7.6 MAPS

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INTRODUCTION

The theory of maps is likely the oldest topic in this volume, going back, not just to the 4-color problem posed in 1852 and to the theory of automorphic functions developed in the late 1800's, but to the Platonic solids dating to antiquity. Among the many contributors to the subject are Archimedes, Kepler, Euler, Poinsot, de Morgan, Hamilton, Dyck, Klein, Heawood, Hurwitz, Steinitz, Whitney, Koebe, Tutte, Coxeter and Grünbaum. General references on maps include [BoLi95], [BrSc97], [CoMo57], [GrTu87], [MoTh01], and [Wh01].

7.6.1 Maps and Polyhedra Maps

Basic notions are introduced: map and polyhedral map, duality, isomorphism, face and edge-width. The existence and uniqueness of a map with a given graph is addressed.

DEFINITIONS

D1: A ***map*** M ***on a surface*** S is a finite cell-complex whose underlying topological space is S. The surface of a map M is denoted $|M|$.

D2: The ***graph of the map*** M is its 1-skeleton. It is denoted $G := G(M)$.

D3: The ***vertices*** and ***edges of a map*** M are the vertices and edges, respectively, of its graph $G(M)$.

D4: The ***faces of a map*** M are the connected components of $|M| \setminus G(M)$.

D5: The ***0-, 1-, and 2-dimensional faces of a map*** M are its vertices, edges and faces, respectively.

D6: The ***dual map*** M^* of a map M on a surface S is a map on the same surface S whose vertex set V^* consists of one point interior to each face of M and whose edge set

E^* consists of, for each edge e of M, an edge e^* crossing e and joining the vertices of V^* that correspond to the faces incident with e. (A more general definition of duality appears in [Vi95], for example.)

D7: A ***polyhedral map*** M, generalizing the notion of a convex polyhedron, is a map whose face boundaries are cycles, and such that any two distinct face boundaries are either disjoint or meet in either a single edge or vertex.

D8: Maps M_1 and M_2 are ***isomorphic***, denoted $M_1 \approx M_2$, if there is a homeomorphism of the respective surfaces that induces an isomorphism of the respective graphs.

D9: The ***face-width of a map*** M, denoted $fw(M)$, is the minimum number of points $|\tau \cap G(M)|$ over all noncontractible simple closed curves τ on the surface.

D10: The ***edge-width of a map*** M, denoted $ew(M)$, is the length of a shortest cycle in $G(M)$ that is noncontractible on the surface.

D11: A ***large-edge-width (LEW)*** map is a map whose edge-width is greater than the number of edges in any face boundary.

EXAMPLES

E1: A map M on the torus and the dual map M^* appear in Figure 7.6.1. (The torus is obtained by identifying like labeled edges on the boundary of the polygon.) Neither M nor M^* is polyhedral.

Figure 7.6.1 A torus map and its dual.

E2: Figure 7.6.2 shows two nonisomorphic maps on the sphere with the same 2-connected, but not 3-connected, graph. The maps are related by a *Whitney flip*. This example is relevant to Fact F6 below.

Figure 7.6.2 Maps on the sphere with the same 2-connected graph.

E3: Figure 7.6.3 shows two polyhedral maps on the projective plane with isomorphic 3-connected graphs. (The projective plane is depicted as a disc with antipodal points identified.) This example shows that the analogy to the Whitney uniquesness theorem (Fact F6) for projective planar graphs fails.

Figure 7.6.3 Maps on the projective plane with the same 3-connected graph

REMARKS

R1: It is equivalent to regard a map as a 2-***cell imbedding*** of a graph G on a surface S, i.e., an imbedding such that the connected components of $S \setminus G$ are 2-cells.

R2: Face-width, introduced in [RoSe88], is a measure of locally planarity, or of how dense the graph is on the surface, or of how well the graph represents the surface.

R3: The concept of map has been extended to cell-complexes whose underlying topological space is a manifold of dimension greater than 2. This includes, in particular, the boundary complex of any polytope. The generalization to higher dimensions, though natural and interesting, is omitted here.

R4: A map M on the sphere S can be drawn in the plane via, for example, stereographic projection from any point of $S \setminus G(M)$.

R5: A map may have multiple edges, self-loops, and vertices of degree 1 or 2. A polyhedral map, however, can have none of these. Moveover, in a polyhedral map, the closure of each face is topologically a closed disc.

FACTS

F1: **Euler's formula** For any map M with f_0 vertices, f_1 edges, f_2 faces and characteristic $c(M)$,

$$f_0 - f_1 + f_2 = c(M)$$

F2: If M is a map, then $(M^*)^* = M$.

F3: If M is a map, then $fw(M^*) = fw(M)$.

F4: Map M is polyhedral if and only if its graph $G(M)$ is 3-connected and $fw(M) \geq 3$. Moreover, M is polyhedral if and only if its dual is polyhedral.

F5: Every connected graph G admits a map. The *rotation scheme* described in §6 gives a systematic method for obtaining all 2-cell imbeddings of G.

F6: [Wh32] **Whitney Uniqueness Theorem:** A 3-connected, planar graph has a unique imbedding on the sphere.

F7: [Th90] A uniqueness theorem for general surfaces: if M_1 and M_2 are LEW maps with the same graph, then $|M_1| = |M_2|$. Moreover, if the graph is 3-connected, then $M_1 \approx M_2$.

REMARKS

R6: According to Fact F5 above, every connected graph has a 2-cell imbedding on a surface. Whether a graph can be imbedding on a surface such that the face boundaries are (simple) cycles is problematic (see the conjectures below).

R7: [SeTh96] gives a uniqueness result similar to Fact F6 for maps with sufficiently large face-width as a function of the genus. However, [Ar92] provides an example, for every pair of integers k, b, of two maps M_1, M_2 with the same k-connected graph such that $fw(M_1), fw(M_2) > b$ and $|M_1| \neq |M_2|$.

CONJECTURES

The Cycle Double Cover Conjecture: Every 2-connected graph contains a set $\mathcal{C}$ of cycles such that every edge is contained in exactly two cycles of $\mathcal{C}$.

The Strong Imbedding Conjecture: Every 2-connected graph can be embedded on a surface so that each face is bounded by a cycle in the graph. The strong imbedding conjecture implies the Cycle Double Cover Conjecture.

