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n-GRAPHS

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During the past few years papers have appeared that take a graph theoretic approach to the investigation of PL-manifolds. These graphs have variously been called crystallizations, gems (graph encoded manifolds) and n-graphs. The basic ideas and major results of this combinatorial method are surveyed in this paper.

1. Introduction

The use of combinatorial methods in topology is certainly not new; simplicial homology is an obvious example. However the graph theoretic approach to be surveyed here is fairly recent, due to Pezzana, Ferri, Gagliardi, Lins, Mandel, Vince and others. This paper is not intended as a complete review of the subject; for this we refer to [7] where an extended list of references can also be found. Instead we present only major results with motivation. Proofs are either sketched or omitted. A reference is given for each result but the proof sketched may not be the original.

Graphs can have multiple edges and $V(G)$ and $E(G)$ denote the point and edge sets of G respectively. Let $[n]$ denote the set $\{0, 1, \ldots, n-1\}$. An *n-graph* is a graph G, regular of degree n, together with an edge coloring $E(G) \to [n]$ such that incident edges are different colors. The motivation for this definition is that an $(n+1)$-graph G encodes an n-dimensional simplicial complex ΔG as follows. For each point v of G let σ_v be an n-simplex whose set of $n+1$ vertices is in bijection with $[n+1]$. Let k be the disjoint union of the σ_v, $v \in V(G)$. For each $i \in [n+1]$ identify the $(n-1)$-face of σ_u colored $[n+1]-\{i\}$ with the $(n-1)$-face of σ_v colored $[n+1]-\{i\}$ if and only if u and v are joined in G by an edge colored i. If $\sim$ is this indentification then $K/\sim$ is denoted ΔG and the underlying topological space of ΔG is denoted $|G|$. Fig. 1 is an example of a 3-graph G and the corresponding 2-dimensional complex ΔG. The underlying space $|G|$ is the 2-sphere.

Throughout this paper manifolds will be PL, compact, and without boundary. The simple, but basic, fact in the theory is that any PL n-manifold M is homeomorphic to $|G|$ for some $(n+1)$-graph G [2, 19, 21]. Such a G is obtained as the appropriately colored dual 1-skeleton of the barycentric subdivision of any triangulation of M. Recall that the dual 1-skeleton of a triangulation is the graph

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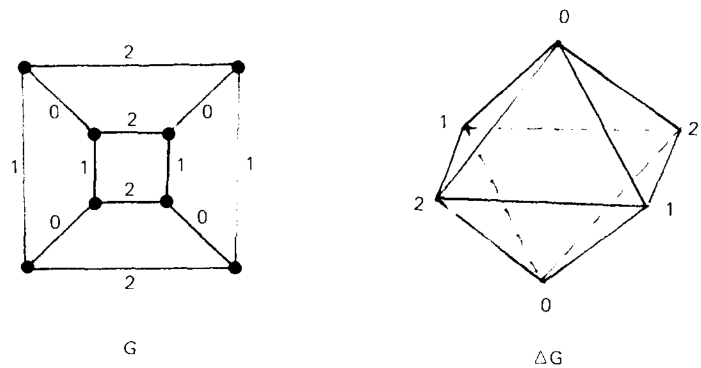

Fig. 1. A 3-graph and the associated simplicial complex ΔG.

whose points are the facets and two points are joined by an edge if and only if the corresponding n-simplices share a codimension 1 simplex. Hereafter $\simeq$ denotes homeomorphism.

Theorem 1. *For any PL n-manifold M there exists an* $(n+1)$*-graph G such that* $|G| \simeq M$.

It is known that every 1, 2, or 3-manifold can be triangulated and hence can be encoded as a 2, 3, or 4-graph. Therefore in these low dimensions the scheme is completely general. In Fig. 2, for example, are graphs representing the sphere product $S^1 \times S^2$ and the non-orientable sphere bundle $S^1 \underset{\sim}{\times} S^2$, which will be useful later. *Our point of view from here on is to regard an n-manifold as an*

