and theorem 7.2), then $t_2 = t_0$ satisfies the statement of the theorem.

So we may assume that there is a t such that A_t has non-empty interior. Let t_2 be the greatest lower bound of those t such that the interior of A_t is non-empty. It is now sufficient to show that, if $0 < t' < t'' < t_0$, and $A_{t'}$ has non-empty interior, then $A_{t''}$ also has non-empty interior. Let x be the fixed point of a function $f_t(x) = tf(x) + q \in F_t$ such that x is not a cusp of A_t. Then there is a cone Y centred at x that is contained in A_t. The cone Y can be chosen small enough so that, by the definition of an attractor, $Y \subset f_{t'}(Y)$. Because the family is semi-linear and f_t is a similarity with fixed point x for all $t < t_0$, we have $Y \subset f_{t'}(Y) \subset f_{t''}(Y) \subset F_{t''}(Y)$. This implies that $F_{t''}(Y) \subset F^2_{t''}(Y)$. Proceeding inductively, $Y \subset F_{t''}(Y) \subset F^2_{t''}(Y) \subset F^3_{t''}(Y) \cdots$.

Because $A_{t''} = \lim_{n\to\infty} F^n_{t''}(Y)$, we have $Y \subset A_{t'}$. Thus $A_{t''}$ has non-empty interior since it contains the (non-empty) interior of cone Y. □

Lemma 7.3. *Let A be the attractor of an IFS F. If there is a ball $B \subseteq \mathbb{R}^d$ and an integer n_0 such that $B \subseteq F^{n_0}(B)$, then A has non-empty interior.*

Proof. If n_0 is such that $B \subseteq F^{n_0}(B)$, then inductively $B \subseteq F^{mn_0}(B)$ for all positive integers m. Since $A = \lim_{n\to\infty} F^n(B)$, it must be the case that $B \subseteq A$. □

Let $F_t = \{tf_i(x) + q_i : 1 \leqslant i \leqslant N\}$ be a semi-linear one-parameter similarity family, and let $F = \{f_1, f_2, \ldots, f_N\}$. Let ρ be the maximum scaling ratio of the similarities in F, and denote by $\widehat{F} = \{\widehat{f}_1, \ \widehat{f}_2, \ldots, \widehat{f}_{\widehat{N}}\}$ the subset of F consisting of those similarities in F with scaling ratio ρ. Let

$$\widehat{F}_t = \{t\,\widehat{f}_i(x) + \hat{q}_i \,:\, i = 1,\ 2, \ldots, \widehat{N}\}$$

denote the corresponding subset of F_t. By proposition 5.4, each function $t_0\widehat{f}_i(x) + \widehat{q}_i \in \widehat{F_{t_0}}$ is an isometry with fixed point $\widehat{q}_i$.

Theorem 7.4. *Let $F_t = \{tf_i(x) + q_i : 1 \leqslant i \leqslant N\}$ be a semi-linear, but not linear, one-parameter similarity family. With notation as above, let H_t be any subset of $\widehat{F}_t$ having the property that the $\widehat{q}_i$ are all equal, i.e., H_t has the form $H_t = \{th_i(x) + q_0 : 1 \leqslant i \leqslant k\}$ for some q_0. If the orbit of a point on the unit sphere centred at q_0 under the action of the set H_{t_0} of isometries is dense on this unit sphere, then there is a $\tau < t_0$ such that the attractor A_t of F_t has non-empty interior for all $t \in (\tau, t_0)$.*

Proof. Referring to remark 4.1, we assume without loss of generality that $t_0 = 1$, which implies that the maximum scaling ratio of among the linear parts of the functions in F_t is 1. We also assume, without loss of generality, that q_0 is the origin. Since F_t is not linear, not all functions in F_t have the same fixed point. Let $g_t \in F_t$ be such that the fixed point, call it p, is not the origin 0. Recall that, by proposition 5.4, the fixed point of g_t is independent of $t \in [0, 1)$ and $g_t(0) = p$. Denote the similarity ratio of g_1 by $0 < s < 1$. Choose $\epsilon > 0$ and $\pi/2 > \theta > 0$ such that

$$2(1-\epsilon)\cos\ \theta > 1 + (1-\epsilon)^2(1-s^2). \tag{7.2}$$

Note that this is possible if ϵ and θ are chosen sufficiently small. Let $\mathcal{C}$ be a finite set of infinite circular cones of central angle θ centred at the origin, such that the union of the cones in $\mathcal{C}$ is $\mathbb{R}^d$.

