Advanced Calculus 2, Dr. Block, Lecture Notes, 3-23-2020

We will go over Exercise 10 in section 6.4 in the text. Our solution is based on the following item in the Chapter 6 Course notes:

Corollary 11. Suppose that $f \in R[a,b]$. Let $\{P_n\}$ be a sequence of partitions of $[a,b]$ whose norm converges to zero. Suppose that for each positive integer n, $S(P_n, f)$ is a Riemann sum associated to the partition P_n. Then

$$\lim_{n\to\infty} S(P_n, f) = \int_a^b f.$$

Recall that we used this to solve Exercise 10(a) from section 6.1 in the text. This is what we proved.

Exercise 10(a), Section 6.2. Suppose that $f \in R[0,1]$. Define a sequence $\{a_n\}$ by

$$a_n = \frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}).$$

Then the sequence $\{a_n\}$ converges to $\int_0^1 f$.

□

Recall that the proof of Exercise 10(a) is based on the following:

Let $a = 0$ and $b = 1$. For each n let P_n be the partition of $[0,1]$ obtained by dividing the interval $[0,1]$ into n equal parts. Then

$$S(P_n, f) = a_n.$$

We now go to Exercise 10 in section 6.4 in the text.

Exercise 10(a), Section 6.4. Evaluate the limit:

$$\lim_{n\to\infty} \frac{1}{n^{16}}\sum_{k=1}^{n} k^{15}$$

Solution. We wish to rewrite this limit so that exercise 10(a), Section 6.2 may be applied, for some function f. We have

$$\lim_{n\to\infty} \frac{1}{n^{16}} \sum_{k=1}^{n} k^{15} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{n^{15}} \sum_{k=1}^{n} k^{15}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k^{15}}{n^{15}} = \int_0^1 x^{15} dx = \frac{1}{16}.$$

□

An important step in the solution is recognizing that if we let $f(x) = x^{15}$, than $f(\frac{k}{n}) = \frac{k^{15}}{n^{15}}$.

We now go to the other parts of this Exercise.

Exercise 10(b), Section 6.4. Evaluate the limit:

$$\lim_{n\to\infty} \frac{1}{n^{15}} \sum_{k=1}^{n} k^{15}$$

Solution. We use the part (a) above. We have

$$\lim_{n\to\infty} \frac{1}{n^{15}} \sum_{k=1}^{n} k^{15} = \lim_{n\to\infty} n \cdot (\frac{1}{n^{16}} \sum_{k=1}^{n} k^{15}) = \infty.$$

Here the limit on the right has the form $\infty \cdot \frac{1}{16}$ (using part (a) above). So we conclude (by a form of the product rule for limits, discussed last semester) that the limit is ∞.

Exercise 10(c), Section 6.4. Evaluate the limit:

$$\lim_{n\to\infty} \frac{1}{n^{17}} \sum_{k=1}^{n} k^{15}$$

Solution. We use the part (a) above. We have

$$\lim_{n\to\infty} \frac{1}{n^{17}} \sum_{k=1}^{n} k^{15} = \lim_{n\to\infty} \frac{1}{n} \cdot (\frac{1}{n^{16}} \sum_{k=1}^{n} k^{15}) = 0.$$

Exercise 10(d), Section 6.4. Evaluate the limit:

$$\lim_{n\to\infty} n^{-(\frac{3}{2})} \sum_{k=1}^{n} \sqrt{k}$$

Solution. Using the function $f(x) = \sqrt{x}$, we have

$$\lim_{n\to\infty} n^{-(\frac{3}{2})} \sum_{k=1}^{n} \sqrt{k} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{\frac{k}{n}} = \int_0^1 \sqrt{x}dx = \frac{2}{3}.$$

Exercise 10(e), Section 6.4. Conclude from part (d) that

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \approx \frac{2}{3} n^{\frac{3}{2}}.$$

Solution. This problem depends on the following definition. We say the two sequences $\{a_n\}$ and $\{b_n\}$ are asymptotic (denoted by $a_n \approx b_n$) if and only if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

Now, using part (d) above, we have

$$\lim_{n\to\infty} \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}}{\frac{2}{3} n^{\frac{3}{2}}} = \frac{3}{2} \cdot \lim_{n\to\infty} \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}}{n^{\frac{3}{2}}}$$

$$= \frac{3}{2} \cdot \lim_{n\to\infty} n^{-(\frac{3}{2})} \sum_{k=1}^{n} \sqrt{k} = \frac{3}{2} \cdot \frac{2}{3} = 1.$$

Exercise 10(e), Section 6.4. Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2 + n^2} = \frac{\pi}{4}.$$

Solution. We have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2 + n^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{n^2}{k^2 + n^2}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{(\frac{k}{n})^2 + 1} = \int_0^1 \frac{1}{x^2 + 1} = \arctan x \Big|_0^1 = \frac{\pi}{4}.$$