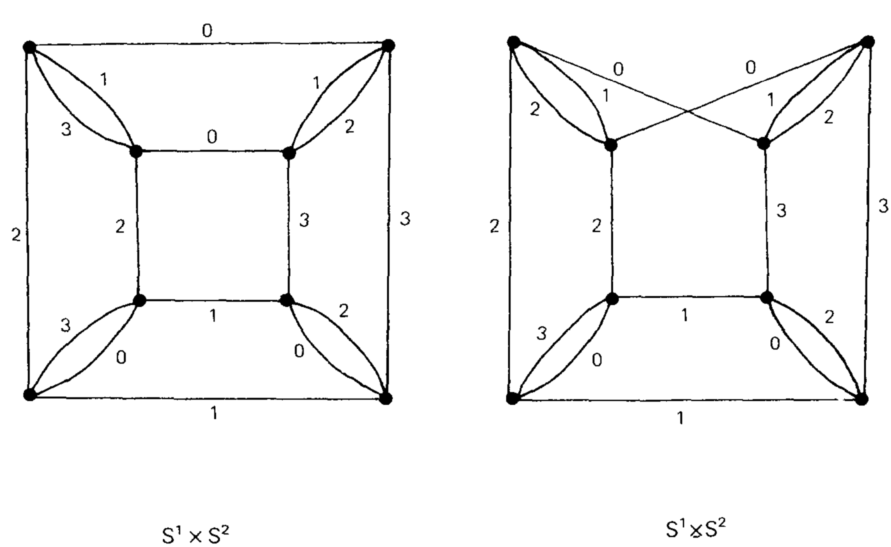

Fig. 2. Encodings of the orientable and non-orientable 2-sphere bundles over S^1.

$(n+1)$*-graph*. The goal is to gain topological insight into the space $|G|$ from the combinatorics of the graph G.

The basic notions about n-graphs addressed in this paper are (1) fusion, (2) canonical forms, (3) fundamental group and (4) regular embedding. Basic results connecting graphs and manifolds are contained in Section 2. In Section 3 equivalence of n-graphs is defined. Equivalence of graphs G_1 and G_2 corresponds to homeomorphism between the topological spaces $|G_1|$ and $|G_2|$. An equivalence step is one of several types of fusion, a basic operation on n-graphs that is also discussed in Section 3. The ideal situation would be to have, for each PL manifold, a unique canonical n-graph. Unfortunately this is known only for $n \leqslant 3$ and is covered in Section 4. Two simple algorithms on an n-graph G are given in Section 5 for determining the fundamental group of $|G|$. Regular embedding of an n-graph on a closed surface provides a new topological invariant for manifolds—the graph theoretic genus. For 2-manifolds this invariant is the ordinary genus of the surface, and for 3-manifolds it is essentially the Heegaard genus. This and other properties of embedding are discussed in Section 6.

2. Graph encoded manifolds

Although proofs concerning n-graphs are often combinatorial, the next result provides the link between the combinatorics of the graph G and the topology of the complex ΔG. For an n-graph G and for $J \subset [n]$ let G_J denote the subgraph of G obtained by deleting all edges with colors not in J. Each connected component of G_J is a $|J|$-graph and is called a *residue of type* J or $|J|$*-residue*. In particular, G itself is the only n-residue; each point of G is a 0-residue; each edge is a 1-residue; 2-colored cycles in G are 2-residues, etc.

Theorem 2. *There is a* 1–1, *inclusion reversing correspondence between the residues of an* $(n+1)$*–graph* G *and the simplices of* ΔG. *For* $i \in [n]$, *i-residues in* G *correspond to* $(n-i)$*-simplices in* ΔG.

By the above theorem, facets (highest dimensional simplices) of ΔG correspond to points of G; codimension 1-faces of ΔG corresponds to edges of G; . . . ; vertices of ΔG correspond to n-residues of G. Hence links of vertices in ΔG are encoded by n-residues; in general, links of i-faces in ΔG are encoded by $(n-i)$-residues of G.

A 2-graph encodes the circle S^1; a 3-graph encodes a surface. For an n-graph, $n \geqslant 3$, the underlying space $|G|$ is not necessarily a manifold because the link of a vertex in ΔG is not necessarily a sphere. As a consequence of Theorem 2 a 4-graph encodes a 3-manifold exactly if each 3-residue encodes a 2-sphere. Moreover, application of the Euler characteristic to 3-residues results in the following necessary and sufficient condition for a 4-graph to encode a manifold [2, 19].

Theorem 3. *Let v denote the number of vertices in a 4-graph G and r_2, r_3 the number of 2 and 3-residues, resp. Then $v \geq r_2 - r_3$ with equality if and only if $|G|$ is a manifold.*

A space $|G|$ is orientable if there is a coherent orientation of the facets in ΔG. In terms of the graph G this is expressed as follows [21].

Theorem 4. *The topological space $|G|$ is orientable if and only if the graph G is bipartite.*

3. Equivalent n-graphs

A basic construction in the theory of n-graphs is fusion. Consider two points u and v in an n-graph G(or in two n-graphs G_1 and G_2) and let G^* be the n-graph obtained by

(1) removing u, v and all edges connecting them;
(2) reconnecting the 'free' edges (previously incident to one of u or v) of like color.

