Advanced Calculus 2, Dr. Block, Lecture Notes, 3-20-2020

We begin by recalling Definition 27 in the Chapter 6 course notes.

Definition 27. Let $a \in \mathbb{R}$, and let $f : [a, \infty) \to \mathbb{R}$. Suppose that f is Riemann integrable on $[a, b]$ for each $b > a$. If $\lim_{b\to\infty} \int_a^b f$ exists and is some real number L, then we say that the improper integral $\int_a^\infty f$ converges to L and write $\int_a^\infty f = L$.

We next look at Theorem 29.

Theorem 29. Let $a \in \mathbb{R}$, and let $f : [a, \infty) \to \mathbb{R}$. Suppose that f is nonnegative, and f is Riemann integrable on $[a, b]$ for each $b > a$. Suppose that there exists $M > 0$ such that $\int_a^b f \leq M$ for all $b > a$. Then $\int_a^\infty f$ converges.

We now give a proof of this theorem.

Proof. Consider the function $F : [a, \infty) \to \mathbb{R}$ defined by $F(x) = \int_a^x f$. By our hypothesis, the set $\{F(x) : x \in [a, \infty)\}$ is bounded above. Hence this set has a least upper bound which we call L. We claim that

$$\lim_{x\to\infty} F(x) = L.$$

We now prove this claim. Let $\epsilon > 0$. There exist some real number $t \in [a, \infty)$ such that $F(t) > L - \epsilon$. This is true because no real number less than L can be an upper bound for $\{F(x) : x \in [a, \infty)\}$.

Now since the function f is nonnegative, we see that the function F is increasing. Suppose that $s > t$. Then

$$L - \epsilon < F(t) \leq F(s) \leq L < L + \epsilon.$$

It now follows from the definition of the limit that

$$\lim_{x\to\infty} F(x) = L.$$

This proves the claim. It follows from Definition 27 that the improper integral $\int_a^\infty f$ converges and $\int_a^\infty f = L$.

$\square$

We next give an important test for determining whether improper integrals converge or diverge.

Theorem 30. (Comparison test) Let $a \in \mathbb{R}$, and let $f, g : [a, \infty) \to \mathbb{R}$. Suppose that both f and g are Riemann integrable on $[a, b]$ for each $b > a$. Suppose that $0 \leq f(x) \leq g(x)$ for all $x \geq a$. If $\int_a^\infty g$ converges, then $\int_a^\infty f$ also converges.

Proof. Set $M = \int_a^\infty g$. Then M is a nonnegative real number, and for all $x \geq a$ we have

$$0 \leq f(x) \leq g(x) \leq M.$$

By Theorem 29, $\int_a^\infty f$ converges.

□

It is important to note that the contrapositive of the conclusion of Theorem 3 is, of course, true. This is the contrapositive:

If $\int_a^\infty f$ diverges, then $\int_a^\infty g$ also diverges.

So the comparison test can be used to show that certain improper integrals converge, and certain improper integrals diverge.

In applying the comparison test to solve a given problem, the difficult part is to find another function to compare the given function to. Sometimes we can use a function of the form $\int_1^\infty \frac{1}{x^p}\,dx$. Recall that this improper integral converges if $p > 1$ and diverges if $p \leq 1$.

Finally, we look at the notion of absolute convergence of an improper integral.

Definition 31. Let $a \in \mathbb{R}$, and let $f : [a, \infty) \to \mathbb{R}$. Suppose that f is Riemann integrable on $[a, b]$ for each $b > a$. We say that $\int_a^\infty f$ converges absolutely if and only if $\int_a^\infty |f|$ converges.

We have the following theorem.

Theorem 32. Let $a \in \mathbb{R}$, and let $f : [a, \infty) \to \mathbb{R}$. Suppose that f is Riemann integrable on $[a, b]$ for each $b > a$. If$\int_a^\infty f$ converges absolutely, then $\int_a^\infty f$ converges.

This theorem and a proof are given in the text (Theorem 6.5.13 on page 269). Also, you should look at, Example 6.5.4 on page 269.

Please work on these exercises from the text. I will be going over several of these, but it is always best if you try to do them first.

Section 6.4, page 262 and 263, exercises 5 (all parts), 6 (all parts) 7 (parts a, b, e, h, j, k), 10 (all parts).

Section 6.5, Page 271 and 272, exercises 9 (parts a, b, c, f, g, i, j, k), 10 (parts a, b, c, d, e, f), 11 (Parts a, b), 12, 14.

Also, note that we will have a take home exam which I will post next Friday. You will have one week to complete the exam submit your work to me. The exam will be on Sections 6.4 and 6.5.

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