Advanced Calculus 2, Dr. Block, Lecture Notes, 3-13-2020

First, we recall the theorem proved last time:

23. Theorem. (Change of Variables) Suppose that $g : [c, d] \to [a, b]$ is differentiable with $g(c) = a$ and $g(d) = b$. Suppose also that $g' \in R[c, d]$. Finally, suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then

$$\int_c^d (f \circ g) \cdot g' = \int_a^b f.$$

We give two examples where this theorem may be useful in evaluating an integral. The first example is Exercise7a, Section 6.4 of the text.

Example 1. Evalutate the integral.

$$\int_0^{\frac{1}{2}} \frac{x^3}{\sqrt{1-x^2}} dx$$

Here, we let

$$a = 0,\ b = \frac{1}{2},\ f(x) = \frac{x^3}{\sqrt{1-x^2}}.$$

So the given integral is $\int_a^b f$. To evaluate this integral we let

$$c = 0,\ d = \frac{\pi}{6},\ g(u) = \sin u.$$

Note that all of the hypotheses of the theorem are satisfied. Also,

$$f(g(u)) = \frac{\sin^3 u}{\sqrt{1-\sin^2 u}}$$

and $g'(u) = \cos u$. So by the theorem we have:

$$\int_0^{\frac{1}{2}} \frac{x^3}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{6}} \frac{\sin^3 u}{\sqrt{1-\sin^2 u}} \cdot \cos u du$$

The integral on the right is equal may be simplified by cancelling to obtain:

$$\int_0^{\frac{\pi}{6}} \sin^3 u du$$

This integral may be evaluated by noting that:

$$\sin^3 u = \sin^2 u \cdot \sin u = (1 - \cos^2 u) \cdot (\sin u)$$

In situations where the Change of Variables Theorem applies, we may think of applying the theorem by making a substitution. In this example the substitution is simply described by writing $x = \sin u$, $dx = \cos u du$.

Example 2. Evaluate the integral:

$$\int_0^2 2x\sqrt{x^2 + 1}dx$$

Here we let:

$$c = 0,\ d = 2,\ a = 1,\ b = 5,\ g(x) = x^2 + 1, f(u) = \sqrt{u}$$

Then the given integral is equal to:

$$\int_c^d (f \circ g) \cdot g'$$

Again, we can check that all of the hypotheses of the Change of Variables theorem are satisfied. So this integral is equal to:

$$\int_a^b f = \int_1^5 \sqrt{u}du$$

The latter integral may be evaluated using the power rule.

Again, we may think of applying the theorem by making a substitution. In this example the substitution is simply described by writing

$$u = x^2 + 1,\ du = 2xdx.$$

We now proceed to Theorem 24 in the Chapter 6 course notes:

24. Theorem. Suppose that $g : [c,d] \to [a,b]$ is differentiable, and $f : [a,b] \to \mathbb{R}$ is continuous. Define $H : [c,d] \to \mathbb{R}$ by

$$H(x) = \int_a^{g(x)} f(t)dt.$$

Then H is differentiable and $H'(x) = (f(g(x))) \cdot g'(x)$.

Before giving the proof of Theorem 24 we recall the following theorem:

22. Theorem. Suppose that $f \in R[a,b]$. Define a function $F : [a,b] \to \mathbb{R}$ by $F(x) = \int_a^x f$. Suppose that $c \in [a,b]$. If f is continuous at c, then F is differentiable at c and $F'(c) = f(c)$.

We now prove Theorem 24.

Proof: Consider the function $F : [a,b] \to \mathbb{R}$ given by $F(x) = \int_a^x f$. By Theorem 22, F is differeintable on $[a,b]$ and $F' = f$. Now, for all $x \in [c,d]$ we have

$$H(x) = \int_a^{g(x)} f(t)dt = F(g(x)).$$

By the Chain rule, H is differentiable and

$$H'(x) = (F'(g(x))) \cdot g'(x) = (f(g(x))) \cdot g'(x).$$

Here is an example.

Example 3. If $H(x) = \int_0^{x^3} \sin^2 t dt$, find $H'(x)$.

We apply Theorem 24 with $g(x) = x^3$ and $f(t) = \sin^2 t$. We have

$$H'(x) = (f(g(x))) \cdot g'(x) = (\sin^2(x^3)) \cdot 3x^2.$$