Kantian Addition: A Phenomenology of Arithmetic

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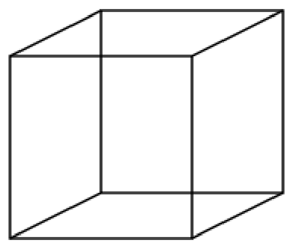
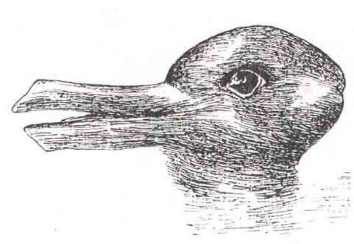
In the following essay we shall argue that the experiential, intuitive, or ”phenomenological” basis for arithmetic propositions such as 2+2=4 is essentially of the same sort as noticing that the following figures can be seen in two different ways:





We should like to make it clear that our intention is not to advocate a strict ”intuitionism” à la Brouwer, or to argue that the only meaningful mathematical statements are those that can be verified through direct intuitive experience. Once the project of mathematics has gotten off of the ground, so to speak, we are able to use our formal machinery to look more and more deeply into mathematical structures, in the same sense that a telescope affords us sights that we could never have seen with the naked eye. It is our intention, however, to indicate what the ground of mathematics might be. The ”formalist” who believes mathematics is an empty game of tautologies with symbols is like an astronomer who assures us that we don't need our eyes, so long as we have an impressive telescope. We hope to provide some indication of why it is that we make just these particular definitions and just these rules of inference; why is it that we are drawn to just this formalism and not some other? We will explore the accounts of arithmetic given by Kant, Hegel, Frege, and Wittgenstein, and shall, with Wittgenstein, defend Kant's account.

We take the following principle of Wittgenstein's as the guiding thread of our investigation:

> There must not be anything hypothetical in our considerations. We must do away with all *explanation*, and description alone must take its place...The problems are solved, not by giving new information, but by arranging what we have always known. Philosophy is a battle against the bewitchment of our intelligence by means of language. (10, p.47)

Describing what is happening when we do addition, or what addition is, is difficult precisely because the synthesizing faculty is assumed in virtually any description. We mean by this synthesis the act whereby we grasp an object as a single thing (and this is implicit in our saying "an" object). This activity is so ubiquitous that it is difficult to go outside it so that we may describe it at a distance. Kant wrote:

> It is true that space and time contain a manifold of pure *a priori* intuition, but they belong also to the conditions of the receptivity of our mind—conditions under which alone it can receive representations of objects, and which therefore must also always affect the concept of these objects. The spontaneity of our thought requires that this manifold, in order to be turned into knowledge, should first be gone through, taken up and combined in a certain manner. This act I call synthesis. By synthesis, in its most general sense, I mean the act of putting different representations together, and of comprehending their manifoldness in one item of knowledge. (6, p.103)

> The combination of a manifold in general can never come to us through the senses...for it is an act of spontaneity by the power of representation...This act we shall call by the general name of synthesis, in order to show that we cannot represent to ourselves anything as combined in the object without having previously combined it ourselves, and that of all representations combination is the only one which cannot be given through objects, but, as an act of the subject's self-activity, can only be carried out by the subject itself...[the] dissolution [of synthesis], that is, analysis, which seems to be its opposite, always presupposes it. For where the understanding has not previously combined, there is nothing for it to dissolve...But the concept of combination includes, besides the concept of the manifold and of its synthesis, also the concept of the unity of the manifold. Combination is the representation of the synthetic unity of the manifold. (6, p.122)

> As every appearance contains a manifold, whereby different perceptions are encountered in the mind indvidually and dispersed, they need to be combined in a way that they cannot be in sense itself. Hence there exists in us an active faculty of the synthesis of this manifold. (6, p.157).

As Kant notes elsewhere, physical objects are always seen to be extended (and how else could they be visible?):

> ...the concept of body serves, according to the unity of the manifold which is thought through it, as a rule for our knowledge of outer appearances...the concept of body, whenever we perceive something outside us, necessitates the representation of extension. (6, p. 137)

This means it is implicit that (at least in our mind's eye) we may divide this "single" object into multiple parts. Thus every perceived unity is already a plurality. We note that the ideal geometric point, which has no spatial parts to speak of, is so defined that it is impossible to ever actually perceive a single, isolated geometric point in the sense that we may perceive a marble. From this discussion it should be apparent that there is no essential difference between grasping an object as a unit, and grasping a collection of objects as a unit; every object is already implicitly a collection of spatially extended parts, and it is really only topological considerations that prejudice us towards calling connected objects "one" and disconnected objects "many".