Let $r = |p|$, and let B_r be the ball of radius r centred at 0. Since the orbit under H_1 of a point on the unit sphere centred at 0 is dense on this unit sphere, there is an integer M such that every cone in $\mathcal{C}$ contains a point of $\bigcup_{k=0}^{M} H_1^k(p)$. Let $\tau < 1$ in the statement of the theorem be the positive real that satisfies $\tau^{M+1} = 1 - \epsilon$. Let t be such that $1 > t > \tau$, in particular $t^{M+1} > 1 - \epsilon$. By lemma 7.3, to prove the theorem it is sufficient to show that $B_r \subseteq F_t^{M+1}(B_r)$.

For any $h_t \in H_t$, note that $h_t^{M+1}(B_r)$ is a ball of radius $rt^{M+1} > r(1-\epsilon)$. Therefore $B_{r(1-\epsilon)} \subset h_t^{M+1}(B_r) \subseteq F_t^{M+1}(B_r)$. It only remains to show that the shell S, the region between the concentric spheres of radius r and radius $r(1-\epsilon)$, is covered by $F_t^{(M+1)}(B_r)$.

For every k such that $0 \leqslant k \leqslant M$, the set $H_t^k g_t H_t^{M-k}(B_r)$ is the union of balls of radius $rst^M > rst^{M+1} > rs(1-\epsilon)$. Denote this set of balls by $\mathcal{B}$. The set of centres of these balls lies in the set $H_t^k g_t H_t^{M-k}(0) = H_t^k g_t(0) = H_t^k(p)$. Therefore, the distance of each centre from the origin is $t^k r$ which satisfies $r \geqslant t^k r \geqslant t^{M+1} > r(1-\epsilon)$. This implies that the centres of all balls in $\mathcal{B}$ lie in the shell S. Also, since each cone in $\mathcal{C}$ contains a point of $\bigcup_{k=0}^{M} H_1^k(p)$, each cone $C \in \mathcal{C}$ contains a centre of a ball $B \in \mathcal{B}$. Since $H_t^k g_t H_t^{M-k}(B_r) \subseteq F_t^{M+1}(B_r)$, it only remains to show

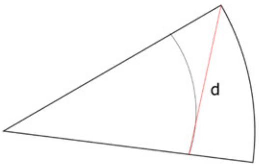

Figure 6. The cone in the proof of theorem 7.4.

that $S \cap C \subset B$. This is insured if, in figure 6, the radius of B is greater than the length d. Using the cosine law, we require

$$r^2 s^2 (1-\epsilon)^2 > d^2 = r^2 + r^2(1-\epsilon)^2 - 2r^2(1-\epsilon)\cos\ \theta,$$

which follows from equation (7.2). □

Corollary 7.1. *Let F_t be a semi-linear, but not linear, similarity family in $\mathbb{R}^2$. If there is a function in F_t whose linear part has maximum scaling ratio among the functions in F_t and whose rotation angle is an irrational multiple of π, then there is a $\tau < t_0$ such that the attractor A_t of F_t has non-empty interior for all $t \in (\tau, t_0)$.*

8. Transition attractors for bounded families

This section concerns the phenomena that can occur at the transition point t_0 between the existence and non-existence of an attractor. Two notions of a transition attractor for a one-parameter family F_t are defined.

Definition 8.1. Call a one-parameter similarity family F_t **bounded** if it is semi-linear and if there is a unique function, call it $\widehat{f}_t(x) = t\widehat{f}(x) + q_* \in F_t$, such that $\widehat{f}$ has maximum scaling ratio. The motivation for the terminology 'bounded' is theorem 8.1 below. By proposition 5.4, q_* is a fixed point of $\widehat{f}_t(x)$ for all $t \in [0, t_0]$. Call $f^* = \widehat{f}_{t_0}$ the **special function** associated with F_t, and call q_* the **special fixed point**. Note that the special function g is an isometry.