7.6.2 The f-vector, v- and p-sequences, and Realizations

Elementary equalities hold among the basic parameters of a map. The two questions addressed in this section are, first, when are these necessary conditions also sufficient for the existence of a map with these parameters and, second, when can the map be embedded in Euclidean space E^3 or E^4 such that the faces are plane convex polygons. The classic results for maps on the sphere are Eberhard's theorem of 1891 and Steinitz's theorem of 1922.

DEFINITIONS

D12: A ***map*** is ***of type*** $\{p, q\}$ if each face has p edge incidences and each vertex has q edge incidences. (No global symmetry is implied; in fact, the automorphism group of the map, as defined in §5, may be trivial.)

D13: The ***cell-distribution vector (f-vector)*** of a map M is the 3-tuple (f_0, f_1, f_2), where f_0, f_1, f_2 are the numbers of vertices, edges, and faces of M, respectively.

D14: The ***face-size sequence (p-sequence)*** of a polyhedral map M is the sequence $\{p_i\}_{i\geq 3}$ where p_i is the number of i-gonal faces in M.

D15: The ***vertex-degree sequence (v-sequence)*** of a polyhedral map M is the sequence $\{v_i\}_{i\geq 3}$ where v_i is the number of vertices of degree i in M.

D16: A polyhedral map M is ***simplicial*** (or a ***triangulation***) if the boundary of each face is a 3-cycle.

D17: A polyhedral map M is ***simple*** if its graph is 3-regular.

D18: A ***geometric realization (realization)*** of a polyhedral map M is an imbedding of M into Euclidean space E^d (no self intersection) such that each face is a plane convex polygon and that adjacent faces are not coplanar.

REMARK

R8: Using a less stringent definition of realization than above, [Mc89] defined the *realization spaces* and studied its topological properties. Also see [BuSt00] and [MoWe00] for the realization space of the torus maps.

EXAMPLES

E4: The map M in Figure 7.6.1 is of type $\{3,6\}$ with face vector $(4,12,8)$. Its dual M^* is of type $\{6,3\}$ with face vector $(8,12,4)$. The maps in Figure 7.6.2 both have v-sequence $(6,3)$, but the first has p-sequence $(0,6,0,1)$ while the second has p-sequence $(1,3,3)$.

E5: Five maps on the sphere and their corresponding 3-dimensional realizations appear in Figure 7.6.4.

Figure 7.6.4 The Platonic solids as realizations of maps.

FACTS

F8: The f-vector, the p-sequence and the v-sequence satisfy the following elementary equalities:

$$\sum p_i = f_2, \qquad \sum v_i = f_0, \qquad \sum i p_i = 2f_1 = \sum i v_i$$

F9: For a map M on an orientable surface of genus g, $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$, Euler's formula implies that

$$\sum (\alpha i - 2) v_i + \sum (\beta i - 2) p_i = 4(g-1) \tag{1}$$

For example, taking $\alpha = 1/3,\ q = 0$, and M simple yields

$$\sum (6-i) p_i = 12 \tag{2}$$

F10: [Eb1891] **Eberhard's Theorem:** Condition (2) above is sufficient for the existence of a sphere map, in the following sense: if a sequence $\{p_i \,|\, i \geq 3,\ i \neq 6\}$ satisfies $\sum_{k \neq 6} (6-k) p_k = 12$, then there exist values of p_6 such that $\{p_i \,|\, i \geq 3\}$ is the p-sequence of a simple polyhedral map on the sphere. For variations on Eberhard's Theorem, see [Gr70] and [Je93]. There is no known generalization of Eberhard's theorem to arbitrary surfaces.

F11: [EdEwKu82] If S is a surface with Euler characteristic $c(S)$, if f_0, f_1, f_2, p, q are positive integers such that $f_0 - f_1 + f_2 = c(S)$, and if $pf_2 = 2f_1 = qf_0$, then there exists a map of type $\{p,q\}$ on S with f-vector (f_0, f_1, f_2), except when S is the projective plane and $\{p,q\} = \{3,3\}, f_0 = f_2 = 2, f_1 = 3$.

F12: [St22] **Steinitz's Theorem:** Every polyhedral map on the sphere is isomorphic to the boundary complex of a 3-dimensional polytope. Thus, any polyhedral map on the sphere has a realization in E^3.

F13: [Al71, Gr67] A simple polyhedral map M cannot be realized in Euclidean space of any dimension unless $|M|$ is the sphere.

F14: [BrSc95] Each simplicial polyhedral map on the torus or projective plane can be realized in E^4.

F15: [BrWi93] On any nonorientable surface N_g, there exists a simplicial map that cannot be realized in E^3. (When $g > 1$, it is an open question whether each simplicial polyhedral map of orientable genus g can be realized in E^3.)

F16: [Gr83] Equation (2) for the torus (with $\alpha = 1/3$) becomes

$$2\sum(i-3)\,v_i + \sum(i-6)\,p_i = 0$$

which leads to the following analogue of Eberhard's theorem for the torus. Given a sequence $\{p_i \,|\, i \geq 3,\, i \neq 6\}$ and a positive integer s, there is a realization in E^3 of some polyhedral map on the torus with p-sequence $\{p_i \,|\, i \geq 3\}$ and $\sum(i-3)v_i = s$ if and only if $\sum_{k\neq 6}(6-k)p_k = 2s$ and $s \geq 6$. Related results appear in [BaGrHö91].

F17: [St06] The vector (f_0, f_1, f_2) is the f-vector of a realization in E^3 of some polyhedral map on the sphere if and only if $f_0 - f_1 + f_2 = 2$, $4 \leq f_0 \leq 2f_2 - 4$, and $4 \leq f_2 \leq 2f_0 - 4$.

F18: [Gri83] The vector (f_0, f_1, f_2) is the f-vector of a realization in E^3 of some polyhedral map on the torus if and only if $f_0 - f_1 + f_2 = 0$, $f_2(11 - f_2)/2 \leq f_0 \leq 2f_2$, $f_0(11 - f_0)/2 \leq f_2 \leq 2f_0$, $2f_1 - 3f_0 \geq 6$, and $f_1 \neq 19$.

F19: [Ko36,An79,Th78] **Koebe-Andreev-Thurston circle packing theorem**: Every simplicial map M admits a circle packing representation, i.e., there exists a Riemannian metric of constant curvature $+1$, 0, or -1 on the surface and a collection of pairwise disjoint open disks on $|M|$ whose boundaries are geodesic circles such that the tangency graph of this collection of circles is $G(M)$. For a generalization to a larger class of maps, see [Mo97]. A circle packing representation of the octahedral map on the sphere appears in Figure 7.6.5.