As Kant notes, the subject (or "observer" in perhaps more modern terms) must combine the disparate sensory information into intelligible objects (though we might say with Kant that "we are scarcely ever conscious" p.104 of this process; for before there are objects, what could we be conscious of?). Furthermore, we must combine our sensory impressions under a particular aspect, as is made obvious from the figures above. We do not believe that the figure is *really* a duck or a rabbit; as a figure it simply takes up a certain area on the page, which can be split into parts having one shade or the other, and these are what we might call objective, physical facts that are indifferent to our manner of grasping them. Nevertheless, it is almost impossible to see the figure without seeing it *as* either a duck or rabbit, and this should help to illuminate the extent of the contribution of the subject observing it.[1] We could, of course, have used any image of either a duck or rabbit to make these points, but images like the "duck-rabbit" above make it much more clear that any picture is subject to multifarious interpretations; while we are engrossed in any particular interpretation we can hardly believe that there could be a radically different way of seeing the picture. The pictures above both have the remarkable quality that our interpretation of them may suddenly restructure, reorganize, (re-present?) the image in a flash. They also may serve as rare examples where we are explicitly "conscious" of the synthetic process.

The sudden transition between our interpretation of the above images has been called the "change of aspect" by Wittgenstein:

> "But surely you would say that the picture is altogether different now!" But what is different: my impression? my point of view?—Can I say? I *describe* the alteration like a perception; quite as if the object had altered before my eyes...(10, p.195)
>
> My visual impression has changed and now I recognize that it has not only shape and colour but also a quite particular "organization". My visual impression has changed;—what was it like before and what is it like now?—If I represent it by means of an exact copy—and isn't that a good representation of it?—no change is shown.(10, p.196)
>
> The expression of a change of aspect is the expression of a new perception and at the same time of the perception's being unchanged...Hence the flashing of an aspect on us seems half visual experience, half thought...(10, p.196-197)
>
> You only 'see the duck and rabbit aspects' if you are already conversant with the shapes of those two animals.(10, p.207)
>
> When the aspect changes parts of the picture go together which before did not. (10, p.208)
>
> The colour of the visual impression corresponds to the colour of the object (this blotting paper looks pink to me, and is pink)—the shape of the visual impression to the shape of the object (it looks rectangular to me, and is rectangular)—but what I perceive in the dawning of an aspect is

[1]If the reader is uncomfortable with bringing the observer into arithmetic, they may want to consider the importance of the observer in both relativistic and quantum physics.

> not a property of the object, but an internal relation between it and other objects...'The echo of a thought in sight'—one would like to say. (10, p.212)

Hegel's description of what he called Force also may be taken as an apt description of the "self-cancelling movement" of the change of aspect:

> ...the 'matters' posited as independent directly pass over into their unity, and their unity directly unfolds its diversity, and this once again reduces itself to unity. But this movement is what is called *Force*...the difference, then, is posited by the Understanding in such a way that, at the same time, it is expressly stated that the difference is not a *difference belonging to the thing itself*. (5, p.81)
>
> What is present here is not merely bare unity in which *no difference* would be posited, but rather a *movement* in which a *distinction is certainly made but*, because it is no distinction, is *again cancelled*. (5, p.95)
>
> It is clear that this movement is nothing else than the movement of perceiving, in which the two sides, the percipient and what is perceived, are...just as much in a unity, as this unity, which appears as the middle term over against the independent extremes, is a perpetual diremption of itself into just these extremes which exist only through this process. (5, p.82)
>
> This...may be called the simple essence of life, the soul of the world, the universal blood, whose omnipresence is neither disturbed nor interupted by any difference, but rather is itself every difference, as also their supersession; it pulsates within itself but does not move, inwardly vibrates, yet is at rest. It is self-identical, for the differences are tautological; they are differences that are none. (5, p.100)

We point out a few observations about the change of aspect: First, it happens instantaneously. There is no delay to speak of, where in between seeing the duck or rabbit we see a "meaningless" figure or an indifferent collection of lines. Furthermore, we do not see anything like a continuous transition of interpretations between the duck and rabbit: rather, the change of aspect appears as a radical discontinuity in interpretation.