Theorem 8.1. *If F_t is a bounded one-parameter family, then there is a ball B such that then $F_t(B) \subset B$ for all $t \in [0, t_0]$.*

Proof. Without loss of generality assume (remark 4.1) that $t_0 = 1$. Using lemma 5.1 and proposition 5.4, let $F_t = \{tL_i(x - q_i) + q_i : i = 1, 2, \ldots, N\}$ be the bounded one-parameter family. The scaling ratio of L_i is denoted r_i. Without loss of generality assume that the special fixed point is 0, the origin. Let B be a ball centred at the origin of radius

$$R > \max\left\{\frac{\max(|(I - L_i)(q_i)|, |q_i|)}{1 - r_i} : i = 1,\ 2, \ldots, N, \quad r_i < 1\right\}.$$

For the unique function $\widehat{f}_t(x) = t\widehat{f}(x) + q_* \in F_t$ whose linear part has scaling ratio 1 we have $|\widehat{f}_t(x)| = t|L_f(x)| \leqslant R$ for all $x \in B$ with equality if and only if $t = 1$.

For any $f_t(x) = tf(x) + q \in F_t$ whose linear part L has scaling ratio $r < 1$, let $h(x) = f(x) + q$. Then $f_t(x) = th(x) + (1-t)q$. For $t \in [0, t_0]$ and for all $x \in B$, we have

$$\begin{aligned} |f_t(x)| &= |th(x) + (1-t)q| \leqslant t|h(x)| + (1-t)\,|q| \leqslant t|f(x)| + t|q| + (1-t)\,|q| \\ &\leqslant t|L(x)| + \big(t\,|(I-L)(q)| + (1-t)|q|\big) \leqslant trR + \max\{|(I-L)(q)|, |q|\} \\ &\leqslant trR + (1-r)R \leqslant trR + (1-tr)R = R. \end{aligned}$$

□

Remark 8.1. It is part of the definition of a *bounded* one-parameter family that there is a unique function whose linear part has maximum scaling ratio. If this requirement is omitted from the definition, then the conclusion of theorem 8.1 may fail to hold. This is the case for the one-parameter family of example 4.1. Likewise if the requirement that the family be semi-linear is omitted, then again the conclusion of theorem 8.1 may fail to hold. Any affine family for which $t_0 = 1$ that contains a function of the form $tx + a$ is an example.

Corollary 8.1. *For a bounded one-parameter family F_t, there is ball B such that the attractor A_t is contained in B for all $t \in [0, t_0)$.*

Proof. This follows immediately from the fact that $A_t = \bigcap_{n \geqslant 1} F_t^n(B)$, where B is the ball in the statement of theorem 8.1. □

For a one-parameter affine family F_t, call a compact set A^* **an upper transition attractor** if A^* is the limit of a sequence $A_{t_k}, \quad k = 1, 2, \ldots,$ of attractors with $t_k \to t_0$.

Proposition 8.1. *A bounded one-parameter family F_t has at least one upper transition attractor. Moreover, F_t has a unique upper transition attractor if and only if the following two equivalent conditions hold:*

(*a*) $\lim_{t \mapsto t_0} A_t$ *exists,*
(*b*) *The function $t \mapsto A_t$ is uniformly continuous on the interval* $(0, t_0)$.

Proof. For a bounded family F_t the closed ball B of corollary 8.1 contains the attractor A_t for all $t \in [0, t_0)$. Since B is compact, it is well know that $\mathbb{H}(B)$ is also compact, hence sequentially compact. Therefore, F_t has at least one upper transition attractor.

A basic result from analysis implies that $\lim_{t \mapsto t_0} A_t$ exists if and only if the function $t \mapsto A_t$ is uniformly continuous on the interval $(0, t_0)$.