Figure 7.6.5 A circle packing representation of the octahedral map.

7.6.3 Map Coloring

The famous results on map coloring are the Four Color Theorem for the sphere and the Heawood Map Coloring Theorem, which is the generalization of the Four Color Theorem to surfaces of higher genus. Also in this section are a few results on coloring densely imbedded graphs.

DEFINITION

D19: The **chromatic number** $\chi(S)$ of a surface S is the least number of colors sufficient to properly color the faces of any map on S. By duality, it is also the least number of colors sufficient to properly color the vertices of any map on S. In this section, coloring will mean vertex coloring.

FACTS

F20: [ApHa76] **Four Color Theorem:** $\chi(S_0) = 4$.

F21: [Fr34] $\chi(N_2) = 6$.

F22: [RiYo68] **Heawood Map Coloring Theorem:** For every surface S except the Klein bottle N_2,

$$\chi(S) = \left\lfloor \frac{7 + \sqrt{49 - 24c}}{2} \right\rfloor$$

where c is the Euler characteristic of S. The right-hand side of the equation is called the ***Heawood formula***.

F23: A map M on the torus with $ew(M) \geq 4$ is 5-colorable. It is not known whether this same statement holds for surfaces of higher genus.

F24: [Th93] Any map M on S_g with $ew(M) \geq 2^{14g+6}$ is 5-colorable.

F25: [Th97] For a fixed surface S, there is a polynomial time algorithm to decide if a map on S can be 5-colored.

F26: Even on the sphere, the problem of deciding whether a map can be 3-colored is NP-complete.

F27: [RSST96] On the sphere, a 4-coloring can be found in $O(n^2)$ steps.

REMARKS

R9: The problem of determining the chromatic number of the sphere appeared in a 1852 letter from Augustus de Morgan to Sir William Hamilton, and was likely due to Francis Guthrie, the brother of a student of de Morgan. The computer dependent proof of Appel and Haken [ApHa76] that four colors suffice was simplified considerably [RSST97] (but still computer dependent).

R10: That the formula in the Heawood Map Coloring Theorem gives an upper bound on $\chi(S)$ was proved by Heawood [He1890]. That there exist graphs that actually require the number colors given by that formula is a consequence of the formula for the genus of complete graphs due to Ringel and Youngs [RiYo68].

R11: Whether there is a polynomial time algorithm for deciding whether a map on an arbitrary surface can be 4-colored is unknown.

EXAMPLES

E6: Figure 7.6.9a is map on the projective plane that requires 6 colors for a proper coloring, and Figure 7.6.6 is map on the torus that requires 7. This shows that $\chi(N_1) \geq 6$ and that $\chi(S_1) \geq 7$. In fact, $\chi(N_1) = 6$ and $\chi(S_1) = 7$, in accordance with Fact F23. (The torus in Figure 7.6.6 is obtained by identifying left and right sides of the rectangle and the top and bottom sides with a 2/7 twist.)

Figure 7.6.6 A map on the torus whose graph is K_7.

E7: An example of Fisk [Fi78] shows that no 4-color analogue of Thomassen's result (Fact F24 above) can hold. See Figure 7.6.7, where the torus is obtained by identifying opposite sides of the square.

Figure 7.6.7 A map M on the torus with exactly two odd-degree vertices is not 4-colorable.

7.6.4 Minimal Maps

A map can be quite "degenerate", for example, the map on the sphere with 2 vertices, 1 edge, and 1 face. Polyhedral maps (and maps with edge-width or face-width bounded from below) cannot be this small. This section concerns maps that are in some sense minimal — either with respect to the number of vertices, or with respect to being polyhedral, or with respect to having edge-width k. Also covered in this section are weakly neighborly polyhedral maps.

DEFINITIONS

D20: A polyhedral map is ***neighborly*** if every pair of distinct vertices is joined by an edge.

D21: A polyhedral map is ***weakly neighborly*** (abbr. a ***wnp-map***) if every two vertices are contained on a face.

D22: The operation of ***edge contraction*** for a triangulation, and its inverse operation ***vertex splitting***, are depicted in Figure 7.6.8. After contracting an edge in a triangulation, the map may no longer be a triangulation, i.e., no longer polyhedral; this occurs if the edge is contained in a 3-cycle that is not a face boundary or if the map is the tetrahedral map.

D23: A ***minimal triangulation*** of a surface S is a triangulation such that the contraction of any edge results in a map that is no longer polyhedral.

D24: A ***k-minimal triangulation*** is a triangulation with edge-width k, such that each edge is contained in a noncontractible k-cycle. (Except on the sphere, minimal and 3-minimal are equivalent.)

Figure 7.6.8 Edge contraction and vertex-splitting in a triangulation.

EXAMPLES

E8: The only wnp-maps on the sphere are the boundary complexes of the pyramids and triangular prism.

E9: There are 5 wnp-maps on S_1 and none on S_2.

E10: The wnp-maps on nonorientable surfaces up to genus 4 appear in [AlBr87].

E11: There is 1 minimal triangulation of the sphere (the tetrahedral map), 2 minimal triangulations of the projective plane (see Figure 7.6.9), 21 of the torus, and 25 of the Klein bottle.

Figure 7.6.9 The minimal triangulations of the projective plane.

FACTS

F28: If the map M with f_0 vertices and Euler characteristic c is polyhedral, then

$$f_0 \geq \left\lceil \frac{7 + \sqrt{49 - 24c}}{2} \right\rceil ,$$

and this lower bound is attained for all surfaces except S_2, N_2 and N_3. By duality the same bound holds for f_2.

F29: The neighborly polyhedral maps attain the bound in Fact F28.

F30: [AlBr86] Each surface admits at most finitely many wnp-maps. (See Example E8.)

F31: [BaEd89] The set of minimal triangulations is finite for every fixed surface. (See Example E11.) In other words, for each surface, there is a finite set of triangulations from which any triangulation on that surface can be generated by vertex splittings.

F32: For any $k \geq 3$, the set of k-minimal graphs on a fixed surface is finite. ([MoTh01] provides a proof.)

