Advanced Calculus 2, Dr. Block, Lecture Notes, 3-16-2020

We begin with Theorem 25 in the Chapter 6 Notes.

25. Theorem. (Integration by parts) Suppose that $f, g : [a, b] \to \mathbb{R}$ are differentiable and $f', g' \in R[a, b]$. Then

$$\int_a^b f \cdot g' = f(b)g(b) - f(a)g(a) - \int_a^b f' \cdot g.$$

We will omit the proof of this theorem. The proof is in the text.

The conclusion of this theorem is sometimes summarized it the following:

$$\int u\,dv = uv - \int v\,du$$

Here is an example. This is Exercise 7k in Section 6.4 of the text.

Example 1. Evalutate the integral.

$$\int_1^2 \ln x dx$$

Solution: We let

$$u = \ln x,\ \ dv = 1\,dx$$
$$du = \frac{1}{x},\ v = x.$$

Then we have

$$\int_1^2 \ln x dx = uv\Big|_1^2 - \int_1^2 1\,dx = u(2)v(2) - u(1)v(1) - 1 = 2\ln 2 - 1.$$

Our next goal is to prove Theorem 26 in the Chapter 6 Course notes. The proof depends on some previous material. We recall this material. First, we call two theorems from the Chapter 6 notes.

Theorem 19. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded, and let $c \in (a, b)$. Then $f \in R[a, b]$ if and only if $f \in R[a, c]$ and $f \in R[c, b]$. In this case

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Theorem 24. Suppose that $g : [c, d] \to [a, b]$ is differentiable, and $f : [a, b] \to \mathbb{R}$ is continuous. Define $H : [c, d] \to \mathbb{R}$ by

$$H(x) = \int_a^{g(x)} f(t)dt.$$

Then H is differentiable and $H'(x) = (f(g(x))) \cdot g'(x)$.

We also recall something discussed in class. Suppose that v and w are real numbers with $v < w$. Suppose also that $f \in R[v, w]$. Let $s, t \in [v, w]$. Then $\int_s^t f$ is defined in all cases as follows.

Case 1. $s < t$. Then $\int_s^t f$ is the ordinary Riemann integral defined at the beginning of the Chapter 6 notes.

Case 2. $s > t$. Then we define $\int_s^t f$ to be $-\int_t^s f$ where the latter integral is the ordinary Riemann integral defined at the beginning of the Chapter 6 notes.

Case 3. $s = t$. Then $\int_s^t f$ is defined to be zero.

With this in mind, we can extend Theorem 19 and Theorem 24 as follows. Note that in the extension of Theorem 19, there is no restriction at all on the order of the points a, b, c on the real line.

Extension of Theorem 19. Suppose that v and w are real numbers with $v < w$. Suppose also that $f \in R[v, w]$. Let $a, b, c \in [v, w]$. Then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Extension of Theorem 24. Suppose that v and w are real numbers with $v < w$, and c and d are real numbers with $c < d$. Suppose that

$g : [c, d] \to [v, w]$ is differentiable, and $f : [v, w] \to \mathbb{R}$ is continuous. Suppose that $a \in [v, w]$. Define $H : [c, d] \to \mathbb{R}$ by

$$H(x) = \int_a^{g(x)} f(t)dt.$$

Then H is differentiable and $H'(x) = (f(g(x))) \cdot g'(x)$.

We now prove Theorem 26.

Theorem 26. If f is continuous on $\mathbb{R}$ and g and h are differentiable on $\mathbb{R}$, then

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x).$$

Proof. Choose $a \in \mathbb{R}$. For example, we may choose $a = 0$. Then by the extension of Theorem 19 we have for all $x \in \mathbb{R}$,

$$\int_a^{h(x)} f(t)dt = \int_a^{g(x)} f(t)dt + \int_{g(x)}^{h(x)} f(t)dt.$$

It follows that

$$\int_{g(x)}^{h(x)} f(t)dt = \int_a^{h(x)} f(t)dt - \int_a^{g(x)} f(t)dt$$

and

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)dt = \frac{d}{dx} \int_a^{h(x)} f(t)dt - \frac{d}{dx} \int_a^{g(x)} f(t)dt.$$

Finally, by the extension of Theorem 24, we have the desired conclusion,

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x).$$

$\square$

For example, we have for all $x \in \mathbb{R}$,

$$\frac{d}{dx} \int_{3x}^{\sin x} e^{t^2} dt = (e^{\sin^2 x})(\cos x) - 3e^{9x^2}.$$