Clearly, $\lim_{t \mapsto t_0} A_t$ exists if and only if the upper transition attractor is unique. □

For a one-parameter affine family F_t, theorem 4.2 states that the map $t \mapsto A_t$ is continuous on $(0, t_0)$, but example 4.1 shows that it may not be uniformly continuous. If, however, we further assume that F_t is bounded, then example 4.1 is no longer a counterexample because the attractors A_t in this example grow in size without bound as $t \to t_0$. This motivates the following main open problem in this paper. Corollary 8.2 below verifies the conjecture in the one-dimensional case, and graphical evidence seems to support it in dimension 2.

Conjecture 8.1. If F_t is a bounded one-parameter family, then there is a unique upper transition attractor.

For a set X, let $\operatorname{conv} X$ denote the convex hull of X.

Lemma 8.1. *Let $F_t, 0 \leqslant t < t_0$, be a bounded one-parameter family, and let A_t denote the attractor of F_t. If $K_t = \operatorname{conv} A_t$, then $K_s \subseteq K_t$ for all $0 \leqslant s \leqslant t < t_0$.*

Proof. Without loss of generality (see remark 4.1), assume that $t_0 = 1$. If B is the ball in theorem 8.1, then $F_t(B) \subset B$ for all $0 \leqslant t < 1$. Let $f_t(x) := tf(x) + q$ be any function in F_t. Since $f_s(x)$ and $f_t(x)$ are both contractive similarity transformations centred at $q \in B$ with the same linear part up to a constant, it must be the case that $f_s(B) \subseteq f_t(B)$ for all $f \in F$. For $0 \leqslant t < 1$, let K_t be the smallest convex set K such that $F_t(K) \subseteq K$, i.e., the intersection of all convex sets with this property. Because all the fixed points of functions in F_t are contained in such a convex set, $F_t(K) \subseteq K$ implies that $F_s(K_t) \subseteq K_t$. Therefore

$$K_s = \bigcap \{K \,:\, F_s(K) \subseteq K, \quad K \text{ convex}\} \subseteq K_t.$$

The facts that the functions are affine and that $F_t(A_t) = A_t$ implies that $F_t(\operatorname{conv} A_t) \subseteq \operatorname{conv} A_t$. Since K_t is the smallest convex set K such that $F_t(K) \subseteq K$, we have $K_t \subseteq \operatorname{conv} A_t$. On the other hand, since $A_t = \bigcap_{n \geqslant 0} F^n(K_t)$, we have $A_t \subset K_t$, which implies that $\operatorname{conv} A_t \subseteq K_t$ because K_t is convex. Therefore $K_t = \operatorname{conv} A_t$. □

Corollary 8.2. *If F_t is a one-dimensional bounded one-parameter family, then $A^* := \lim_{t \to t_0} A_t$ exists.*

Proof. By theorem 6.2 the attractor A_t is connected, hence is a point or a line segment for t sufficiently close to t_0. Therefore, $A_t = K_t$ for those values of t. By lemma 8.1, $A_s \subseteq A_t$ for $s \leqslant t$, so that $A^* := \lim_{t \to t_0} A_t$, as a limit of bounded nested intervals (or a point if F_t is linear), is itself an interval (or a point). □

Let $F_t, 0 \leqslant t < t_0$, be a bounded one-parameter family, and let A_t denote the attractor of F_t. Let $K_t = \operatorname{conv} A_t$ and

$$K^* = \overline{\bigcup_{t<t_0} K_t} = \lim_{t \to t_0} K_t,$$

where the bar denotes the closure. The union is nested by lemma 8.1 and is bounded by theorem 8.1. Therefore K^* is compact and convex. Call K^* the **transition hull** of F_t.

For a bounded family $F_t = \{tf_i(x) + q_i : 1 \leqslant i \leqslant N\}$, let

$$F^* := F_{t_0} = \{t_0\, f_i(x) + q_i \,:\, 1 \leqslant i \leqslant N\}.$$

From definition 8.1, the special function f^*, with fixed point q_* is the unique isometry in F^*. A compact set X will be called (F^*, q_*)**-invariant** if $F^*(X) = X$ and $q_* \in X$.

Definition 8.2. For a bounded one-parameter family

$$A_* = \bigcap \{A \in \mathbb{H}(\mathbb{R}^d) \,:\, A \text{ is } (F^*, q_*)\text{-invariant}\}$$

will be called the **lower transition attractor** of F_t. The terms 'upper' and 'lower' are used because, for any upper transition attractor A^*, it is part of theorem 8.2 below that $A_* \subseteq A^*$. Subsequent examples in this section show that they are not necessarily equal.