REMARK

R12: [Br90] has provided a (non-tight) lower bound for f_1, for a polyhedral map of Euler characteristic c.

7.6.5 Automorphisms and Coverings

Every map M has a universal cover that is a classical tiling of the sphere, Euclidean plane, or hyperbolic plane (unit disc). This fact and its consequences are the subject of this section. Also addressed is the relation between a group acting as automorphisms of a map and a group acting as homeomorphisms of the surface.

In this section the classical Euclidean and hyperbolic tessellations are regarded as infinite maps, even though, by our definition, a map is a *finite* cell complex. For an expository article on connections between maps, Galois groups and Grothendieck's *dessins d'enfants*, see [JoSi96].

DEFINITIONS

D25: An ***automorphism*** of a map M is an isomorphism of M onto itself. The automorphisms form a group $\mathcal{A}ut(M)$ under composition.

D26: A ***map covering*** $f : M_1 \to M_2$ is a topological covering (see §7.2) of the respective surfaces that takes the graph of M_1 onto the graph of M_2, with ramification points possible only at vertices and face centers.

D27: The ***tessellation*** $\{p, q\}$ is the unique tesselation of the sphere or plane into regular p-gons, q incident at each vertex. This is a tiling of the sphere if $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$, of the Euclidean plane if $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, or of the hyperbolic plane (unit disc) if $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$.

D28: The ***triangle group*** $(p, q, 2)$ is the symmetry group of the tessellation $\{p, q\}$.

D29: The ***Coxeter group*** $W(p, q)$ is the group with presentation by three generators ρ_0, ρ_1, ρ_2 and the relations

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_2\rho_0)^2 = 1 \tag{3}$$

EXAMPLES

E12: Both torus maps M and M^* in Figure 7.6.1 are coverings of the tetrahedral map in Figure 7.6.4. The covering by M is ramified at vertices and the covering by M^* is ramified at face centers. Both are 2-fold coverings, that is, each unramified point of the sphere is covered by two points of the torus.

E13: Figure 7.6.10 shows all the (hyperbolic) mirors of reflection symmetries of the tessellation $\{6, 4\}$ (or $\{4, 6\}$). These lines form a subdivision (called the ***Coxeter complex***) of the hyperbolic plane into triangles (called ***flags***).

Figure 7.6.10 Reflection symmetries of the hyperbolic tessellation $\{6,4\}$.

FACTS

F33: The symmetry group $(p,q,2)$ of the tessellation $\{p,q\}$ is isomorphic to the Coxeter group $W(p,q)$. The Coxeter generators ρ_0, ρ_1, ρ_2 (and their conjugates) correspond to reflections in the three sides of a flag, as described in example E12; the products $\rho_1\rho_2, \rho_0\rho_2, \rho_0\rho_1$ (and their conjugates) correspond to rotations about vertices, midpoints of edges, and face centers, respectively.

F34: Every map M has a covering by a tessellation $\{p,q\}$ for some p,q. In other words, every map M is the quotient of a tesselation $\{p,q\}$ by a subgroup H_M of the Coxeter group $W(p,q)$.

F35: [Vi83a] The automorphism group $\mathcal{A}ut(M)$ of any map M is isomorphic to the quotient $N_W(H_M)/H_M$, where N_W denotes the normalizer and where H_M is the subgroup of Fact 34.

F36: Every map M of type $\{p,q\}$ has an unramified covering by the tessellation $\{p,q\}$. For example, the map on N_5 of type $\{5,5\}$ in Figure 7.6.11, is covered by the tessellation $\{5,5\}$ of the hyperbolic plane. (The map is obtained by identifying like labeled edges in the figure.)

Figure 7.6.11 The regular self-dual map $\{5,5\}_3$ and its universal cover $\{5,5\}$.

F37: [Bi72] The automorphism group of an orientable map of genus $g > 1$ can be faithfully represented in the group of $2g \times 2g$ symplectic matrices with integral entries. From this it can be proved, for example, that if prime p divides $|\mathcal{A}ut(M)|$, then the genus of the map M is either $1,2$ or at least $\frac{1}{2}(p-1)$.

F38: [Hu1892] ***Hurwitz formula:*** If a group Γ acts on a surface of Euler characteristic $c < 0$, then $|\Gamma| \leq -84c$.

F39: A map M with Euler characteristic $c < 0$ satisfies $|\mathcal{A}ut(M)| \leq -84c$ with equality if and only if M is a regular map of type $\{3,7\}$ or $\{7,3\}$ (see §7 for the definition of regular). This is a direct consequence of the Hurwitz formula.

F40: [Tu83] If a group Γ acts on an orientable surface S, then some Cayley graph G of Γ embeds in S, and the natural action of Γ on G (by left multiplication) extends to an action of Γ on S.

REMARK

R13: [JoSi78] Fact F35 implies that the surface of any map M can be assumed to be a Riemann surface such that $\mathcal{A}ut(M)$ acts as a group of conformal homeomorphisms. The edges of $G(M)$ are geodesics of equal length with respect to a Riemannian metric of constant curvature (defined everywhere except perhaps at finitely many ramification points located at vertices and face centers) and the angles formed by successive edges incident with a vertex are equal.

7.6.6 Combinatorial Schemes

The definition of map in §1 as a cell complex is topological. A strictly combinatorial description, although less intuitive, is often easier to apply. Three such schemes are described: rotation scheme, permutation scheme, and graph encoded map.

DEFINITIONS

D30: A ***rotation scheme*** (G, ρ) consists of a graph G and a set $\rho = \{\rho_v\}_{v \in V(G)}$, where ρ_v is a cyclic permutation of the edges incident to v. This scheme [Ed60] encodes any map with graph G embedded on an orientable surface (and can be extended to include nonorientable imbeddings).

D31: The ***map of a rotation scheme*** is obtained as follows. Given a directed edge $e_1 = (v_0, v_1)$ of G, consider the cycle consisting of successive directed edges $e_1 e_2 \ldots e_m = e_1$, where $e_i = (v_{i-1}, v_i)$ and

$$e_{i+1} = \rho_{v_i}(e_i)$$

Each (undirected) edge lies on exactly two such cycles. Regarding each cycle as the boundary of a polygonal 2-cell and gluing together 2-cells along paired edges results in an orientable surface in which G is embedded. Conversely, the ***rotation scheme of a map*** M on an orientable surface is (G, ρ), where G is the graph of M and ρ_v is the cyclic permutation of the edge incidence on vertex v induced by the orientation of the surface, say clockwise.

