Advanced Calculus, Dr. Block, Chapter 1 notes, 9-10-2019

1. In this class, we adopt an informal approach to set theory. A set is a collection of things called elements. We use the notation $x \in A$ to denote that x is an element of the set A. We use the notation $x \notin A$ to denote that x is not an element of the set A. Two sets are equal if and only if they contain exactly the same elements. A set S may be either finite or infinite. If S is a finite set, the cardinality of S is the number of elements in S.

2. The unique set with cardinality zero is called the empty set and denoted ϕ.

3. We let $\mathbb{R}$ denote the set of real numbers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, and $\mathbb{N}$ the set of positive integers, $\mathbb{N} = \{1, 2, 3, \dots\}$. Note that
$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

4. We often use set builder notation to define a set. For example, the set of rational numbers is given by
$$\mathbb{Q} = \{x \in \mathbb{R} | x = \frac{p}{q} \text{ for some}\, p, q \in \mathbb{Z}\, \text{with}\, q \neq 0\}.$$

5. Suppose that A and B are sets. We let $A \times B$ denote the set of ordered pairs (a, b) such that $a \in A$ and $b \in B$. Two ordered pairs (c, d) and (v, w) are equal if and only if $c = v$ and $d = w$.

6. More generally, if n is a positive integer and $A_1, A_2, \dots A_n$ are sets, we define the Cartesian poroduct of these sets by
$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \dots, x_n) : x_1 \in A, x_2 \in A_2, \dots, x_n \in A_n\}.$$
The expression $(x_1, x_2, \dots, x_n)$ is called an ordered n-tuple. Two ordered n-tuples $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ are equal if and only if $x_i = y_i$ for each $i = 1, 2, \dots, n$. Note the meaning of $\dots$ (dots).

7. If A is a set and n is a positive integer we define the Cartesian power A^n by
$$A^n = A_1 \times A_2 \times \cdots \times A_n,$$
where $A_i = A$ for each $i = 1, 2, \dots, n$.

8. Suppose that A and B are sets. We say that A is a subset of B, denoted $A \subseteq B$, if and only if every element of A is also an element of B. Note that two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$. Using this to prove that two sets are equal is sometimes called the method of double containment.

9. Note that for any set A, we have $\phi \subseteq A$.

10. Suppose that A and B are sets. There exist sets $A \cap B$, $A \cup B$, and $A \setminus B$ given by

    $x \in A \cap B$ if and only if $x \in A$ and $x \in B$,

    $x \in A \cup B$ if and only if $x \in A$ or $x \in B$,

    $x \in A \setminus B$ if and only if $x \in A$ and $x \notin B$.

    The set $A \cap B$ is called the intersection of the sets A and B. The set $A \cup B$ is called the union of the sets A and B. The set $A \setminus B$ is called the complement of B in A.

11. Suppose that S is a set, and for each $s \in S$, a set A_s is defined. We assume that there are sets denoted by $\bigcup_{s \in S} A_s$ and $\bigcap_{s \in S} A_s$ such that $x \in \bigcup_{s \in S} A_s$ if and only if there exists $s \in S$ with $x \in A_s$, and $x \in \bigcap_{s \in S} A_s$ if and only if for every $s \in S$ we have $x \in A_s$.

    The set S is called an index set, the family of sets A_s is called an indexed family of sets, the set $\bigcup_{s \in S} A_s$ is called the union of the indexed family of sets, and the set $\bigcap_{s \in S} A_s$ is called the intersection of the indexed family of sets.

    If $S = \{1, 2, \dots, n\}$, instead of $\bigcup_{s \in S} A_s$ we often write $\bigcup_{i=1}^{n} A_i$ or $A_1 \cup A_2 \cup \cdots \cup A_n$.

    If $S = \mathbb{N}$, instead of $\bigcup_{s \in S} A_s$ we often use the notation $\bigcup_{i=1}^{\infty} A_i$ or $A_1 \cup A_2 \cup \dots$.

    The same is true for $\bigcap_{s \in S} A_s$.

12. Definition. Suppose that A and B are sets. A relation from A to B is just a subset of $A \times B$.

13. Suppose that f is a relation from A to B. We say that f is a function from A to B if and only if for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$. We use the notation $f : A \to B$ to indicate that f is a function from A to B. Also, if $a \in A$, we let $f(a)$ denote the unique $b \in B$ such that $(a, b) \in f$. The set A is called the domain of the function. The set B is called the target space of the function. The range of the function is the set of all $y \in B$ such that there exists $x \in X$ with $f(x) = y$.