From proposition 8.1, there exists at least one upper transition attractor. The following theorem states several properties of the lower transition attractor A_* and any upper transition attractor A^*, and provides some relationships between A_* and A^*.

Theorem 8.2. *If $F_t, 0 \leqslant t < t_0$, is a bounded family, A_* its lower transition attractor and A^* any upper transition attractor, then the following hold:*

(*a*) *A^* is (F^*, q_*)-invariant;*

(*b*) A_* *is the unique minimal* (*with respect to inclusion*) (F^*, q_*)*-invariant set;*
(*c*) $A_* \subseteq A^*$*;*
(*d*) K^* *is the unique minimal* (F^*, q_*)*-invariant convex set;*
(*e*) $\operatorname{conv} A_* = \operatorname{conv} A^* = K^*$*;*
(*f*) $A_* = \overline{\bigcup_{n\geqslant 0} F^{*n}(q_*)}$.

Proof. Let $A^* = \lim_{k\to\infty} A_{t_k}$ be an upper transition attractor, where $t_k \to t_0$.

Concerning statement (a), we know that $q_* \in A^*$ because all the fixed points of the functions in F_t are contained in A^*. We must show that $F^*(A^*) = A^*$. From the fact that $F_t(A_t) = A_t$ for all $t \in [0, t_0)$, we have

$$\lim_{k\to\infty} F_{t_k}(A_{t_k}) = \lim_{k\to\infty} A_{t_k} = A^*.$$

Therefore it is sufficient to show that $F^*(A^*) = \lim_{k\to\infty} F_{t_k}(A_{t_k})$, or $f_{t_0}(A^*) = \lim_{k\to\infty} f_{t_k}(A_{t_k})$ for all $f_{t_k} \in F_{t_k}$. For all $f_{t_k} \in F_{t_k}$, we have

$$f_{t_k}(A_{t_k}) - f_{t_0}(A^*) = \left(f_{t_k}(A_{t_k}) - f_{t_k}(A^*)\right) + \left(f_{t_k}(A^*) - f_{t_0}(A^*)\right).$$

The result then follows from $A^* = \lim_{k\to\infty} A_{t_k}$ and the fact that $t \mapsto f_t$ is continuous on $[0, t_0]$.

Concerning statement (b), clearly $q_* \in A_*$. To show that A_* is F^*-invariant, in one direction, that $F^*(A) = A$ for all A that are (F^*, q_*)-invariant, implies that

$$\begin{aligned} F^*(A_*) &= F^*\left(\bigcap\{A : A \text{ is } (F^*, q_*)\text{-invariant}\}\right) \subseteq \bigcap\{F^*(A) : A \text{ is } (F^*, q_*)\text{-invariant}\} \\ &= \bigcap\{A : A \text{ is } (F^*, q_*)\text{-invariant}\} = A_*. \end{aligned}$$

In the other direction, $F^*(F^*(A_*)) \subset F^*(A_*)$ implies that

$$F^*(A_*) \in \{A \in \mathbb{R}^d : q_* \in A \text{ and } F^*(A) \subseteq A\},$$

which in turn implies that $A_* \subseteq F^*(A_*)$. Since A_* is the intersection of all F^*-invariant sets, it is the minimum.

Statement (c) follows immediately from statements (a) and (b).

Concerning the equality $\operatorname{conv} A^* = K^*$ in statement (e), the fact that $A_t \subseteq K_t \subseteq K^*$ implies that $A^* \subseteq K^*$. Now, from lemma 8.1,

$$\operatorname{conv} A^* = \operatorname{conv}\left(\lim_{k\to\infty} A_{t_k}\right) = \lim_{k\to\infty} \operatorname{conv} A_{t_k} = K^*. \tag{8.1}$$

Concerning the equality $F^*(K^*) = K^*$ in statement (d), using equation (8.1),

$$\begin{aligned} F^*(K^*) &= F^*\left(\operatorname{conv} A^*\right) = \bigcup_{g\in F^*} g\left(\operatorname{conv} A^*\right) \subseteq \bigcup_{g\in F^*} \operatorname{conv} g(A^*) \\ &\subseteq \operatorname{conv} F^*(A^*) = \operatorname{conv} A^* = K^*. \end{aligned}$$

Then $K^* \subseteq F^*(K^*)$ because one function in F^* is an isometry.