D32: A ***permutation scheme*** (π, σ) ***on a finite set*** X consists of permutations π and σ acting on X, such that each orbit of π has length 2 and such that the permutation group $H\langle \pi, \sigma \rangle$ generated by σ and π is transitive on X.

D33: The ***vertices, edges*** and ***faces*** of the permutation scheme (π, σ) are the cycles of σ, π and $\sigma \circ \pi$, respectively.

D34: Two faces (of any dimension) of the permutation scheme (π, σ) on a set X are ***incident*** if the corresponding cycles have an element of X in common.

D35: The ***permutation scheme of a map*** M has as the elements of its object set X the "half edges" of M (see example E14). Each cycle of σ is the cyclic (say clockwise) order of the half edges incident to a given vertex on the surface $|M|$, and each cycle of π is the two "half edges" at a midpoint of an edge. In a permutation scheme for a map, the graph is not explicitly part of the data.

D36: A ***graph encoded map*** (abbr. ***GEM***) is a connected, finite graph $\mathcal{G}$, regular of degree 3, together with a proper 3-coloring of the edges in the color set $I = \{0, 1, 2\}$, and with subgraphs $\mathcal{G}_i$, each induced by all edges not colored i, such that the connected components of $\mathcal{G}_1$ are 4-cycles.

D37: The ***vertices, edges*** and ***faces of a GEM*** $\mathcal{G}$ are the connected components of $\mathcal{G}_0, \mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

D38: Two faces (of any dimension) of a GEM are ***incident faces*** if the corresponding subgraphs have non-empty intersection.

D39: The ***graph encoding*** $\mathcal{G}$ ***of a given map*** M, is obtained from the barycentric subdivision Δ of M, by giving each vertex v of Δ the label $0, 1$, or 2, according to the dimension of the face in M that vertex v represents; then $\mathcal{G}$ is the dual graph of Δ, with color i assigned to edge e if and only if the two endpoints of the edge of Δ that e crosses are not labeled i.

REMARKS

R14: Permutation schemes can represent any map on an orientable surface (and can be extended to include maps on nonorientable surfaces). They have been used by [Ja68], [Co75], [Tu79], [JoSi78], [Wa75], and [St80].

R15: GEM's were introduced (in arbitrary dimension) as "combinatorial maps" by [Vi83] and as "crystallizations" by [Fe76] and [Ga79] in a topological context. The terminology "graph encoded map" is due to [Li82]. A graph encoding can represent any orientable or nonorientable map.

R16: Lifting the restriction that each orbit of π in a permutation scheme must have length 2 or that each component of $\mathcal{G}_1$ in a graph encoded map must be a 4-cycle, results in the concept of ***hypermap*** or ***hypergraph imbedding***.

EXAMPLE

E14: On the left in Figure 7.6.12 is a map on the sphere, and on the right is the corresponding graph encoded map. A rotation scheme for this map is given by the three cyclic permutations of edges incident to each of the three vertices:

$$\rho_1 = (1\ 2\ 3), \qquad \rho_2 = (1\ 4\ 5), \qquad \rho_3 = (5\ 4\ 3\ 2)$$

A permutation scheme for this same map is given by two permutations:

$$\pi = (1^+\ 1^-)(2^+\ 2^-)(3^+\ 3^-)(4^+\ 4^-)(5^+\ 5^-) \quad \sigma = (1^+\ 2^+\ 3^+)(1^-\ 4^+\ 5^+)(5^-\ 4^-\ 3^-\ 2^-)$$

Figure 7.6.12 A map on the sphere and its graph encoding.

FACTS

F41: The notion of an automorphism of a map M can be translated into terms of each of the above schemes:

If a map M is given as as a rotation scheme (G, ρ), then an automorphism of M corresponds to an isomorphism $f : G \to G$ that preserves the cyclic permutations, i.e., $f \circ \rho_v = \rho_{f(v)} \circ f$ for all $v \in V(G)$.

If M is given as a permutation scheme (π, σ) on a set X, then an automorphism of M corresponds to a bijection $f : X \to X$ such that $f \circ \phi = \phi \circ f$ for all $\phi \in H\langle \pi, \sigma \rangle$.

If M is given as a GEM $\mathcal{G}$, an automorphism of M corresponds to a color preserving graph isomorphisms of $\mathcal{G}$.

F42: The notion of duality can also be easily translated:

If M is given as a permutation scheme (π, σ) acting on the set X, the dual map M^* is given by the permutation scheme $(\pi, \sigma \circ \pi)$ acting on the same set X.

If M is given as a GEM $\mathcal{G}$, the dual map M^* is encoded by the same graph $\mathcal{G}$ with edge colors 0 and 2 interchanged.

F43: The surface of a map is orientable if and only if the corresponding GEM is bipartite.

7.6.7 Symmetry of Maps

Regular maps, those enjoying the greatest symmetry, are the surface analogues of the Platonic solids. Also discussed are symmetrical and vertex-transitive maps.

DEFINITIONS

D40: A ***flag*** of a map M is an ordered triple (F_0, F_1, F_2) of mutually incident faces of dimensions $0, 1$ and 2, respectively.

D41: A map M is a ***regular map*** if $\mathcal{A}ut(M)$ acts transitively on the set of flags.

D42: A map M is a ***symmetrical map*** if $\mathcal{A}ut(M)$ has at most two orbits in its action on the set of flags.

D43: A map is a ***chiral map*** if it is symmetrical, but not regular.

D44: A ***Cayley map for a group*** Γ with generator set Δ, is an imbedding of the Cayley graph $G_\Delta\Gamma$, using a rotation scheme as defined in §6. The cyclic permutation on the edges $\Delta^* = \Delta \cup \Delta^{-1}$ incident at each vertex must be the same at each vertex (see Example E21).

REMARK

R17: For a symmetrical map M, the automorphism group $\mathcal{A}ut(M)$ acts transitively on the set of vertices, on the set of edges, and on the set of faces.

R18: If a map M is given in terms of a GEM $\mathcal{G}$, then the flags of M are in bijective correspondence with the vertices of $\mathcal{G}$. Therefore the map M is regular if and only if the (graph) automorphism group of $\mathcal{G}$ is vertex-transitive.