Then G^* is said to be obtained from G by *fusion* on u and v. If there are $m \geq 1$ edges connecting u and v then the graph removed in step (1) is called an *m-dipole* and the fusion is called *removing a dipole.* The inverse operation is called *adding a dipole.* If J denotes the set of colors of a dipole D of an n-graph G and if u and v lie in the same residue of type $[n]-J$, then D is called *degenerate.* Otherwise D is *non-degenerate.*

Call two n-graphs *equivalent* if one can be obtained from the other by a sequence of adding or removing non-degenerate dipoles. Fig. 3 shows three equivalent 3-graphs. First dipole d_1 is added and then dipole d_2 is removed. Removing a non-degenerate dipole in G corresponds in $|G|$ to removing a ball and identifying two hemispheres on the boundary in the natural way. This makes the 'if' part of the following theorem reasonable. What is surprising is that the converse is also true [4].

Theorem 5. $|G_1| \simeq |G_2|$ *if and only if G_1 and G_2 are equivalent.*

The graph of Fig. 3c encodes the Klein bottle (see Section 4). Hence, by Theorem 5, Fig. 3a also encodes the Klein bottle.

The topological consequences of other types of fusion are summarized in the following theorem. If points u and v lie in distinct n-graphs G_1 and G_2 then fusion is denoted by $G_1 \overset{uv}{\#} G_2$. Recall that the connected sum $M_1 \# M_2$ of two manifolds is obtained by removing an open ball from each and identifying the two spherical boundaries via a homeomorphism. Also $\otimes$ will stand for the orientable (ordinary product) or non-orientable bundle. Parts (a, b, c, d) of the theorem are found in [4, 17, 12, 8], respectively.

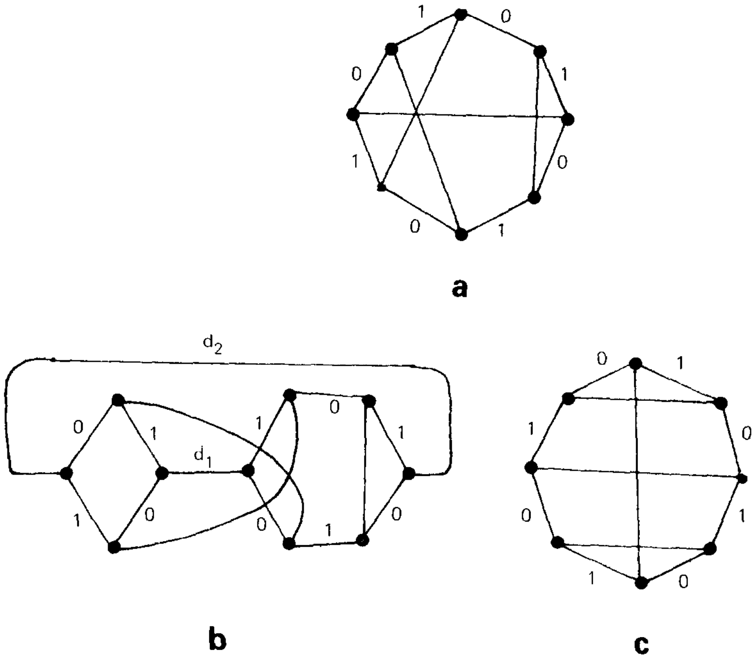

Fig. 3. Equivalent 3-graphs.

Theorem 6. *For* 4-*graphs that encode* 3-*manifolds*

(a) $|G_1 \overset{uv}{\#} G_2| \simeq |G_1| \# |G_2|$.

(b) *If u and v are not contained in the same* 3-*residue of G and G' is obtained from G by fusion on u and v, then* $|G'| \simeq |G| \# (S^1 \otimes S^2)$.

(c) *If G' is obtained from G by removing a degenerate* 2-*dipole then* $|G| \simeq |G'| \# (S^1 \otimes S^2)$.

(d) *If G' is obtained from G by removing a degenerate* 1-*dipole then one of following cases holds*:

(i) $|G| \simeq |G'|$

(ii) $|G| \simeq |G'| \# (S^1 \otimes S^2)$.

(iii) $|G| \simeq |G'| \# (S^1 \otimes S^2) \# (S^1 \otimes S^2)$

(iv) $|G| \simeq |G'_1| \# |G'_2| \# (S^1 \otimes S^2)$

depending on whether the endpoints u and v of the dipole are both contained on exactly (i) *one* 2-*residue with colors other than that of edge* $\{u, v\}$; (ii) *two such* 2-*residues*; (iii) *three such* 2-*residues and G' is connected*; *or* (iv) *three such* 2-*residues and G' has two connected components G'_1 and G'_2.*

Examples of parts (a) and (c) are shown in Figs 4 and 5 respectively. In Fig. 4, G_1 and G_2 both encode $S^1 \times S^2$ and $G_1 \overset{uv}{\#} G_2$ encodes the connected sum.

