1. In this class, we adopt an informal approach to set theory. A set is a collection of things called elements. We use the notation $x \in A$ to denote that $x$ is an element of the set $A$. We use the notation $x \notin A$ to denote that $x$ is not an element of the set $A$. Two sets are equal if and only if they contain exactly the same elements. A set $S$ may be either finite or infinite. If $S$ is a finite set, the cardinality of $S$ is the number of elements in $S$.

2. The unique set with cardinality zero is called the empty set and denoted $\emptyset$.

3. We let $\mathbb{R}$ denote the set of real numbers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$, and $\mathbb{N}$ the set of positive integers, $\mathbb{N} = \{1, 2, 3, \ldots \}$. Note that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

4. We often use set builder notation to define a set. For example, the set of rational numbers is given by

$$\mathbb{Q} = \{x \in \mathbb{R} | x = \frac{p}{q} \text{ for some } p, q \in \mathbb{Z} \text{ with } q \neq 0\}.$$

5. Suppose that $A$ and $B$ are sets. We let $A \times B$ denote the set of ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$. Two ordered pairs $(c, d)$ and $(v, w)$ are equal if and only if $c = v$ and $d = w$.

6. More generally, if $n$ is a positive integer and $A_1, A_2, \ldots A_n$ are sets, we define the Cartesian product of these sets by

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) : x_1 \in A, x_2 \in A_2, \ldots, x_n \in A_n\}.$$

The expression $(x_1, x_2, \ldots, x_n)$ is called an ordered $n$-tuple. Two ordered $n$-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are equal if and only if $x_i = y_i$ for each $i = 1, 2, \ldots, n$. Note the meaning of $\ldots$ (dots).
7. If $A$ is a set and $n$ is a positive integer we define the Cartesian power $A^n$ by

$$A^n = A_1 \times A_2 \times \cdots \times A_n,$$

where $A_i = A$ for each $i = 1, 2, \ldots, n$.

8. Suppose that $A$ and $B$ are sets. We say that $A$ is a subset of $B$, denoted $A \subseteq B$, if and only if every element of $A$ is also an element of $B$. Note that two sets $A$ and $B$ are equal if and only if $A \subseteq B$ and $B \subseteq A$. Using this to prove that two sets are equal is sometimes called the method of double containment.

9. Note that for any set $A$, we have $\emptyset \subseteq A$.

10. Suppose that $A$ and $B$ are sets. There exist sets $A \cap B$, $A \cup B$, and $A \setminus B$ given by

$x \in A \cap B$ if and only if $x \in A$ and $x \in B$,

$x \in A \cup B$ if and only if $x \in A$ or $x \in B$,

$x \in A \setminus B$ if and only if $x \in A$ and $x \notin B$.

The set $A \cap B$ is called the intersection of the sets $A$ and $B$. The set $A \cup B$ is called the union of the sets $A$ and $B$. The set $A \setminus B$ is called the complement of $B$ in $A$.

11. Suppose that $S$ is a set, and for each $s \in S$, a set $A_s$ is defined. We assume that there are sets denoted by $\bigcup_{s \in S} A_s$ and $\bigcap_{s \in S} A_s$ such that $x \in \bigcup_{s \in S} A_s$ if and only if there exists $s \in S$ with $x \in A_s$, and $x \in \bigcap_{s \in S} A_s$ if and only if for every $s \in S$ we have $x \in A_s$.

The set $S$ is called an index set, the family of sets $A_s$ is called an indexed family of sets, the set $\bigcup_{s \in S} A_s$ is called the union of the indexed family of sets, and the set $\bigcap_{s \in S} A_s$ is called the intersection of the indexed family of sets.

If $S = \{1, 2, \ldots, n\}$, instead of $\bigcup_{s \in S} A_s$ we often write $\bigcup_{i=1}^n A_i$ or $A_1 \cup A_2 \cup \cdots \cup A_n$.