EXAMPLES

E15: The regular maps on the sphere are the boundary complexes of the five ***Platonic solids*** (see Figure 7.6.4) which have types $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$, $\{5,3\}$, respectively, plus the infinite families of (non-polyhedral) maps of types $\{p,2\},\{2,p\}$, $p>0$.

E16: Since every map on the projective plane has a 2-fold covering by a map on the sphere (Fact F51), it follows from Example 15 that there are four regular maps on the projective plane of types $\{3,4\},\{4,3\},\{3,5\},\{5,3\}$ and infinite families of types $\{p,2\}$ and $\{2,p\}$, where $p \equiv 2 \bmod 4$.

E17: There are three infinite families of regular torus maps of types $\{3,6\},\{6,3\}$ and $\{4,4\}$. For example, in the notation of Fact F60, the maps in Figure 7.6.1, are $\{3,6\}_4$ and $\{6,3\}_4$.

E18: [CoDo01] used the bijection in Fact F57 and a network of computers to determine all regular maps on orientable surfaces of genus 2 to 15 and all regular maps on nonorientable surfaces from genus 4 to 30.

E19: The ***Kepler-Poinsot regular star polyhedra***, shown in Figure 7.6.13, are self-intersecting realizations of regular maps. In the notation of Fact F60 below, these maps are $\{5,5\,|\,3\}$ (twelve pentagons on a surface of genus 4 - great dodecahedron and small stellated dodecahedron), $\{5,3\}_{10}$ (twelve pentagons on the torus - great stellated dodecahedron) and $\{3,5\}_{10}$ (20 triangles on the torus - great icosahedron).

Figure 7.6.13 **Star polyhedra.**

E20: [ScWi85, ScWi86] From the history of automorphic functions come two regular maps of genus 3, the 1879 ***Klein map*** $\{7,3\}_8$ composed of 24 heptagons with automorphism group $PGL(2,7)$, and the 1880 ***Dyck map*** $\{8,3\}_6$ composed of 12 octagons (shown in dual form in Figure 7.6.14). The ***Coxeter regular skew polyhedra*** in E^4 also provide examples of regular maps; they are $\{4,6\,|\,3\}$, $\{6,4\,|\,3\}$, $\{4,8\,|\,3\}$ and $\{8,4\,|\,3\}$. The Klein, Dyck, and Coxeter maps all have realizations in E^3.

Figure 7.6.14 Dyck's map $\{3,7\}_8$.

E21: Figure 7.6.15 is a chiral map on the torus. Opposite sides of the square are to be identified. This map is presented as the Cayley map of the cyclic group $Z_5 = \{0,1,2,3,4\}$ with generating set $\Delta = \{1,2\}$ and cycle $\pi = (-1\ 2\ 1\ -2)$ on Δ^*.

Figure 7.6.15 A chiral map on the torus given an a Cayley map of Z_5.

Denoting an edge by a pair of vertices and a face by its four vertices, the flags $(1,12,1234)$ and $(2,12,1234)$ are in two different orbits under the action of the automorphism group acting on the set of flags. There is no automorphism that leaves edge 12 and face 1234 fixed and takes vertex 1 to vertex 2.

E22: Coxeter and others noticed that regular maps frequently occur as coverings of smaller regular maps on other surfaces. For example, the regular torus maps $\{3,6\}_4$ and $\{6,3\}_4$ in Figure 7.6.1 are 2-fold coverings of the tetrahedral map $\{3,3\}$ on the sphere. Constructions of families of regular maps using coverings appear in [JoSu00], [Si00], [Vi84], and [Wi78] among others.

E23: There are various group theoretical constructions of regular maps, for example [Mc91], [McMoWe93]. McMullen constructed a family of maps related to the Klein map $\{7,3\}_8$. For each odd prime p there is a regular map of type $\{p,3\}$ with $\frac{1}{2}(p^2-1)$ faces and orientation preserving automorphism group $\mathrm{PSL}\,(2,p)$.

E24: The vertex-transitive maps on the sphere, classified by [FlIm79], consist of the regular spherical maps and the boundary complexes of the Archimedean solids (semi-regular polyhedra), of the prisms and antiprisms. [Ba91] gave a classification of the vertex-transitive maps on the Klein bottle.

FACTS

F44: A map M with f_1 edges has exactly $4f_1$ flags.

F45: In $\mathcal{A}ut(M)$, the stabilizer of any flag is trivial.

F46: For any map M with f_1 edges, the two immediately preceding facts imply that

$|\mathcal{A}ut(M)| \leq 4f_1$, with equality if and only if M is regular. In this sense, the regular maps have the largest possible automorphism group.

F47: On each orientable surface there is a regular map.

F48: For a regular map on an orientable surface, half the automoprhisms act as orientation preserving homeomorphisms of the surface and half as orientation reversing.

F49: Not every nonorientable surface has a regular map; for example, there are no regular maps on the surfaces with nonorientable genus 2 and 3.

F50: [Vi83b], [Wi78a] Every nonorientable regular map has a unique 2-fold unramified covering by a regular orientable map.

F51: No chiral map exists on a nonorientable surface.

F52: For any surface S with Euler characteristic $c(S) < 0$, there are at most finitely many regular maps. This follows from the Hurwitz formula in §5.

F53: [Vi83b] For any pair (p, q) such that $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$, there are infinitely many regular maps of type $\{p, q\}$. [NeŠk01] subsequently showed that these maps may be chosen to have arbitrarily large face-width.

F54: [Wi89] There is a regular map with complete graph K_n if and only if $n = 2, 3, 4, 6$.

F55: [Bi71] There is a symmetrical map with complete graph K_n if and only if n is a prime power and, for each prime power, the symmetrical map is unique.

F56: [Vi83a, 83b] The regular maps M of type $\{p, q\}$ are in bijection with the conjugacy classes of normal subgroups N of finite index in the Coxeter group $W(p, q)$.

F57: A regular map M is the quotient of the tessellation $\{p, q\}$ by the corresponding normal subgroup N of symmetries of $\{p, q\}$; moreover $\mathcal{A}ut(M) \approx W(p, q)/N$.

F58: According to Fact 57, $\mathcal{A}ut(M)$, for a regular map M, has a presentation with three generators, the same relations as given for the Coxeter group $W(p, q)$ in Equation (3) together with some additional relations (except no additional relations in the case of a regular spherical map).