14. Remark. Suppose that f and g are functions from A to B. Then $f = g$ if and only if for all $a \in A$ we have $f(a) = g(a)$.

15. Definition. Suppose that $f : A \to B$ and $g : B \to C$. The composition $g \circ f : A \to C$ is defined as follows: For $a \in A$ set $(g \circ f)(a) = g(f(a))$.

16. Definition and Remark. Suppose that $f : A \to B$. We say that f is injective or one-to-one if and only if for all $a_1 \in A$ and $a_2 \in A$ if $f(a_1) = f(a_2)$ then $a_1 = a_2$. We say that f is surjective or onto if and only if for every $b \in B$ there exists $a \in A$ with $f(a) = b$. Note that f is onto if and only if B is the range of f.

17. Definition and Remark. Suppose that $f : A \to B$. Suppose that $D \subseteq A$. The image of D under f is given by

$$f(D) = \{y \in B | \, \exists x \in D \, \mathrm{with} \, f(x) = y\}.$$

Here, the symbol $\exists$ means "there exists". Note that the image of A under f is the range of f.

18. Definition and Remark. Suppose that $f : A \to B$. Suppose that $E \subseteq B$. The inverse image of E under f is given by

$$f^{-1}(E) = \{x \in A | f(x) \in E\}.$$

Note that the inverse image of a set under f is defined for all functions f, and is independent of the existence of an inverse function.

19. Axiom. Every nonempty subset of $\mathbb{N}$ has a smallest element.

20. Theorem. (Mathematical Induction). Suppose $j \in \mathbb{N}$. Suppose that $P(x)$ is a statement for each $x \in \mathbb{N}$. Suppose that

    1. $P(j)$ and

    2. For all $k \in \mathbb{N}$ with $k \geq j$ if $P(k)$ holds then $P(k+1)$ also holds.

    Then for all $n \in \mathbb{N}$ with $n \geq j$ we have $P(n)$.

21. Remark. Similar to Mathematical Induction, we sometimes use recusive definitions. We may define a function f with domain $\mathbb{N}$ by defining $f(0)$ and defining $f(k+1)$ in terms of $f(k)$. For example, if x is a real number we may define x^n by

    $x^0 = 1$ and $x^{(k+1)} = xx^k$.

22. Theorem. (Mathematical Induction, Strong Form). Suppose $j \in \mathbb{N}$. Suppose that $P(x)$ is a statement for each $x \in \mathbb{N}$. Suppose that

    1. $P(j)$ and

    2. For all $k \in \mathbb{N}$ with $k \geq j$ if P(s) holds for all $s \in \mathbb{N}$ with $j \leq s \leq k$ then $P(k+1)$ also holds.

    Then for all $n \in \mathbb{N}$ with $n \geq j$ we have $P(n)$.

23. Theorem. (Binomial Theorem). Suppose that $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

24. Definition. Suppose that R is a relation from A to B. We define the inverse relation R^{-1} from B to A by

$$R^{-1} = \{(y,x) \in B \times A | (x,y) \in R\}.$$

25. Definition. Suppose that S is a set. The **identity function** on S is the function $i_S : S \to S$ given by $i_S(x) = x$ for all $x \in S$.

26. Remark and Theorem. Suppose that $f : A \to B$. Then f is also a relation from A to B. So the inverse relation f^{-1} is defined and is a relation from B to A. We have the following theorem: f^{-1} is a function from B to A if and only if f is one-to-one and onto. Moreover, in this case we have $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

27. Theorem: Suppose that $f : A \to B$ and $g : B \to A$. Suppose also that $g \circ f = i_A$ and $f \circ g = i_B$. Then $g = f^{-1}$.

28. Definition. A **field** is a triple, $(\mathbb{F}, +, \cdot)$, where $\mathbb{F}$ is a set and $+$ and $\cdot$ are binary operations (functions from $\mathbb{F} \times \mathbb{F}$ to $\mathbb{F}$) called addition and multiplication respectively satisfying the following:

    For every $x, y \in \mathbb{F}$ we have $x + y = y + x$ and $xy = yx$;

    For every $x, y, z \in \mathbb{F}$ we have $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$;

    There is an element $0 \in \mathbb{F}$ such that $0 + w = w$ for every $w \in \mathbb{F}$;

    There is an element $1 \in \mathbb{F}$, distinct from 0, such that $1w = w$ for every $w \in \mathbb{F}$;

    For each $x \in \mathbb{F}$ there is an element $-x \in \mathbb{F}$ such that $x + (-x) = 0$;

    For each $x \neq 0$ in $\mathbb{F}$, there is an element $x^{-1} \in \mathbb{F}$ such that $x \cdot x^{-1} = 1$;

    For every $x, y \in \mathbb{F}$ we have $(x + y)z = xz + yz$.