To show that K^* is the minimum such set in statement (d), we claim that $K^* := \bigcap\{K : K \text{ compact convex}, F^*(K) \subseteq K, q_* \in K\}$. Let

$$K^{**} := \bigcap\{K : K \text{ compact convex}, \ F^*(K) \subseteq K, \ q_* \in K\}.$$

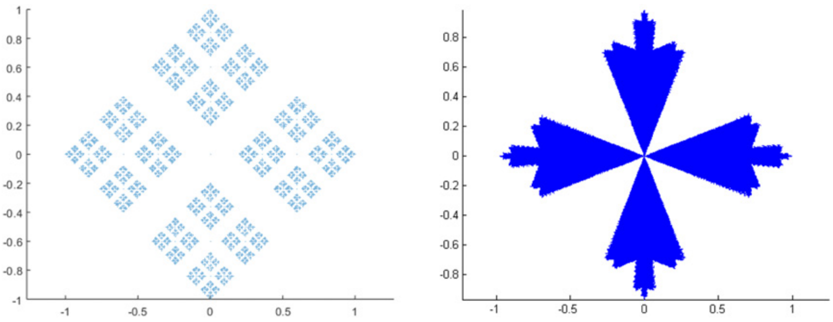

Figure 7. The lower and upper transition attractors of example 8.2.

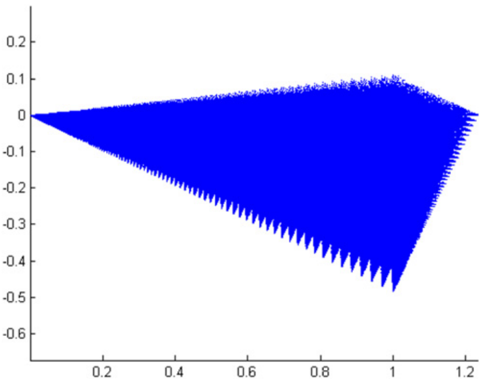

Figure 8. The upper transition attractor A^* of example 8.3.

Since K^* is one set in this intersection, clearly $K^{**} \subseteq K^*$. In the other direction, let K be any element of K^{**}, i.e., K is a compact convex set such that $F^*(K) \subseteq K$ and $q_* \in K$. Since K is compact, $F_*(K) \subseteq K$ and $q_* \in K$, we have that $\lim_{n\to\infty} \widehat{f}_t^n(q_*) \in K$ for all $\widehat{f} \in F^*$ which have similarity ratio less than 1. But the limit converges to the fixed point of $\widehat{f}$. Since the special fixed point q_* of the special function $f^* \in F^*$ is contained in K by assumption, all fixed points of functions in F^*, and hence all fixed points of functions in F_t for all $t \in [0, t_0)$, are contained in K. Because K is a convex set containing all fixed points of the similarity functions in $F_t, t \in [0, t_0)$, the fact that $F^*(K) \subseteq K$ implies that $F_t(K) \subseteq K$ for all $t \in [0, t_0)$. Since this is true for all $K \in K^{**}$, we have $F_t(K^{**}) \subseteq K^{**}$. Now $A_t = \lim_{n\to\infty} F_t^n(K^{**}) \subseteq K^{**}$, and hence $K^* = \operatorname{conv} A^* \subseteq K^{**}$.

Concerning the equality $\operatorname{conv} A_* = K^*$ in statement (e), $\operatorname{conv} A_*$ is (F^*, q_*)-invariant because A_* is (F^*, q_*)-invariant by statement (b) and F is affine. Therefore, by statement (d), we have $K^* = \bigcap\{K : K \text{ convex}, F^*(K) \subseteq K, q_* \in K\} \subseteq \operatorname{conv} A_*$. In the other direction, we have $A_* \subseteq A^*$ because A_* is the smallest (F^*, q_*)-invariant set and A^* is (F^*, q_*)-invariant by statement (a). Therefore $\operatorname{conv} A_* \subseteq \operatorname{conv} A^* = K^*$ by the first equality in statement (e).