F59: Two special cases have received particular attention, the regular maps $\{p, q\}_r$ where the single relation $(\rho_0\rho_1\rho_2)^r$ has been added and the regular maps $\{p, q \,|\, m\}$ where the single relation $(\rho_0\rho_1\rho_2\rho_1)^m$ has been added. Coxeter and Moser [CoMo57] have provided partial tables of parameters p, q, r and p, q, m for which a *finite* regular map with those parameters exists. Figure 7.6.11 shows the regular map $\{5, 5\}_3$.

F60: Any Cayley map of a group Γ is vertex transitive, Γ acting as a group of automorphism of the Cayley map by left multiplication.

F61: The double torus S_2 has the interesting property that only finitely many groups act (as a group of homeomorphisms) on S_2, but there are infinitely many vertex-transitive (Cayley graphs) with genus 2.

F62: [Th91],[Ba91] For each $g \geq 3$, there are only finitely many vertex-transitive graphs of orientable genus g while there are infinitely many of genus $0, 1$ and 2.

7.6.8 Enumeration

W. T. Tutte [Tu68] pioneered map enumeration in the 1960's. Explicit results for maps on the sphere appear below. Results on generating functions and asymptotics for the number of such maps on general surfaces can be found in the texts [GoJa83], [Ya99] and the references therein. A connection between map enumeration, matrix integrals and 2-dimensional quantum gravity is explained in [Zv97].

DEFINITIONS

D45: A **rooted map** is a map in which a flag has been distinguished.

D46: A rooted map is a ***near triangulation*** if every nonroot face is a 3-gon.

EXAMPLES

E25: For the sphere, the 2-connected rooted maps with 4 edges are shown in the first row of Figure 7.6.16. The first four of these comprise all 2-connected rooted maps with 3 vertices and 3 faces. The root face is the outer face, the root vertex and edge are in boldface.

E26: On the second row of Figure 7.6.16 are the rooted near triangulations with 4 inner faces and a root face with 2 edges. The root face is the outer face; the root edge is the bottom edge; and the root vertex is in boldface.

Figure 7.6.16 Counting maps on the sphere.

FACTS

F63: [Tut63] The number of rooted maps on the sphere with $n \geq 0$ edges is

$$g(n) = \frac{2 \cdot 3^n (2n)!}{n!(n+2)!}$$

F64: [Tut63] The number of 2-connected rooted maps on the sphere with $n \geq 1$ edges is

$$\frac{2(3n-3)!}{n!(2n-1)!}$$

F65: [N. Wormald] (see [GoJa83]) The number of 2-edge-connected rooted maps on the sphere with $n \geq 0$ edges is

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}$$

F66: [BrTu64] The number of 2-connected rooted maps on the sphere with $n \geq 1$ vertices and $k \geq 2$ faces is

$$\frac{(2n+k-5)!(2k+n-5)!}{(n-1)!(k-1)!(2n-3)!(2k-3)!}$$

F67: [Br63] The number of rooted near triangulations of the sphere with $n+2j$ inner faces and $n \geq 2$ edges on the root face is

$$\frac{2^{j+2}(2n+3j-1)!(2n-3)!}{(j+1)!(2n+2j)!((n-2)!)^2},\ j \geq -1$$

7.6.9 Paths and Cycles in Maps

This section covers three topics involving paths and cycles: the Lipton-Tarjan separator theorem, the existence of nonrevisiting paths in polyhedral maps, and the decomposition of maps along cycles in the graph. The third topic is related to a result of Robertson and Seymour on minors.

DEFINITIONS

D47: A path p in the graph of a map M is said to be ***nonrevisiting*** if $p \cap F$ is connected for each face F of M.

D48: A surface S has the ***nonrevisiting path property*** if, for any polyhedral map M on S, any two vertices of M are joined by a nonrevisiting path.

D49: A map M is a ***map minor*** of a map M' if M can be obtained from M' by a sequence of edge contractions and deletions. The operations of edge deletion and edge contraction on a graph can be extended to a surface imbedding of the graph in an obvious way.

EXAMPLE

E27: A polyhedral map on the surface S_2 that fails to have the nonrevisiting path property appears in Figure 7.6.17 below. There is no nonrerevisiting path from x to y. (The map is obtained by gluing along like labeled edges.)

FACTS

F68: [LiTa79] **Planar Separator Theorem**: A planar graph with n vertices has a set of at most $2\sqrt{2n}$ vertices whose removal leaves no component with more than $2n/3$ vertices.

F69: [AlSeTh94] Let M be a loopless map on the sphere with n vertices. Then there is a simple closed curve τ on the surface of the sphere passing through at most $k \leq 3\sqrt{2n}/2$ vertices (and no other points of the graph) such that each of the two open disks bounded by τ contain at most $2n/3 - k/2$ vertices. This result slightly improves the Lipton-Tarjan separator theorem.

Figure 7.6.17 **A map on S_2 that does not satisfy the non-revisiting path property.**

F70: [GiHuTa84] A map of genus g contains a set of at most $O(\sqrt{gn})$ vertices whose removal leaves no component of the graph with more than $2n/3$ vertices. This generalizes the Lipton-Tarjan theorem to maps on orientable surfaces of higher genus.

F71: [PuVi98] For polyhedral maps, the nonrevisiting path property holds for the sphere, torus, projective plane and Klein bottle. It fails for all other surfaces except possibly the nonorientable surface of genus 3 (see [PuVi96] and example E27).

F72: The nonrevisiting path property holds for every polyhedral map with face-width at least 4.

F73: [RoSe88] Let M_0 be a map on a surface S other than the sphere. There exists a constant k such that, for any map M on S with $fw(M) \geq k$, M_0 is a map minor of M. The following two results provide values for the constant k when the given M_0 consists of certain sets of disjoint cycles.

F74: [Sc93] A map M on the torus with face-width w contains $\lfloor 3w/4 \rfloor$ disjoint noncontractible cycles.

F75: [BrMoRi96] For general surfaces there exist $\lfloor w/2 \rfloor$ pairwise disjoint contractible cycles in the graph of any map M, all containing a particular face, $\lfloor (w-1)/2 \rfloor$ pairwise disjoint, pairwise homotopic, surface nonseparating cycles, and $\lfloor (w-1)/8 \rfloor - 1$ pairwise disjoint, pairwise homotopic, surface separating, noncontractible cycles. (It is unknown whether any map of orientable genus $g \geq 2$ with face-width at least 3 must contain a noncontractible surface separating cycle.)