Some comments on Theorem 6 are in order. First, the graph G' always encodes a 3-manifold. In part (a) $|G_1 \overset{uv}{\#} G_2|$ is independent of u and v unless both G_1 and

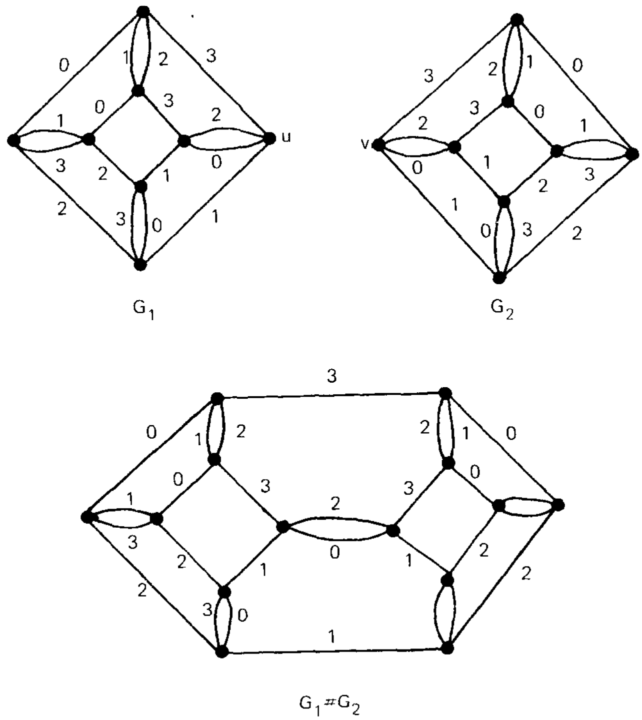

Fig. 4. Connected sum.

G_2 are bipartite. In this case both $|G_1|$ and $|G_2|$ are orientable (by Theorem 4) and there are two possible topological connected sums (orientable and non-orientable); which one depends on the partite sets in which u and v belong. In part (b) $\otimes$ is the orientable or non-orientable sphere bundle depending on whether or not there is a path from u to v with an odd number of edges. Likewise in (c) and (d) the graph G determines the type of bundle in a straightforward way, but we omit the details here and refer the reader to the original sources for these results. The only case of fusion not listed in Theorem 6 is that of a

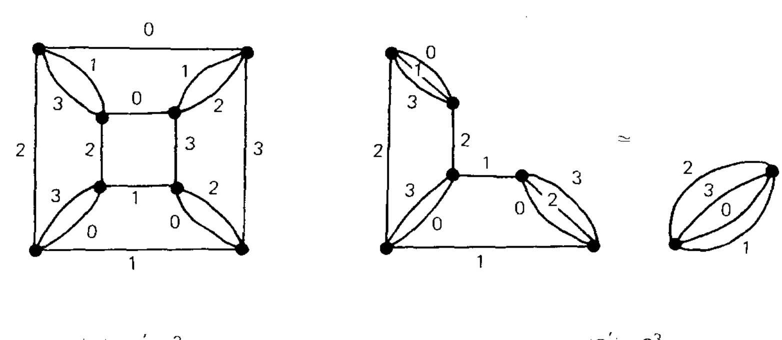

Fig. 5. Removing a handle.

degenerate 1-dipole where u and v are contained in *no* 2-residue (of type not containing the color of $\{u, v\}$). Unlike the cases covered in Theorem 6, there is no known characterization of the topological space corresponding to the 4-graph after the removal of such a dipole. Unfortunately, in a large class of 4-graphs, all dipoles are of this type [8].

Proof sketch of Theorem 6. In part (a) the operation $\overset{uv}{\#}$ corresponds, via Theorem 2, to removing a facet from ΔG_1 and ΔG_2 and identifying boundaries, i.e. a connected sum. Likewise in part (b) the fusion corresponds to removing two disjoint facets from ΔG and identifying boundaries. This is usually referred to as 'adding a handle', which is equivalent to a connected sum as in part (b). For part (c) a 4-graph G^* exists (see [12] for the construction) such that (1) G^* is equivalent to G' (by removing two nondegenerate dipoles of G^*) and (2) G is obtained from G^* by fusion of two points of G^* not in the same 3-residue. The result then follows from part (b) and Theorem 5. Part (d) is proved in a similar fashion using part (c). □