Concerning statement (f), since $q_* \in A_*$ and A_* is (F^*, q_*)-invariant, it must be the case that $\overline{\bigcup_{n\geqslant 0} F^{*n}(q_*)} \subseteq A_*$. But since $\overline{\bigcup_{n\geqslant 0} F^{*n}(q_*)}$ is (F^*, q_*)-invariant and A_* is the minimum (F^*, q_*)-invariant compact set, $A_* \subseteq \overline{\bigcup_{n\geqslant 0} F^{*n}(q_*)}$. □

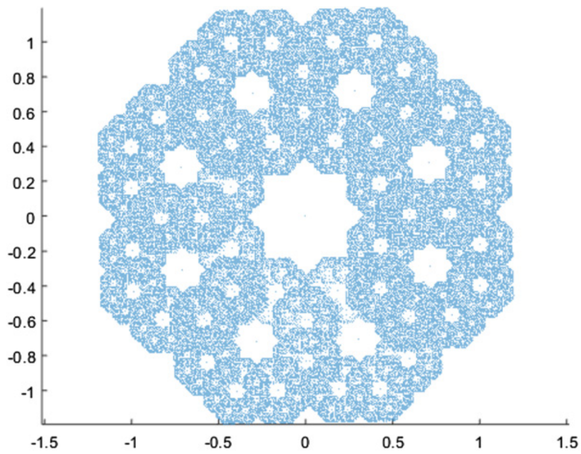

Figure 9. The transition attractor A_* of example 8.4.

Example 8.1 (**A bounded family for which** $A^* \neq A_*$). For the bounded one-parameter family $F_t = \{(t/4)x, t(x-1)+1\}$ on the real line $\mathbb{R}$, the threshold $t_0 = 1$. For $t < 1$ sufficiently close to 1, the attractor is the unit interval. Hence $A^* = \lim_{t\to t_0} A_t = [0,1]$. However, by statement (e) of theorem 8.2, $A_* = \{0\} \cup \{(1/4)^n\}_{n=0}^{\infty}$, a countable set.

Example 8.2. Consider the bounded family $F_t = \{tf_1(x), tf_2(x) + (1,0)\}$, where

$$f_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \qquad f_2\begin{pmatrix}x\\y\end{pmatrix} = \alpha\begin{pmatrix}x\\y\end{pmatrix} - \alpha\begin{pmatrix}1\\0\end{pmatrix},$$

and $0 < \alpha < 1 = t_0$. The lower transition attractor A_* is shown on the left in figure 7, with $\alpha = 0.4$. This figure was computed using statement (e) of theorem 8.2. The transition hull is the square with vertices $(1,0), (0,1), (-1,0), (0,-1)$. Computer graphics indicate that the upper transition attractor $A^* = \lim_{t\to 1} A_t$ exists and appears as on the right in figure 7 with $\alpha = 0.3$. This figure actually shows $A_{0.99}$, computed using the chaos game algorithm with a higher probability of choosing the first function to compensate for the value of t close to 1.

Example 8.3. Consider the bounded family $F_t = \{tf_1(x), tf_2(x) + (1,0)\}$, where

$$f_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \qquad f_2\begin{pmatrix}x\\y\end{pmatrix} = \alpha\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\left(\begin{pmatrix}x\\y\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right),$$

and $0 < \alpha < 1 = t_0$. The lower transition attractor A_* is the countable set of points

$$\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \bigcup \left\{\alpha^n \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}^n \begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} : n \geqslant 0\right\},$$

that spiral around and approach the point $(1,0)$. The upper transition attractor $A^* = \lim_{t\to 1} A_t$ appears to exist and is approximated in figure 8.