F76: [Bar88] Every polyhedral map on the torus (projective plane, Klein bottle) is isomorphic to the complex obtained by identifying the boundaries of two faces of a 3-polytope (cross identifying one face of a 3-polytope, cross identifying two faces of a 3-polytope).

F77: [Yu97] (see also [Th93]) If d is a positive integer and M is a map on S_g of face-width at least $8(d+1)(2^g - 1)$, then the graph of M contains a collection of induced cycles $C_1, C_2, \ldots, C_g$ such that the distance between distinct cycles is at least d and cutting along the cycles results in a map on the sphere. This generalizes Fact 76.

F78: [Sc91] Schrijver proved necessary and sufficient conditions (conjecteure by Lovasz and Seymour) for the existence of pairwise disjoint cycles $\tilde{C}_1, \ldots, \tilde{C}_k$ on the graph of a map M homotopic to given closed curves $C_1, \ldots, C_k$ on the surface.

REMARK

R19: The Lipton-Tarjan separator theorem has applications to divide-and-conquer algorithms. Nonrevisiting paths arise in complexity issues for edge following linear programming algorithms like the simplex method.

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GLOSSARY

2-cell imbedding: an imbedding of a graph G on a surface S such that the components of $S \setminus G$ are open 2-cells.

Archimedean solid: semi-regular polyhedron - regular polygons as faces and the same configuration of faces at each vertex.

automorphism of a map: an isomorphism of the map onto itself.

Cayley map: an imbedding of a Cayley graph on a surface using a rotation scheme as described in the text.

chiral map: a map that is symmetrical, but not regular.

chromatic number of a surface S: the least number of colors sufficient to properly color the faces (or vertices) of any map on S.

covering $f : M_1 \to M_2$: a topological covering of the respective surfaces that takes the graph of M_1 onto the graph of M_2, with ramification points only at vertices and faces centers.

Coxeter group (of rank 3): a group with presentation by three generators ρ_0, ρ_1, ρ_2 and the relations $\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_2\rho_0)^r = 1$.

Coxeter complex: the barycentric subdivision of the tessellation $\{p, q\}$, formed by all mirrors of reflection symmetries.

dual map: the geometric dual of the graph imbedded on a surface.

edge-width: the length of a shortest cycle in the imbedded graph that is noncontractible on the surface.

Euler characteristic of a map: $f_0 - f_1 + f_2$, where f_i denotes the number of i-dimensional faces of the map.

face: (also 2-face) a connected component of $S \setminus G$ where graph G is 2-cell imbedded on surface S.

face boundary: the edges incident to the face (with repetitions possible) ordered cyclically according to the rotation scheme of the map.

face-width: the minimum number of points $|\gamma \cap G|$ over all noncontractible simple closed curves γ on the surface on which the graph G is imbedded.

flag: an ordered triple (F_0, F_1, F_2) of pairwise incident faces of a map of dimensions $0, 1$ and 2, respectively.

f-vector: the triple (f_0, f_1, f_2) where f_i is the number of i-dimensional faces of the map.

genus of a surface: the number of handles for an orientable surface and the number of cross caps for a nonorientable surface.

graph encoded map: a particular system for describing a map using colored graphs.

hypermap: a generalization of graph imbedded on a surface to hypergraph imbedded on a surface.

isomorphism of maps: a homeomorhism of the respective surfaces that induces a graph isomorphism of the respective graphs.

Klein bottle: the nonorientable surface of genus 2.

large-edge-width map: a map whose edge-width is greater than the number of edges in any face boundary.

map: a 2-cell imbedding of a graph on a surface.

map minor of M: a map $\overline{M}$ obtained from map M by deleting and/or contracting edges.

minimal triangulation: a simplicial polyhedral map for which the contraction of any edge results in a map that is no longer polyhedral.

near triangulation: a rooted map in which every nonroot face is a 3-gon.

neighborly polyhedral map: a polyhedral map in which every pair of distinct vertices is joined by an edge.

noncontractible cycle: a cycle in the imbedded graph that is noncontractible on the surface.

nonseparating cycle: a cycle in the imbedded graph whose removal separates the surface.

nonrevisiting path: a path p in the graph of a map M such that $p \cap F$ is connected for each face F of M.

permutation scheme: a particular system for describing a map using a pair of permutations.

polyhedral map: a map M whose face boundaries are cycles, and such that any two distinct face boundaries are either disjoint or meet in either a single edge or vertex.

projective plane: the nonorientable surface of genus 1.

p**-sequence**: $\{p_i\}$, where p_i is the number of i-gonal face in a polyhedral map.

ramification point of a covering: a point of the surface at which the covering is not a local homeomorphism.

realization: an imbedding of a map into Euclidean space $\mathbf{E}^d$ such that each face is a plane convex polygon and adjacent faces are not coplanar.

regular map: a map whose automorphism group acts transitively on the set of flags.

rooted map: a map in which a flag has been distinguished.

rotation scheme: a particular system for describing an imbedding of a graph G on a surface using a cyclic permutation at each vertex of G.

simple map: A map in which each vertex has degree 3.

simplicial map: (triangulation) a map where each face boundary is a 3-cycle.

skew polyhedron: a realization of a polyhedral map in $\mathbf{E}^d$, $d > 3$.

star polyhedron: polyhedron allowing faces to intersect.

surface: a compact, connected, 2-dimensional manifold without boundary

symmetrical map: a map with at most two orbits under the action of the automorphism group on the set of flags.

tessellation $\{p, q\}$: the classical tiling of the sphere, Euclidean plane, or hyperbolic plane into p-gons, q incident at each vertex.

torus: the orientable surface of genus 1.

triangle group: the symmetry group of the tessellation of type $\{p, q\}$.

triangulation of a surface: (simplicial map) a map where each face boundary is a 3-cycle.

type $\{p, q\}$ **map**: a map with p edges incident with each vertex and q edges incident with each face.

vertex splitting: an operation on a map inverse to edge contraction — a single vertex is replaced by two vertices joined by an edge.

vertex-transitive map: a map whose automorphism group acts transitively on the set of vertices.

v**-sequence**: $\{v_i\}$, where v_i is the number of vertices of degree i in a polyhedral map.

weakly neighborly polyhedral map: a polyhedral map for which every pair of vertices is contained on a face.