4. Canonical *n*-graphs

One of the major goals in the theory of n-graphs is to obtain canonical forms from PL-manifolds. Various notions of 'canonical' appear in the literature. If M is a PL-manifold, then G is called *minimum* for M if G is an n-graph with minimum number of points that encodes M [19]. For example, the minimum 4-graph for the 3-sphere is given in Fig. 6a (two 3-balls with their boundaries identified). The minimum number of $(n-1)$-residues in an n-graph is n. An n-graph that achieves this minimum is called *simple* [19]. A simple n-graph that encodes a manifold is also referred to in the literature as a *crystallization* [2]. We call an n-graph with no non-degenerate dipoles *reduced.* It is easy to check that if G is reduced, then G is simple. The converse is not true, as in Fig. 6b. Also it is obvious that if G is minimum, then G is reduced. Again the converse is not true, as in Fig. 6c [15]. By removing nondegenerate dipoles until no longer possible, Theorems 1 and 5 imply:

Corollary 7. *For any manifold M there is a reduced n-graph G such that $|G| \simeq M$.*

A strong form of Theorem 5 for simple n-graphs is proved in [4].

Theorem 8. *Let G_1 and G_2 be n-crystallizations. Then $|G_1| \simeq |G_2|$ if and only if G_2 can be obtained from G_1 by a finite sequence of the following moves:*

(a) *adding or removing a non-degenerate m-dipole, $n-1>m>1$.*

(b) *addition of a non-degenerate 1-dipole followed by the removal of another non-degenerate 1-dipole.*

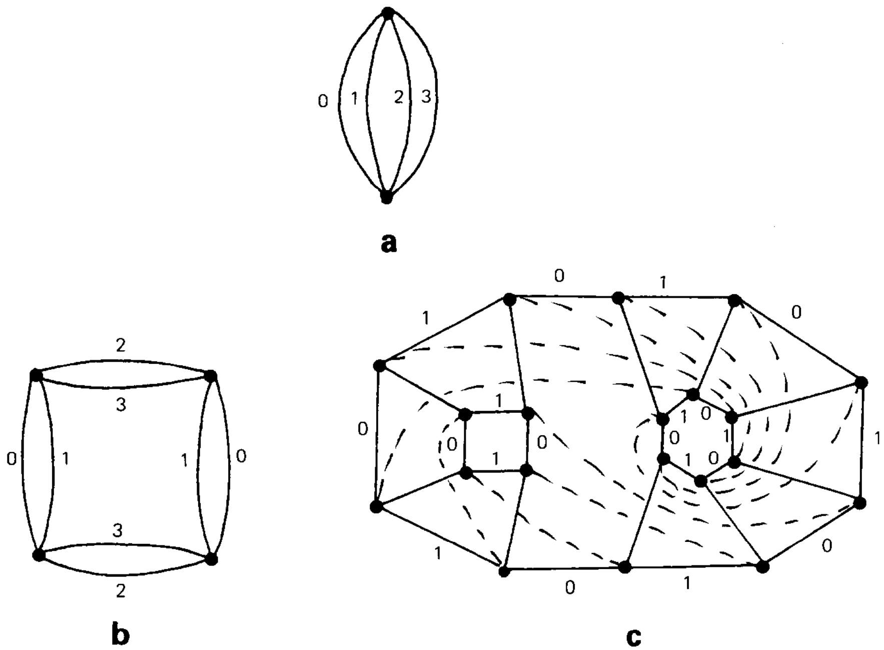

Fig. 6. Some 4-graphs encoding S^3.

Note that in each step in the process of Theorem 8, the intermediate n-graph is also a crystallization. Also for any $i \in [n]$, the moves (a) and (b) can be taken so that either all the dipoles contain color i or none contain color i [3].

The ideal situation would be to have a unique reduced graph to represent each equivalence class of n-graph. Unfortunately the only non-trivial case for which this is known is $n = 3$. To classify the 3-graphs up to equivalence let C_{2m} be the cycle with $2m$ points $\{0, 1, \ldots, 2m-1\}$ where edge $\{2i-1, 2i\}$ is colored 0 and $\{2i, 2i+1\}$ is colored 1 for all $i \pmod{2m}$. Let H_m be the 3-graph obtained from C_{2m}, m odd, $m \geq 1$, by adding edges $\{i, i+m\}$ colored 2 for all $i \pmod{2m}$. Likewise let L_m be obtained from C_{2m}, $m \geq 2$, by adding edges $\{i, 2m-i\}$ and $\{0, m\}$ colored 2 for all $i \pmod{2m}$. The graphs H_5 and L_5 are shown in Fig. 7.