29. Definition. An ordered field $(\mathbb{F}, +, \cdot, <)$ consists of a field $(\mathbb{F}, +, \cdot)$ and a relation $<$ on $\mathbb{F}$ such that,

    For each $x, y \in \mathbb{F}$, exactly one of the following hold,

    $$x < y, \quad y < x, \quad x = y;$$

    If $x, y, z \in \mathbb{F}$ satisfy $x < y$ and $y < z$, then also $x < z$.

    If $x, y, z \in \mathbb{F}$ and $x < y$, then $x + z < y + z$;

    If $x, y, z \in \mathbb{F}$ satisfy $x < y$ and $0 < z$, then $xz < yz$.

30. Remark. Both $\mathbb{R}$ and $\mathbb{Q}$ with the usual addition, multiplication, and ordering are ordered fields.

31. Remark. In an odered field, from the symbol $<$ we define the symbols $\leq$, $>$ and $\geq$ in the usual way.

32. Remark. Other properties of ordered fields which can be proved from the definition may be found in the text.

33. Definition. Let S be a subset of an ordered field $\mathbb{F}$. We say that S is **bounded above** if and only if there is a $b \in \mathbb{F}$ such that $b \geq s$ for all $s \in S$. Any $b \in \mathbb{F}$ such that $b \geq s$ for all $s \in S$ is called an **upper bound** for S.

34. Lemma. Let S be a subset of an ordered field $\mathbb{F}$ and suppose that both b and b' are upper bounds for S. If b and b' both have the property that if $c \in \mathbb{F}$ is an upper bound for S, then $c \geq b$ and $c \geq b'$, then $b = b'$.

35. Definition. The **least upper bound** or **supremum** of a subset S of an ordered field $\mathbb{F}$, if it exists, is a $b \in \mathbb{F}$ such that b is an upper bound for S; and if $c \in \mathbb{F}$ is an upper bound for S, then $c \geq b$.

36. Remark. The Lemma above justifies the use of the phrase "the least upper bound" as opposed to "a least upper bound."

37. Definition. An ordered field $\mathbb{F}$ is call a **complete** ordered field if and only if every nonempty subset of $\mathbb{F}$ which is bounded above has a least upper bound.

38. Axiom. The set of real numbers $\mathbb{R}$ (with the usual addition, multiplication, and ordering) is a complete ordered field.

39. Theorem. Suppose that S is a nonempty subset of $\mathbb{R}$ and k is an upper bound of S. Then k is the least upper bound of S if and only if for every $\epsilon > 0$ there exists $s \in S$ such that $k - \epsilon < s$.

40. Definition. Let S be a subset of an ordered field $\mathbb{F}$. We say that S is **bounded below** if and only if there is a $b \in \mathbb{F}$ such that $b \leq s$ for all $s \in S$. Any $b \in \mathbb{F}$ such that $b \leq s$ for all $s \in S$ is called a **lower bound** for S.

41. Lemma. Let S be a subset of an ordered field $\mathbb{F}$ and suppose that both b and b' are lower bounds for S. If b and b' both have the property that if $c \in \mathbb{F}$ is a lower bound for S, then $c \leq b$ and $c \leq b'$, then $b = b'$.

42. Definition. The **greatest lower bound** or **infimum** of a subset S of an ordered field $\mathbb{F}$, if it exists, is a $b \in \mathbb{F}$ such that b is a lower bound for S; and if $c \in \mathbb{F}$ is an upper bound for S, then $c \leq b$.

43. Remark. The Lemma above justifies the use of the phrase "the greatest lower bound" as opposed to "a greatest lower bound."

44. Theorem. Every nonempty subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.

45. Theorem. Suppose that a and b are real numbers with $a < b$. Then the open interval (a, b) contains both a rational number and an irrational number.

46. Theorem. (Archimedean Order Property of $\mathbb{R}$). If $x \in \mathbb{R}$, then there is a natural number greater that x.

47. Definition. Let $x \in \mathbb{R}$. The absolute value of x is denoted $|x|$ and defined by:

    $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

48. Theorem. (Properties of Absolute Value). Suppose that $x, y \in \mathbb{R}$. Then:

    $|x| \geq 0$;

    $|x| < y$ if and only if $y > 0$ and $-y < x < y$;

    $|x| \geq y$ if and only if $y \leq 0$ or $x \leq -y$ or $x \geq y$;

    $|x \cdot y| = |x| \cdot |y|$;

    $|x + y| \leq |x| + |y|$.