We also point out a certain analogy between the interpretations of a figure and the group of rotational symmetries of a figure. Both the change of aspect and a rotational symmetry, in a sense, leave the figure unchanged: the change of aspect leaves the figure completely physically untouched, while applying a rotational symmetry leaves the subset of space occupied by the figure unchanged (though in general a rotational symmetry will, for instance, permute the vertices of a figure, and so change its orientation). Just as a figure must have some initial orientation with respect to an observer, so it must be seen under some interpretation, some "aspect".[2] To what extent we may speak of a "group of interpretations" of an object is not yet clear.

There is also a clear connection between the change of aspect and what Frege distinguished as "sense" and "reference":

> It is natural...to think of there being connected with a sign (name, combination of words, letter), besides that to which the sign refers, which may be called the reference of the sign, also what I should like to call the *sense* of the sign, wherein the mode of presentation is contained...the reference of 'evening star' would be the same as that of 'morning star', but not the sense. (3, p.207-208)

Thus one could refer to the figure as either "the figure depicting the duck" or "the figure depicting the rabbit". One should keep in mind, however, that this purely linguistic difference in "sense" (which Frege was concerned with) is based on a much deeper "perceptual" change of sense ("mode of presentation") of the figure. We could just as easily call the change of aspect a change of sense.[3]

Finally, we note an analogy between what we have refered to as "the figure itself" (that which we see persists through the change of aspect) and what Kant called the

[2]Conversely, any depiction of a rabbit must take on some fixed physical form, i.e. as portrayed from *some* angle with *some* posture.

[3]Note we are using an aspect of the figure in order to refer to it. Thus a change in aspect of the figure corresponds to a change in the sense we might use to refer to it. It might be misleading though to think there is a complete analogy when we substitute "sense" for "aspect of the figure", and "reference" for the "figure itself", because we would not normally say that the rabbit aspect "refers" to the figure itself; rather we would be inclined to say that the figure itself may have been intended to refer to a rabbit. Several geometrically distinct figures may very well be used to refer to the same rabbit from different points of view. Thus it's necessary to say whether you want to take the figure itself or what the figure depicts to be the reference in our analogy. Of course if we wanted to treat everything on the same footing we should have to deal with the fact that being able to be "seen as words" is merely an aspect of the squiggles on this page.

"thing-in-itself" (things as they are apart from our experience of them). While the thing-in-itself is by definition such that it could never enter experience, the figure itself seems to "hide in plain sight" behind its aspects. It is as if we must always look "through" one aspect or another to see the figure, and once we become conscious of this we may feel that the "figure itself" is just as elusive as the "thing-in-itself"[4]. All of this is in complete analogy with the fact that the reference (in Frege's terminology) functions as a *linguistic* thing-in-itself, because the reference must always appear under the guise of some sense, or "mode of presentation". It may be useful to think of references as equivalence classes of the senses which represent them.

We now come to the point: that basic arithmetic propositions such as 2+2=4 are intuited through nothing other than the change of aspect. When contemplating any collection of objects, the "mind's eye" is in the constant process of partitioning and repartitioning the collection into sub-collections and collections of collections, each partition transitioning instantly into the next, exactly in the manner that the rabbit transitions into the duck, or as the figure of the cube changes its orientation. We see that a collection of four objects may also be seen as a collection of two sub-collections consisting of two objects each. 2+2 and 4 represent two different ways to "synthesize", in the Kantian sense, the same collection, and we know the collection is the same because, though the aspect of the collection has changed, the collection itself has remained exactly as it was.[5]

It may be objected that 2+2=4 is true in some ideal sense, independently of the

[4]We note that objects exhibit a "double hiding": In Kant's terminology, the rabbit-in-itself hides behind its appearances given to us in intuition, while (by our discussion above) in turn each of these appearances in isolation hides behind its rabbit aspect (among other aspects). In a formula, the rabbit hides behind its appearances, while each of these appearances hides behind our interpretation of it *as* a rabbit. This is why there is some ambiguity as to whether we should take the figure itself or its rabbit aspect to be the reference in our analogy (see footnote 3).

