Advanced Calculus 2, Dr. Block, Lecture Notes, 3-30-2020

We begin with the basic definition for Chapter 7.

Definition 1. Let $\{a_n\}_{n=p}^{\infty}$ be a sequence. We define a new sequence $\{S_j\}_{j=1}^{\infty}$ as follows:

$$S_1 = a_p,\ S_2 = a_p + a_{p+1},\ S_3 = a_p + a_{p+1} + a_{p+2},$$

and in general

$$S_j = a_p + a_{p+1} + a_{p+2} + \cdots + a_{p+(j-1)}.$$

This new sequence is called the sequence of partial sums.

We say that the series $\sum_{n=p}^{\infty} a_n$ converges if and only if the sequence $\{S_j\}_{j=1}^{\infty}$ converges to a real number S. In this case we call S the sum of the series and write $\sum_{n=p}^{\infty} a_n = S$.

Example 2. Consider the sequence $\{a_n\}_{n=p}^{\infty}$ where $p = 1$ and $a_n = \frac{1}{n}$. The first four terms of the sequence of partial sums for this sequence are

$$S_1 = \frac{1}{1}$$

$$S_2 = \frac{1}{1} + \frac{1}{2}$$

$$S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3}$$

$$S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

So the sequence of partial sums is formed from the sequence $\{a_n\}_{n=p}^{\infty}$ but is a different sequence. It is important to keep this distinction in mind in this Chapter. Note in particular that in this example, the sequence $\{a_n\}_{n=p}^{\infty}$ is decreasing, while the sequence of partial sums is increasing. Also, while the sequence $\{a_n\}_{n=p}^{\infty}$ converges to zero, it is not obvious at all that the sequence of partial sums converges. In fact, we will see that in this case the sequence of partial sums diverges.

Remark 3. I am calling the first term in the sequence or partial sums, S_1. In the definition in the text, they call the first term in the sequence or partial sums S_p. This will not make any difference.

In proving theorems about whether series converge or not, we may assume without loss of generality that $p = 1$ in Definition 1.

Theorem 4. **(geometric series)** Suppose that p is a nonnegative integer, a is a real constant, and $-1 < r < 1$. Then the series $\sum_{k=p}^{\infty} ar^k$ converges. The sum of the series is $\frac{ar^p}{1-r}$.

Proof. We apply the definition and consider the sequence of partial sums. For each positive integer j, we have

$$S_j = ar^p + ar^{p+1} + \cdots + ar^{p+(j-1)}.$$

Multiplying by $1 - r$ we have

$$\begin{aligned}(1-r)S_j &= ar^p + ar^{p+1} + \cdots + ar^{p+(j-1)} - ar^{p+1} - ar^{p+2} - \cdots - ar^{p+j} \\ &= ar^p - ar^{p+j}.\end{aligned}$$

It follows that

$$S_j = \frac{ar^p - ar^{p+j}}{1-r}.$$

Now, as $-1 < r < 1$, we know that

$$\lim_{j\to\infty} r^j = 0.$$

So we have

$$\lim_{j\to\infty} S_j = \lim_{j\to\infty} \frac{ar^p - ar^{p+j}}{1-r} = \frac{ar^p}{1-r}.$$

Remark 5. Note that in a geometric series, each term after the first term is formed by multiplying the previous term by r. So, r is sometimes called the **ratio** of the geometric series. So the sum of the series can be expressed informally as "the first term of the series divided by one minus the ratio." Note that for the geometric series, we were able to write the j-th partial sum as an expression without the dots, ..., that appear in Definition 1. This is sometimes called a **closed form** for the j-th partial sum. For most series this is not possible.

Let's look at an example.

Problem 6. Determing whether the series converges or diverges. Find the sum if the series converges.

$$\sum_{k=2}^{\infty} \frac{(-2)^{k+1}}{3^k}.$$

Solution. We observe that this is geometric series with first term $\frac{-8}{9}$ and ratio $\frac{-2}{3}$. So the series converges, and the sum is $\frac{-8}{15}$.

□

In the next problem, we see another special type of series where we can get a closed form for the j-th partial sum.

Problem 7. Determing whether the series converges or diverges. Find the sum if the series converges.

$$\sum_{k=1}^{\infty} \frac{1}{4k^2-1}.$$

Solution. We see if the technique of partial fractions might help. We write

$$\frac{1}{4k^2-1} = \frac{A}{2k-1} + \frac{B}{2k+1}.$$

We solve for A and B and obtain $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. So,

$$\frac{1}{4k^2-1} = \frac{\frac{1}{2}}{2k-1} - \frac{\frac{1}{2}}{2k+1}.$$

We use this formula to write the j-th partial sum as follows:

$$S_j = (\frac{\frac{1}{2}}{1} - \frac{\frac{1}{2}}{3}) + (\frac{\frac{1}{2}}{3} - \frac{\frac{1}{2}}{5}) + (\frac{\frac{1}{2}}{7} - \frac{\frac{1}{2}}{5}) + \cdots + (\frac{\frac{1}{2}}{2j+1} - \frac{\frac{1}{2}}{2j-1}).$$

We see that we can simplify by canceling, and we obtain

$$S_j = \frac{\frac{1}{2}}{1} - \frac{\frac{1}{2}}{2j+1}.$$

Thus,

$$\lim_{j\to\infty} S_j = \lim_{j\to\infty}(\frac{\frac{1}{2}}{1} - \frac{\frac{1}{2}}{2j-1})) = \frac{1}{2}.$$

It follows from Definition 1 that the series converges and the sum is $\frac{1}{2}$.

□

Remark 5. Series such as the one in the previous problem are called telescoping.

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