Advanced Calculus 2, Dr. Block, Corrected Lecture Notes, 4-1-2020

We continue discussing material from Section 7.1 of the text.

We begin with an important result.

Theorem 9. (Test for Divergence) If a series $\sum_{n=p}^{\infty} a_n$ converges, then the sequence $\{a_n\}$ converges to zero.

Proof. Suppose that the series $\sum_{n=p}^{\infty} a_n$ converges. Then the sequence of partial sums $\{S_j\}$ converges to some real number S. Suppose that j is a positive integer. We have:

$$S_j = a_p + a_{p+1} + a_{p+2} + \cdots + a_{p+(j-1)}.$$

Also we have

$$S_{j+1} = a_p + a_{p+1} + a_{p+2} + \cdots + a_{p+(j-1)} + a_{p+j}.$$

It follows that

$$a_{p+j} = S_{j+1} - S_j.$$

Since the sequence $\{S_j\}$ converges to S, the sequence S_{j+1} also converges to S. So

$$\lim_{j\to\infty} a_{p+j} = \lim_{j\to\infty} S_{j+1} - \lim_{j\to\infty} S_j = S - S = 0.$$

It follows that the sequence $\{a_n\}$ converges to zero.

□

Recall that if a statement is true, the contrapositive of the statement is also true. The contrapositive of a statement of the form "If P is true, then Q is true" is the statement "If Q is false, then P is false. So we can restate the previous theorem as follows:

Theorem 10. (Test for Divergence, contrapositive form) If a sequence $\{a_n\}$ does not converge to zero, then the series $\sum_{n=p}^{\infty} a_n$ diverges.

This test can be used as follows. Suppose we are asked to determine whether a series $\sum_{n=p}^{\infty} a_n$ converges or diverges. We can evaluate the limit

$$\lim_{n\to\infty} a_n.$$

If the limit does not exist or the limit exists but is not zero, we can conclude that the given series diverges. On the other hand, if

$$\lim_{n\to\infty} a_n = 0,$$

the test gives no information. **The test for divergence can never be used to show convergence.**

It is important to remember that the following statement if false:

False statement If a sequence $\{a_n\}$ converges to zero, then the series $\sum_{n=p}^{\infty} a_n$ converges.

To see that the statement is false, we need to give an example of a sequence $\{a_n\}$ which converges to zero, while the corresponding series $\sum_{n=p}^{\infty} a_n$ diverges. The next theorem does exactly that.

Theorem 11. (harmonic series) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Please see the proof of this theorem in the text on page 297.

Theorem 12. Suppose that $\sum_{n=p}^{\infty} a_n = A$ and $\sum_{n=p}^{\infty} b_n = B$, where A and B are real numbers. Then

$$\sum_{n=p}^{\infty}(a_n + b_n) = A + B$$

and

$$\sum_{n=p}^{\infty}(a_n - b_n) = A - B.$$

Also, if c is a real constant then,

$$\sum_{n=p}^{\infty} ca_n = cA.$$

Proof. We prove the first statement. Let S_j denote the j-th partial sum for the series $\sum_{n=p}^{\infty}(a_n + b_n)$.

Let V_j denote the j-th partial sum for the series $\sum_{n=p}^{\infty} a_n$.

Let W_j denote the j-th partial sum for the series $\sum_{n=p}^{\infty} b_n$.

Then $S_j = V_j + W_j$. It follows that

$$\lim_{j\to\infty} S_j = \lim_{j\to\infty} V_j + \lim_{j\to\infty} W_j = A + B.$$

It now follows from the definition that

$$\sum_{n=p}^{\infty}(a_n + b_n) = A + B.$$

Remark 13. Let $\{a_n\}_{n=p}^{\infty}$ be a sequence. We form a new sequence $\{b_n\}_{n=1}^{\infty}$ as follows: We set

$$b_1 = a_p$$

$$b_2 = a_{p+1}$$

and for each positive integer j,

$$b_j = a_{p+j-1}.$$

Suppose that L is a real number. It is easy to verify using the definition that the sequence $\{a_n\}_{n=p}^{\infty}$ converges L if and only if the sequence $\{b_n\}_{n=1}^{\infty}$ converges to L. Moreover, the sequences of partial sums corresponding to these two sequences are the same:

$$S_1 = b_1 = a_p$$

$$S_2 = b_1 + b_2 = a_p + a_{p+1}$$

and so on.

So the series $\sum_{n=p}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} b_n$ converges. So, we could replace the one series by the other, when proving something about convergence of series. Therefore, in most of the Theorems and Exercises this Chapter we can assume, without loss of generality, that for the sequence $\{a_n\}_{n=p}^{\infty}$ and the corresponding series $\sum_{n=p}^{\infty} a_n$ that we have $p = 1$. So we only need consider sequences and series of the form $\{a_n\}_{n=1}^{\infty}$ and the corresponding series $\sum_{n=1}^{\infty} a_n$.

□