[5]If our treatment seems too dependent on the faculty of sight, we could easily produce examples of synthesis with the other senses; e.g. we can hear a chord as a single musical object or as being made up of a plurality of distinct notes, etc.

world and our minds, for some ideal entities "2" and"4", but the ideality of arithmetic consists precisely in the fact that it does not matter which objects[6] we use to intuit arithmetic truths (and in fact we might define as objects anything that we can apply the rules of arithmetic to). The concept of a number is no more and no less ideal than the concept of an object. The change of aspect, and thus arithmetic, may be thought of as the place where subjectivity and objectivity coincide (see the third Hegel quote above).[7]

To point out apparent failures of arithmetic, such as where "adding" one heap to another heap produces just another single heap (and so would be an apparent demonstration that 1+1=1), is only to point out that the union of two sets is itself a single set, and that in turn the sum of two numbers is a single number. It is not surprising that in a textbook of Set Theory, before defining the sum of the cardinal numbers of two disjoint sets A and B to be the cardinality of their union $\#(A \cup B)$, we find the following preamble:

> Our definitions require no comment; they correspond in the most natural way to our intuitive understanding of the process of adding and multiplying whole numbers. (7, p.152)

Rather than being problematic, the fact that one thing can be seen as many and that many can be seen as one is the soul of arithmetic (if not the world). Plato knew this:

> 'But surely sight of it does involve this, and in no small degree, he said:

[6] Of course these could even be imagined (visualized, etc) objects.

[7] We feel no need to introduce a separate "Platonic realm of ideal forms", which would be a sort of mathematical counterpart to Kant's thing-in-itself: it is supposed to be where the "true" or "unconditioned" circle and number 2 exist, while we only ever encounter "conditioned" representations of them. The author must confess that he has no conception of what sort of existence the number 2 has "in itself". The only experience we have with the number 2 is that intuition given with two *objects*.

> for we do see the same thing at the same time as one and as unlimited in multitude.'
>
> 'Then since one, I replied, the whole of number is also affected in this same way?'
>
> 'Of course.' (8, p.241)

We must indicate that, contra Plato above, we do not really see both aspects "at the same time": this is obvious from the two figures above. If we saw both at once, there would be no perception of a *change* in aspect, or transition in sense, etc. The change of aspect appears to obey the law of non-contradiction in its own manner: firstly, because the object remains as it is; and secondly, because we do not see the two different interpretations at the exact same time. There would be no sense of novelty if we saw all the aspects at once. Wittgenstein wrote:

> If I see that a figure possesses an organization which previously I hadn't noticed, I now see a different figure.Thus I can see |||||| as a special case of || || || or of ||| ||| or of | |||| | etc...
>
> Understanding a Gregorian mode doesn't mean getting used to the sequence of notes in the sense in which I can get used to a smell and after a while cease to find it unpleasant. No, it means hearing something new, which I haven't heard before, much in the same way—in fact it's a complete analogy—as it would be if I were suddenly able to see 10 strokes ||||||||||, which I had hitherto only been able to see as twice five strokes, as a characteristic whole. Or suddenly seeing the picture of a cube as a 3-dimensional when I had previously only been able to see it as a flat pattern. (8, p.281)

It is this novelty that may shed light on why Kant insisted that arithmetic statements such as 5+7=12 are *synthetic*, as opposed to being *analytic*:

> Analytic judgments are...those in which the connection of the predicate with the subject is thought through identity, while those in which this connection is thought without identity should be called synthetic...in the former nothing is added through the predicate to the concept of the subject, and the concept