Example 8.4. Consider the bounded family $F_t = \{tf_1(x), tf_2(x) + (1,0)\}$, where

$$f_1\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}, \quad f_2\begin{pmatrix}x\\y\end{pmatrix} = \alpha\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\left(\begin{pmatrix}x\\y\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right),$$

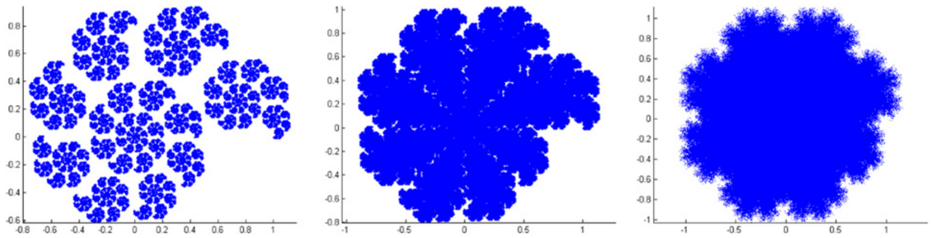

Figure 10. The transition attractor A^* of example 8.4.

$0 < \alpha < 1 = t_0$. The lower transition attractor A_* is shown in figure 9, with $\alpha = 0.4$. The upper transition attractor $A^* = \lim_{t \to 1} A_t$ appears to exist and is approximated in figure 10 with $\alpha = 0.4$. The figure on the left is $A_{0.9}$; the middle figure is $A_{0.94}$; and the figure on the right is $A_{0.98}$.

References

[1] Bandt C 2002 On the Mandelbrot set for pairs of linear maps *Nonlinearity* **15** 1127–47
[2] Bandt C and Hung N V 2008 Fractal n-gons and their Mandelbrot sets *Nonlinearity* **21** 2653–70
[3] Barnsley M and Harrington A N 1985 A Mandelbrot set for pairs of linear maps *Phys.* D **15** 421–32
[4] Barnlsey M 1993 *Fractals Everywhere* 2nd edn (Boston, MA: Academic)
[5] Barnsley M F and Vince A 2011 The eigenvalue problem for linear and affine iterated function systems *Linear Algebra Appl.* **435** 3124–38
[6] Barnsley M F and Vince A 2014 Fractal tiling for an iterated function system *Discrete Comput. Geom.* **51** 729–52
[7] Barnsley M F and Vince A 2017 Self-similar polygonal tiling *Amer. Math. Monthly* **124** 905–21
[8] Berger M A and Wang Y 1992 Bounded semigroups of matrices *Linear Algebra Appl.* **66** 21–7
[9] Bousch T 1992 Sur quelques problèmes de dynamique holomorphe *PhD Thesis* Orsay
[10] Daróczy Z and Kátai I 1988 Generalized number systems in the complex plane *Acta Math. Hung.* **51** 409–16
[11] Daubechies I and Lagarias J C 1992 Sets of matrices all infinite products of which converge *Linear Algebra Appl.* **161** 227–63
[12] Hare K and Sidorov N 2016 Two-dimensional self-affine sets with interior points, and the set of uniqueness *Nonlinearity* **29** 1–26
[13] Hare K and Sidorov N 2017 *Ergod. Th. Dynam. Sys.* **37** 193–227
[14] Hutchinson J 1981 Fractals and self-similarity *Indiana Univ. Math. J.* **30** 713–47
[15] Luo J, Wang L and Lian 2018 Topological properties of self-similar fractals with one parameter *J. Math. Anal. Appl.* **457** 396–409
[16] Li H, Luo J and Yin J 2007 The properties of a family of tiles with a parameter *J. Math. Anal. Appl.* **335** 1383–96
[17] Rota G-C and Strang W G 1960 A note on the joint spectral radius *Nederl. Akad. Wet. Proc., Ser.* A **63** 379–81
[18] Solomyak B 2004 On the 'Mandelbrot set' for pairs of linear maps: the local geometry *Anal. Theory Appl.* **20** 149–57
[19] Solomyak B 2005 On the 'Mandelbrot set' for pairs of linear maps: asymptotic self-similarity *Nonlinearity* **18** 1927–43
[20] Vince A 2000 Digit tiling of Euclidean space *Directions in Mathematical Quasicrystals* ed M Baake and R Moody (Providence, RI: American Mathematical Society) pp 329–70