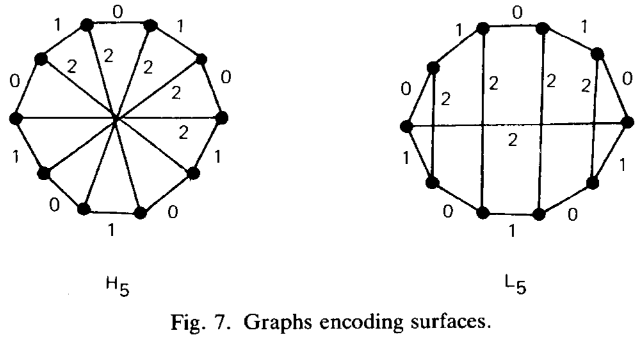

Fig. 7. Graphs encoding surfaces.

Theorem 9. *Each 3-graph is equivalent to a unique graph H_m, $m \geqslant 1$, m odd or L_m, $m \geqslant 2$.*

Corollary 10. *Two reduced 3-graphs are equivalent if and only if they have the same number of points and are both bipartite (or not bipartite).*

Theorem 9 is just a graph theoretic version of the well known classification of closed surfaces. Using Theorem 2 (to compute the Euler characteristic) and Theorem 4, it is easy to check that $|H_{3-\chi}|$ and $|L_{3-\chi}|$ are the orientable and nonorientable surfaces, resp., with Euler characteristic χ. So $|H_1|$ is homeomorphic to the 2-sphere, $|H_3|$ the torus, $|L_2|$ the projective plane, $|L_3|$ the Klein bottle, etc. Concerning Corollary 10, the number of points and the property of being bipartite are invariants of reduced 3-graphs because all steps in Theorem 8 must be of type (a). Each step of type (a) leaves the number of points of G and the properties of being reduced and bipartite invariant. In the other direction the corollary follows from Theorem 9. If G and G' are reduced, have an equal number of points and are both bipartite (or not bipartite), then this is also true of B and B', the graphs to which, by Theorem 9, they are equivalent. Then necessarily $B = B'$.

Any 3-manifold can be encoded by a 4-graph, so naturally the classification problem for 4-graphs becomes drastically more complex than for 3-graphs. For example, by Theorem 5 the 4-graphs in Fig. 6c and 6a must be equivalent, but there is no known algorithm, in general, for obtaining the correct sequence of dipole additions and removals. Certain n-graphs with additional structure, called normal crystallizations, have been shown to encode any 3-manifold [1, 14], but these also have not led to a classification. (So the somewhat technical definition is omitted here.) The determination of a set of moves to get, algorithmically, from any n-graph to any equivalent one would, of course, be a remarkable discovery. For more on this problem see [24].

5. The fundamental group

The encoding of a manifold M by an n-graph provides easy algorithms for finding a presentation of the fundamental group of M. We give two such algorithms. Here $\langle X \mid R \rangle$ denotes a presentation of a group with generators X and relations R. Let H be a subgraph of an n-graph G and e an edge in $E(G) - E(H)$. Call e *dependent* on H if there is a k-colored cycle, $k < n$, containing e, all of whose other edges lie in H. Let $H = H_0, H_1, \ldots, H_m = H^*$ be a sequence of subgraphs of G such that

(1) $H_{i+1} = H_i + e$ where e is dependent on H_i and

(2) there is no edge in G dependent on H^*.

Then H^* is called the *closure* of H in G.

In both algorithms the input is a manifold encoded by an n-graph and the output is a presentation for $\pi_1(|G|)$.

Algorithm 1 [24].

(0) Remove non-degenerate dipoles until G is reduced.
(1) Construct a spanning tree T in G.
(2) Determine the closure T^* of T in G.
(3) Arbitrarily assign an orientation to each edge in $X = E(G) - E(T^*)$.
(4) For each 2-residue τ let r_τ be the sequence of edges in X around the cycle τ, each with exponent $+1$ or -1 depending on the orientation. Let $R = \{r_\tau \mid \tau \text{ is a 2-residue}\}$.
(5) Then $\langle X \mid R \rangle$ is a presentation for $\pi_1(|G|)$.

Algorithm 2 [18].

(0) Remove non-degenerate dipoles until G is reduced.
(1) Let $X = V(G)$, the point set of G.
(2) Choose two colors $j, k \in [n]$ and let C denote the set of all 2-residues (cycles) of type $\{j, k\}$.
(3) For any cycle $\sigma \in C$ of length $2m$ let r_σ denote the word $x_1 x_2^{-1} \cdots x_{2m-1} x_{2m}^{-1}$ where $x_i \in X$ is the i-point of σ. The initial point x_1 and the direction around σ is arbitrary, except that the edge connecting x_1 and x_2 should be colored j. Let $R_1 = \{r_\sigma \mid \sigma \in C\}$.
(4) Let $R_2 = \{xy^{-1} \mid x, y \in X$ and x, y belong to the same residue of type $[n] - \{j, k\}\}$.
(5) If x_0 is an arbitrary point of G then $\langle X \mid R_1 \cup R_2 \cup \{x_0\} \rangle$ is a presentation of $\pi_1(|G|)$.