> is only analysed and broken up into its constituent concepts which had all along been thought in it...while the latter add to the concept of the subject a predicate that had not been thought in it at all, and that could not be extracted from it by any analysis. (6, p.43).
>
> The propositions that if equals be added to equals the wholes are equal, and if equals be taken from equals the remainders are equal, are analytic propositions, because I am immediately conscious of the identity of the production of the one magnitude with the production of the other.
>
> 7+5=12 is not an analytic proposition. For neither in the representation of 7, nor in that of 5, nor in that of the combination of both, do I think the number 12 (That I am meant to think this number in the addition of the two is not the point here; for in an analytic proposition the question is only whether I actually think the predicate in the subject). (6, p.194)
>
> That 5 should be added to 7, this I had no doubt already thought in the concept of a sum =7+5, but not that this sum be equal to the number 12. An arithmetical proposition is therefore always synthetic, which is seen more easily still when we take larger numbers, where it is evident that, turn and twist our concepts as we may, we could never, by means of the mere analysis of our concepts and without the help of intuition, arrive at the sum. (6, p.47)

The fact that Kant requires us to be "immediately conscious" of the truth of an analytic proposition suggests that the change of aspect is not an analytic process: for we do not see the two aspects "immediately"; that is, at the same time. When Kant says it's entirely possible to consider 5+7 without thinking of 12, it is totally analogous to only seeing, for instance, the rabbit aspect of the figure, without noticing that it also resembles a duck. The fact that it is possible to see something new in 7+5=12 is what makes Kant say this proposition is synthetic. Along these lines, we might suggest modifying Kant's definition of *a priori* knowledge ("knowledge absolutely independent of all experience") to be "knowledge absolutely independent of any *particular* experience" in the case of *synthetic a priori* knowledge. Though a mathematical result is *a priori*, we still need the "experience" of *doing* the proof, the construction, the calculation, i.e. the synthesis, in order to know the result. Because Kant was not clear enough about his terms, arguments over just which judgments are to be considered analytic and which synthetic continue to this day, and it was not long before Hegel defined the terms to his own liking and argued contra Kant:

> ...the science of analysis possesses not so much *theorems* as *problems*. The analytical theorem contains the problem as already solved for it...the first side [of 7+5=12] *demands* that 5 and 7 shall be combined in *one* expression...just as 5 is the result of a counting up in which the counting was quite arbitrarily broken off and could just have well been continued, so now, in just the same way, the counting is to be continued with the condition that the ones to be added shall be seven. The 12 is therefore a result of 5 and 7 and of an operation which is already posited and in its nature is an act...devoid of any thought, so that it can be performed even by a machine...it is a mere continuation, that is, *repetition*, of the same operation that produced 5 and 7...the *proof* of a theorem of this kind—and it would require a proof if it were a synthetic proposition—would consist merely in the operation of counting on from 5 for a further 7 ones and in discerning the agreement of the result of this counting with what is otherwise called 12. (4, p.791)

The belief that Kant was wrong about his own terminology persisted all the way to the work of Frege:

> The conclusions we draw from it [arithmetic] extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic[8]. The truth is that they are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house. (2, p.101)

Kant's own example of an analytic proposition, "All bodies are extended", indicates that "analysis", in the Kantian sense, follows from a definition: a body, in Kant's view,

[8] To be fair to Frege, he later states that "I do not claim to have made the analytic character of mathematical propositions more than probable, because it can still always be doubted that they are deducible from purely logical laws, or whether some other type of premiss is not involved at some point in their proof without our noticing it...On these lines what is synthetic and based on intuition cannot be sharply separated from what is analytic. Nor shall we succeed in compiling with certainty a complete set of axioms of intuition, such that from them alone we can derive, by means of the laws of logic, every proof in mathematics."(2, p.102-103) Gödel would later show that even arithmetic cannot be derived this way, let alone "every proof in mathematics".

is by definition extended, so that the proposition "All bodies are extended" should be taken neither as the announcement of an empirical discovery nor an a priori synthesis wherein we learn something new about "bodies". That "bodies are extended" is merely to repeat what we laid down in advance. The point is that, though the sum of 7+5 is determined *a priori* "in advance", we ourselves *do not know that the sum is =12 until we do the synthesis.*

This is related to the confused notion that arithmetic propositions are "true by definition"; it is clear, however, that 2+2=4 cannot "follow from definitions" in the sense that "all bachelors are unmarried" or "all bodies are extended" do. When we define "bachelor" to denote an unmarried man, we have given meaning to what was before an empty sound, "bachelor". Thus when we say that we "define" numbers inductively by 1+1=2, 1+2=3, 1+3=4, etc, all we can have accomplished is to define which numbers the *numeral signs* '1', '2', '3'...etc, shall stand for (and so we give meaning to the empty signs '1', '2', '3'). Such definitions would be of no use to someone who didn't understand what we mean by "numbers". It is senseless to think that we could be in possession of the concept of a number without also having to place it in a number *system* (or that we could imagine any extension without thereby bringing the whole of Euclidean space into play), and it is to this extent that arithmetic propositions are true by the very "meaning" of the terms appearing in them.