If $S = \mathbb{N}$, instead of $\bigcup_{s \in S} A_s$ we often use the notation $\bigcup_{i=1}^\infty A_i$ or $A_1 \cup A_2 \cup \ldots$.

The same is true for $\bigcap_{s \in S} A_s$. 
12. Definition. Suppose that $A$ and $B$ are sets. A relation from $A$ to $B$ is just a subset of $A \times B$.

13. Suppose that $f$ is a relation from $A$ to $B$. We say that $f$ is a function from $A$ to $B$ if and only if for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$. We use the notation $f : A \to B$ to indicate that $f$ is a function from $A$ to $B$. Also, if $a \in A$, we let $f(a)$ denote the unique $b \in B$ such that $(a, b) \in f$. The set $A$ is called the domain of the function. The set $B$ is called the target space of the function. The range of the function is the set of all $y \in B$ such that there exists $x \in X$ with $f(x) = y$.

14. Remark. Suppose that $f$ and $g$ are functions from $A$ to $B$. Then $f = g$ if and only if for all $a \in A$ we have $f(a) = g(a)$.

15. Definition. Suppose that $f : A \to B$ and $g : B \to C$. The composition $g \circ f : A \to C$ is defined as follows: For $a \in A$ set $(g \circ f)(a) = g(f(a))$.

16. Definition and Remark. Suppose that $f : A \to B$. We say that $f$ is injective or one-to-one if and only if for all $a_1, a_2 \in A$ if $f(a_1) = f(a_2)$ then $a_1 = a_2$. We say that $f$ is surjective or onto if and only if for every $b \in B$ there exists $a \in A$ with $f(a) = b$. Note that $f$ is onto if and only if $B$ is the range of $f$.

17. Definition and Remark. Suppose that $f : A \to B$. Suppose that $D \subseteq A$. The image of $D$ under $f$ is given by

$$f(D) = \{y \in B | \exists x \in D \text{ with } f(x) = y\}.$$ 

Here, the symbol $\exists$ means ”there exists”. Note that the image of $A$ under $f$ is the range of $f$.

18. Definition and Remark. Suppose that $f : A \to B$. Suppose that $E \subseteq B$. The inverse image of $E$ under $f$ is given by

$$f^{-1}(E) = \{x \in A | f(x) \in E\}.$$ 

Note that the inverse image of a set under $f$ is defined for all functions $f$, and is independent of the existence of an inverse function.
19. Axiom. Every nonempty subset of \( \mathbb{N} \) has a smallest element.

20. Theorem. (Mathematical Induction). Suppose \( j \in \mathbb{N} \). Suppose that 
   \( P(x) \) is a statement for each \( x \in \mathbb{N} \). Suppose that
   1. \( P(j) \) and
   2. For all \( k \in \mathbb{N} \) with \( k \geq j \) if \( P(k) \) holds then \( P(k + 1) \) also holds.
   Then for all \( n \in \mathbb{N} \) with \( n \geq j \) we have \( P(n) \).

21. Remark. Similar to Mathematical Induction, we sometimes use recursive definitions. We may define a function \( f \) with domain \( \mathbb{N} \) by defining \( f(0) \) and defining \( f(k + 1) \) in terms of \( f(k) \). For example, if \( x \) is a real number we may define \( x^n \) by 
   \( x^0 = 1 \) and \( x^{(k+1)} = xx^k \).

22. Theorem. (Mathematical Induction, Strong Form). Suppose \( j \in \mathbb{N} \). Suppose that \( P(x) \) is a statement for each \( x \in \mathbb{N} \). Suppose that
   1. \( P(j) \) and
   2. For all \( k \in \mathbb{N} \) with \( k \geq j \) if \( P(s) \) holds for all \( s \in \mathbb{N} \) with \( j \leq s \leq k \) then \( P(k + 1) \) also holds.
   Then for all \( n \in \mathbb{N} \) with \( n \geq j \) we have \( P(n) \).