As an example, consider the 3-graph in Fig. 8, which encodes the Klein bottle K^2. (Those edges not colored 0 or 1 are understood to be colored 2.) Using Algorithm 1 (Fig. 8a), the spanning tree T consists of all edges except one in the unique residue of type $\{0, 1\}$ and T^* is the entire (darkened) cycle. Then the

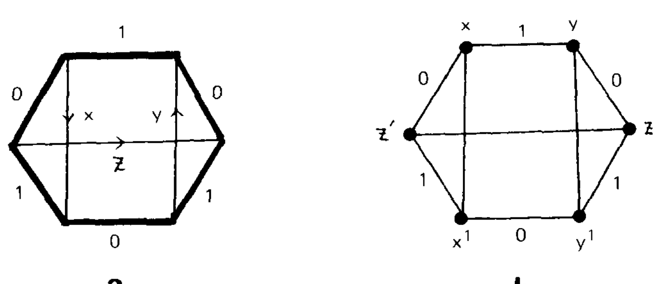

Fig. 8. Computing the fundamental group of $|G|$.

presentation of $\pi_1(K^2)$ is $\langle x, y, z \mid xyz^{-1} = xzy = 1\rangle$ which, by removing generator z, is equivalent to $\langle x, y \mid x^2y^2 = 1\rangle$.

Choosing the pair of colors $\{0, 1\}$ for Algorithm 2 (Fig. 8b) the presentation is

$$\langle x, y, z, x', y', z' \mid x = x', y = y', z = z', xy^{-1}zy'^{-1}x'z'^{-1} = 1, z = 1\rangle$$

which simplifies after removing generators x', y', z', z to

$$\langle x, y \mid xy^{-1}y^{-1}x = 1\rangle = \langle x, y \mid x^2y^2 = 1\rangle.$$

Proof sketch of Algorithms 1 and 2. The first algorithm is based on the fact that G is the dual 1-skeleton of ΔG. Then up to homotopy, each based loop of $\pi_1(|G|)$ is represented by a based closed edge path in G. Each edge $e \in E(G) - E(T)$ represents the path in G that is the unique cycle in $T + e$. An edge in T^* represents a null homotopic loop in $|G|$; hence X is the generating set for $\pi_1(|G|)$. Each two colored cycle in G also corresponds to a null homotopic loop in $|G|$, hence the relations R.

The idea of Algorithm 2 is dual to that of Algorithm 1, in that loops in $|G|$ are represented by based edge paths in ΔG rather than in G. The fundamental group $\pi_1(|G|)$ is isomorphic to the standard edge path group $E(\Delta G)$ whose elements are closed edge paths in the 1-skeleton of ΔG, and edge paths around 2-simplices are null homotopic. It is the main theorem of [13] that a presentation $\langle X \mid R\rangle$ for $E(\Delta G)$ can be obtained by taking X as the set of all edges of type $\{j, k\}$ except one (recall that the vertices of ΔG are colored), and the relations are read around the links of simplices of type $[n] - \{j, k\}$. Translating this presentation, via Theorem 2, from ΔG to the graph G results in Algorithm 2. □

6. Regular embedding

Let F be a closed surface. An embedding of an n-graph G on F is called *regular* (strongly regular in [10]) if

(a) The components of $F - G$ are 2-cells.

(b) For any adjacent pair of points u and v of G, the cyclic permutation τ_u of the color set $[n]$ induced by the edges about u is the inverse of the cyclic permutation τ_v induced by the edges about v.

Up to inverse the cyclic permutation of the edge colors on F is the same at each point. So by counting the number of such cyclic permutations, it is easy to see that there is at most one regular embedding of a 3-graph and at most three regular embeddings of a 4-graph. Note that each 2-cell of F must be bounded by a 2-residue. Conversely, by spanning each of these (disjoint) 2-residues by a 2-cell and identifying pairs of edges that are the same in G, each of the regular embeddings above can be constructed. In general this argument shows that there are exactly $\frac{1}{2}(n-1)!$ regular embeddings of an n-graph.

In this section a graph theoretic invariant of a PL manifold is defined that simultaneously generalizes the ordinary genus of a surface and the Heegaard genus of a 3-manifold. For an n-graph G let the *genus* $\rho(G)$ be the smallest integer g such that G regularly embeds in a surface of genus g. (For an orientable surface M the genus $g(M)$ is the number of torus factors in a connected sum (handles), while for a non-orientable surface it is the number of projective plane factors (cross caps). Then for a PL manifold M let

$$\rho(M) = \min\{\rho(G) \mid |G| \simeq M\}$$

be the minimum genus of any encoding of M. Using Theorem 4 it is not difficult to show that the surface F on which G is embedded is orientable if and only if $|G|$ is orientable. Hence $\rho(M)$ is the orientable or non-orientable genus depending on whether or not M is orientable.