When we determine the truth tables for logical connectives, however, it seems that there is nothing but the *meaning*[9] of the terms "and", "not", "or", etc, that can lead us to posit that (for instance) the conjunction of two truths should itself be true. Though evaluating the truth value of a complicated truth function may take some effort (and so may appear to have the same synthetic quality as an arithmetic calculation), the

[9]Wittgenstein might have objected that truth tables provide the "rules of usage" for logical connectives, and that their meaning is "constituted" by these rules, but we are still left with the question: why is it that *just these rules* turn out to be so useful in practice?

difference is to be seen in how we obtain the basic statements (e.g. 1+1=2 and not-T=F) on which we base all our other calculations. The determination of the truth tables of logical connectives does not seem to depend upon a change of aspect or a synthesis in the way that arithmetic does: nothing *new* is to be found in a formula like not-T=F (and similarly for the tables of "and", "or", etc). While it would be tempting to assert that the entire truth function calculus is a formalization of the way we "synthesize" the truth values of the component propositions of a statement into a single truth value, at each step our judgment is *analytic*. Sensuous intuition seems to have far less a role to play here than in arithmetic; hence the feeling that logic is "empty". This is not to say that logic is a mere game symbols: once the calculus has been laid down, we can of course treat it as a "game with symbols", but it would be disingenuous to pretend that the "rules" of the game were invented in as arbitrary a manner as, for instance, the rules of checkers.[10]

Tautological and contradictory truth-functional schemata, such as "p or not-p" and "p and not-p" respectively, can be characterized as "constant" truth functions: they are invariant under a change in the truth values of their component propositions. They are "trivial" in the sense that they are only apparently functions of the truth values of their component propositions, in the way that $f(x) = x - x = 0$ is only apparently a function of x. Wittgenstein eventually argued against the view that arithmetic propositions are derived from logical tautologies:

> If you write '(E |||||) and (E |||||||) implies (E ||||||||||||)' [call it the proposition A], you may be in doubt as to how I obtained the numerical sign in the right-hand bracket if I dont know that it is the result of adding the two left-hand

[10]In logic and mathematics there is the temptation to assert, in addition to our "rules" such as not-T=F, that these rules describe something objectively *true*. This temptation never arises with the rules of a mere game: it would be very strange to insist, after telling someone the rules of chess, that these rules are also "true". This is because we do not think of the rules we stipulate for games as *judgments* about anything.

> signs. I believe that makes it clear that this expression is only an application of 5+7=12 but doesn't represent this equation itself.
>
> Suppose I wrote out the proposition A but put the wrong number of strokes in the right-hand bracket, then you would and could only come upon this mistake by comparing the structures, not by applying theorems of logic. If I asked you how do you know that this number of strokes in the right-hand bracket is correct, I can only justify it by a comparison of the structures...For if I look at it as a tautology I merely perceive features of its structure and can now perceive the addition theorem *in* them, while disregarding other characteristics that are essential to it as a proposition. The addition theorem is in this way to be recognized *in* it (among other places), not *by means* of it. (8, p.126-127)

He then dismisses an argument very similar to that of Hegel's against Kant (though we do not know if Wittgenstein was aware of Hegel's argument):

