Advanced Calculus 2, Dr. Block, Lecture Notes, 3-27-2020

Here are some comments regarding Section 6.5 Exercises.

Exercise 9(f), Section 6.5. Evaluate the improper integral:

$$\int_0^{\frac{\pi}{2}} (\sec^2 x - \sec x\ \tan x)dx.$$

We might consider separating the improper integral into two improper integrals and writing:

$$\int_0^{\frac{\pi}{2}} (\sec^2 x - \sec x\ \tan x)dx = \int_0^{\frac{\pi}{2}} \sec^2 x dx - \int_0^{\frac{\pi}{2}} \sec x\ \tan x dx.$$

It is true (and not hard to prove) that if the two integrals on the right converge then this would be valid. However, if the two integrals on the right diverge, then the integral on the left could converge or diverge. So, we will not separate the integral into two integrals as above. We have:

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} (\sec^2 x - \sec x\ \tan x)dx \\
&= \lim_{b\to(\frac{\pi}{2})^-} \int_0^{b} (\sec^2 x - \sec x\ \tan x)dx \\
&= \lim_{b\to(\frac{\pi}{2})^-} \left((\tan x - \sec x)\Big|_0^b \right) \\
&= \lim_{b\to(\frac{\pi}{2})^-} (\tan b - \sec b + 1).
\end{aligned}$$

Now consider

$$\lim_{b\to(\frac{\pi}{2})^-} (\tan b - \sec b).$$

This limit has the form $\infty - \infty$. This is an indeterminate form. We evaluate this limit using L'Hopital's rule as follows:

$$\lim_{b \to (\frac{\pi}{2})^-} (\tan b - \sec b)$$

$$= \lim_{b \to (\frac{\pi}{2})^-} \frac{\sin b - 1}{\cos b} = \lim_{b \to (\frac{\pi}{2})^-} \frac{\cos b}{-\sin b} = 0.$$

It follows that

$$\lim_{b \to (\frac{\pi}{2})^-} (\tan b - \sec b + 1) = 1.$$

We conclude that

$$\int_0^{\frac{\pi}{2}} (\sec^2 x - \sec x \, \tan x) dx = 1.$$

□

We look at another exercise.

Exercise 10(c), Section 6.5. Determine whether the improper integral converges or diverges.

$$\int_1^\infty \frac{1}{\sqrt{1+x^2}} dx$$

We use comparison. Let's try to compare the given improper integral with the the improper integral $\int_1^\infty \frac{1}{x}$, which we know diverges. We see that for $x \geq 1$ we have:

$$1 + x^2 \geq x^2.$$
$$\sqrt{1+x^2} \geq x.$$
$$\frac{1}{\sqrt{1+x^2}} \leq \frac{1}{x}.$$

We can not draw any conclusion from this. We need to have $\geq$ instead of $\leq$ in the last inequality to draw a conclusion. So we modify the above inequalities as follows:

$$1 + x^2 \leq x^2 + x^2 = 2x^2.$$

$$\sqrt{1+x^2} \leq \sqrt{2} \cdot x.$$
$$\frac{1}{\sqrt{1+x^2}} \geq \frac{1}{\sqrt{2}} \cdot \frac{1}{x}.$$

Since the integral $\int_1^\infty \frac{1}{x} dx$ diverges, the integral $\int_1^\infty \frac{1}{\sqrt{2}} \cdot \frac{1}{x} dx$ also diverges. We conclude that the improper integral

$$\int_1^\infty \frac{1}{\sqrt{1+x^2}} dx$$

diverges by comparison.

$\square$