The definition above and the results that follow appear in [11]. Recall that a handlebody of genus g is a 3-manifold (with boundary) obtained by identifying in pairs $2g$ disjoint 2-cells on the boundary of a 3-ball [16]. A *Heegaard splitting* of genus g of a 3-manifold M is a pair H, H' of handlebodies such that $H \cap H' = \partial H = \partial H'$ and $H \cup H' = M$. The common boundary $\partial H = \partial H'$ is called the *Heegaard surface*, which is a surface of genus g if orientable or $2g$ if not. The *Heegaard genus* $h(M)$ is the smallest integer g such that M admits a Heegaard splitting of genus g.

Theorem 11. (a) *If M is a 2-manifold then $\rho(M) = g(M)$.*

(b) *If M is a 3-manifold then*

$$\rho(M) = \begin{cases} h(M) & \text{if } M \text{ is orientable} \\ 2h(M) & \text{if } M \text{ is non-orientable.} \end{cases}$$

Proof sketch of Theorem 11. Part (a) of Theorem 11 is easy to prove because G, as the dual graph of the triangulation ΔG, is regularly embedded on $|G|$. Part (b) is proved in two parts. Given a 4-graph G regularly embedded on a surface F, assume, without loss of generality, that for each cyclic permutation τ_u we have $\tau_u^2(0) = 1$. Then consider the surface S that is the union of the 4-sided cells embedded in the facets of ΔG (see Fig. 9) such that there is a vertex of a 4-sided

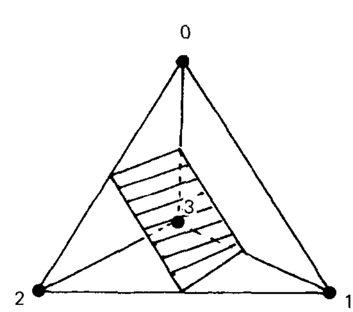

Fig. 9. Portion of the Heegaard surface.

cells at the midpoint of each edge of a facet except the edges 01 and 23. Now S splits $|G|$ into two handlebodies that are regular neighborhoods of the subgraphs of G induced by the edges colored 01 and 23, resp. Hence S is a Heegaard surface. The graph G is embedded in S as the dual graph of this cell division into squares such that the cyclic permutations about the vertices of this embedding are the same as for the embedding of G in F. Therefore S is homeomorphic to F.

Conversely the Heegaard splitting of genus g determines two sets of pairwise disjoint curves on the Heegaard surface F. Each such set consists of the boundaries of a set of pairwise disjoint 2-cells that cut the handlebody H or H' into a 3-ball. These systems of curves, called a Heegaard diagram, can be altered slightly to obtain the desired 4-graph embedding on the surface F. □

In [6] estimates are made on the genus of some 4-manifolds; here T_g, U_h denote the orientable surface of genus g and the non-orientable surface of genus h, respectively:

$$\rho(S^1 \times T_g) = 2g + 1$$
$$\rho(S^1 \times U_h) = 2h + 2$$
$$2(g + g') \leq \rho(T_g \times T_{g'}) \leq 2(6gg' + 3(g + g') + 2)$$
$$2(2g + h) \leq \rho(T_g \times U_h) \leq 2(6gh + 3(2g + h) + 4)$$
$$2(h + h') \leq \rho(U_h \times U_{h'}) \leq 2(3(hh' + h + h') + 4)$$

A corollary of Theorem 11 is that a 4-graph of genus 0 encodes a manifold $|G|$ that has Heegaard genus 0, and hence is a sphere. This is generalized in [5].

Theorem 12. *For an $(n+1)$-graph G, $\rho(G) = 0$ if and only if $|G| \simeq S^n$.*

The proof of Theorem 12 is mainly topological. A stronger statement than this theorem is the following. If true, it should have a completely combinatorial proof. Here G_n^2 is the n-graph with exactly two points and n edges joining them. It is the minimum n-graph for S^{n-1}, (G_4^2 is shown in Fig. 6a).

Conjecture. *If G is a reduced n-graph and $\rho(G) = 0$ then $G = G_n^2$.*

7. Conclusion

The subject of n-graphs is new. Three-manifold theory is well developed, whereas the theory of 4-graphs is not. The hope is that the graph theoretic method holds some potential for combinatorial insight into topological questions. As the subject of n-graphs has developed the proofs have become less topological and more combinatorial. When it is not necessary to refer to the complex ΔG, via Theorem 2, the proofs (in the biased estimation of the author) take on a clear graph theoretic character.

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