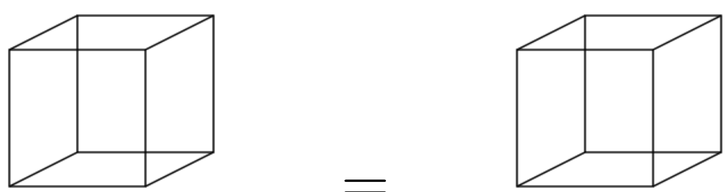
> You could reply:....I combine the first five strokes of the of the right-hand bracket, which stand in 1-1 correspondence with the five in one of the left-hand brackets, with the remaining 7 strokes, which stand in 1-1 correspondence with the seven in the other left-hand bracket, to make 12 strokes...But even if I followed this train of thought, the fundamental insight would still remain, that the 5 strokes and the 7 combine *precisely to make 12* (and so for example to make the same structure as do 4 and 4 and 4). It is always only insight into the internal relations of the structures and not some proposition or other or some logical consideration which tells us this. And, as far as this insight is concerned, everything in the tautology apart from the numerical structures is mere decoration; they are all that matters for the arithmetical proposition (Everything else belongs to the *application* of the arithmetical proposition). (8, p.127)

He then vindicates Kant in an uncharacteristic appeal to "direct insight"[11]:

> No investigation of concepts, only direct insight can tell us that 3+2=5...What I said earlier about the nature of arithmetical equations and about an equation's not being replaceable by a tautology explains—I believe—what Kant means when he insists that 7+5=12 is not an analytic proposition, but synthetic *a priori*. (8, p.127-129)

[11]In our edition there is an editorial note indicating there was 'a mark of dissatisfaction under the words "direct insight"'.

Any attempt to explain arithmetic judgments by recourse to a one-to-one correspondence completely obscures the fact that we see directly that 2+2=4: during the change of aspect, there are not two collections before us, one consisting of 4 objects and the other consisting of 2+2, that we ”match up” in order to draw the conclusion. In fact, one-to-one correspondence in itself can only tell us that two collections have the *same* number; it cannot tell us *what* that number is. The one-to-one correspondence in the change of aspect, if there is any to speak of at all, consists only in the fact that the object itself has remained completely as it was. In other words, the one-to-one correspondence reduces here to the law of identity: $X = X$. If we attempted to describe the ”equivalence” of the two orientations of the cube in this form we should have the useless equation[12]:



which fails spectacularly in expressing the change of aspect, and gives some indication that the change of aspect cannot itself be depicted.[13] It is clear that I cannot force you to see the aspect I intend when I draw a picture. I cannot put the aspect I intend ”in” the picture, in the same sense that if I am forced to communicate with you by a specific code, it would be of no use for me to tell you how to decipher the code *from within the language of that code*.

If (as Frege noted) our equations are to have any content, if '$A = B$' is to express anything other than that B is defined as A, we must work on the level of the ”sense” rather than the ”reference”. The 'X' in '$X = X$' above refers to ”the figure itself”,

[12]See the second of Wittgenstein's remarks about the change of aspect above.

[13]Compare this to Wittgenstein in the *Tractatus*:”What a picture must have in common with reality, in order to be able to depict it–correctly or incorrectly—in the way it does, is its pictoral form. A picture cannot, however, depict its pictorial form: it displays it.”

which we see remains the same. If, however, by 'A' and 'B' we denote the different aspects, it is tempting to write $A = X = B$, and think of the figure X as "mediating" between its different aspects A and B. This notation, however, would lead us to confuse the aspects of the figure with the figure itself. The figure is not literally the duck aspect; as Wittgenstein notes, if you are not familiar with ducks, you will not see the figure as a duck. It may be more useful to think of the "group of aspects" as the "preimage of X under our interpretation". We could even consider X as a subset of the space Y it is situated in, and think of the group of aspects of X to be the group of aspects $F(X)$ associated to X by the "interpretation sheaf F on Y". This way of speaking keeps senses on another level than their reference, and lets us think of interpretation as a "bundle projection" from senses to their reference (from aspects to the object they belong to), or as a sheaf associating to subspaces of some space their groups of aspects.[14]

[14]Recently an (admittedly incredible) attempt to apply sheaf-theoretic ideas to phenomenology has been carried out by Badiou (who, by the way, also claims to have shown Kant was wrong about 7+5=12 being synthetic); it is too bad the entire weight of his system depends on his

> ...function of appearing, which, given two elements of that world, measures their degree of identity...But what are the values of the function of appearing? What measures the degree of identity between two appearances of multiplicities? Here...there is no general or totalizing answer. (1, p.156)

With this concession we have already left the realm of phenomenology and have put our faith into a hypothetical formalism.

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