23. Theorem. (Binomial Theorem). Suppose that \( a, b \in \mathbb{R} \) and \( n \in \mathbb{N} \). Then
   \[
   (a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,
   \]
   where
   \[
   \binom{n}{k} = \frac{n!}{k!(n-k)!}.
   \]

24. Definition. Suppose that \( R \) is a relation from \( A \) to \( B \). We define the inverse relation \( R^{-1} \) from \( B \) to \( A \) by
   \[
   R^{-1} = \{(y, x) \in B \times A | (x, y) \in R\}.
   \]

25. Definition. Suppose that \( S \) is a set. The identity function on \( S \) is the function \( i_S : S \to S \) given by \( i_S(x) = x \) for all \( x \in S \).
26. Remark and Theorem. Suppose that \( f : A \rightarrow B \). Then \( f \) is also a relation from \( A \) to \( B \). So the inverse relation \( f^{-1} \) is defined and is a relation from \( B \) to \( A \). We have the following theorem: \( f^{-1} \) is a function from \( B \) to \( A \) if and only if \( f \) is one-to-one and onto. Moreover, in this case we have \( f^{-1} \circ f = i_A \) and \( f \circ f^{-1} = i_B \).

27. Theorem: Suppose that \( f : A \rightarrow B \) and \( g : B \rightarrow A \). Suppose also that \( g \circ f = i_A \) and \( f \circ g = i_B \). Then \( g = f^{-1} \).

28. Definition. A field is a triple, \((\mathbb{F}, +, \cdot)\), where \( \mathbb{F} \) is a set and \( + \) and \( \cdot \) are binary operations (functions from \( \mathbb{F} \times \mathbb{F} \) to \( \mathbb{F} \)) called addition and multiplication respectively satisfying the following:

For every \( x, y \in \mathbb{F} \) we have \( x + y = y + x \) and \( xy = yx \);

For every \( x, y, z \in \mathbb{F} \) we have \( (x + y) + z = x + (y + z) \) and \( (xy)z = x(yz) \);

There is an element \( 0 \in \mathbb{F} \) such that \( 0 + w = w \) for every \( w \in \mathbb{F} \);

There is an element \( 1 \in \mathbb{F} \), distinct from \( 0 \), such that \( 1w = w \) for every \( w \in \mathbb{F} \);

For each \( x \in \mathbb{F} \) there is an element \( -x \in \mathbb{F} \) such that \( x + (-x) = 0 \);

For each \( x \neq 0 \) in \( \mathbb{F} \), there is an element \( x^{-1} \in \mathbb{F} \) such that \( x \cdot x^{-1} = 1 \);

For every \( x, y \in \mathbb{F} \) we have \( (x + y)z = xz + yz \).

29. Definition. An ordered field \((\mathbb{F}, +, \cdot, <)\) consists of a field \((\mathbb{F}, +, \cdot)\) and a relation \(<\) on \( \mathbb{F} \) such that,

For each \( x, y \in \mathbb{F} \), exactly one of the following hold,

\[ x < y, \quad y < x, \quad x = y; \]

If \( x, y, z \in \mathbb{F} \) satisfy \( x < y \) and \( y < z \), then also \( x < z \).

If \( x, y, z \in \mathbb{F} \) and \( x < y \), then \( x + z < y + z \);

If \( x, y, z \in \mathbb{F} \) satisfy \( x < y \) and \( 0 < z \), then \( xz < yz \).

30. Remark. Both \( \mathbb{R} \) and \( \mathbb{Q} \) with the usual addition, multiplication, and ordering are ordered fields.
31. Remark. In an ordered field, from the symbol $<$ we define the symbols $\leq$, $>$ and $\geq$ in the usual way.

32. Remark. Other properties of ordered fields which can be proved from the definition may be found in the text.

33. Definition. Let $S$ be a subset of an ordered field $\mathbb{F}$. We say that $S$ is **bounded above** if and only if there is an $b \in \mathbb{F}$ such that $b \geq s$ for all $s \in S$. Any $b \in \mathbb{F}$ such that $b \geq s$ for all $s \in S$ is called an **upper bound** for $S$.

34. Lemma. Let $S$ be a subset of an ordered field $\mathbb{F}$ and suppose that both $b$ and $b'$ are upper bounds for $S$. If $b$ and $b'$ both have the property that if $c \in \mathbb{F}$ is an upper bound for $S$, then $c \geq b$ and $c \geq b'$, then $b = b'$.

35. Definition. The **least upper bound** or **supremum** of a subset $S$ of an ordered field $\mathbb{F}$, if it exists, is a $b \in \mathbb{F}$ such that $b$ is an upper bound for $S$; and if $c \in \mathbb{F}$ is an upper bound for $S$, then $c \geq b$.

36. Remark. The Lemma above justifies the use of the phrase ”the least upper bound” as opposed to ”a least upper bound.”

37. Definition. An ordered field $\mathbb{F}$ is called a **complete** ordered field if and only if every nonempty subset of $\mathbb{F}$ which is bounded above has a least upper bound.

38. Axiom. The set of real numbers $\mathbb{R}$ (with the usual addition, multiplication, and ordering) is a complete ordered field.

39. Theorem. Suppose that $S$ is a nonempty subset of $\mathbb{R}$ and $k$ is an upper bound of $S$. Then $k$ is the least upper bound of $S$ if and only if for every $\epsilon > 0$ there exists $s \in S$ such that $k - \epsilon < s$.

40. Definition. Let $S$ be a subset of an ordered field $\mathbb{F}$. We say that $S$ is **bounded below** if and only if there is an $b \in \mathbb{F}$ such that $b \leq s$ for all $s \in S$. Any $b \in \mathbb{F}$ such that $b \leq s$ for all $s \in S$ is called a **lower bound** for $S$. 
41. Lemma. Let \( S \) be a subset of an ordered field \( \mathbb{F} \) and suppose that both \( b \) and \( b' \) are lower bounds for \( S \). If \( b \) and \( b' \) both have the property that if \( c \in \mathbb{F} \) is a lower bound for \( S \), then \( c \leq b \) and \( c \leq b' \), then \( b = b' \).

42. Definition. The **greatest lower bound** or **infimum** of a subset \( S \) of an ordered field \( \mathbb{F} \), if it exists, is a \( b \in \mathbb{F} \) such that \( b \) is a lower bound for \( S \); and if \( c \in \mathbb{F} \) is an upper bound for \( S \), then \( c \leq b \).

43. Remark. The Lemma above justifies the use of the phrase "the greatest lower bound" as opposed to "a greatest lower bound."

44. Theorem. Every nonempty subset of \( \mathbb{R} \) which is bounded below has a greatest lower bound.

45. Theorem. Suppose that \( a \) and \( b \) are real numbers with \( a < b \). Then the open interval \((a,b)\) contains both a rational number and an irrational number.

46. Theorem. (Archimedean Order Property of \( \mathbb{R} \)). If \( x \in \mathbb{R} \), then there is a natural number greater that \( x \).

47. Definition. Let \( x \in \mathbb{R} \). The absolute value of \( x \) is denoted \( |x| \) and defined by:
\[
|x| = x \text{ if } x \geq 0 \text{ and } |x| = -x \text{ if } x < 0.
\]

48. Theorem. (Properties of Absolute Value). Suppose that \( x, y \in \mathbb{R} \). Then:
\[
|x| \geq 0;
\]
\[
|x| < y \text{ if and only if } y > 0 \text{ and } -y < x < y;
\]
\[
|x| \geq y \text{ if and only if } y \leq 0 \text{ or } x \leq -y \text{ or } x \geq y;
\]
\[
|x \cdot y| = |x| \cdot |y|;
\]
\[
|x + y| \leq |x| + |y|.
\]